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THEOREMS OF THE RIESZ TYPE FOR CO-FINE CLUSTER SETS OF HARMONIC MORPHISMS

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Introduction

For analytic mappings between Riemann surfaces a theorem of the Riesz type concerning so-called fine neighbourhood filters of Martin boundary points was proved by Constantinescu and Cornea in [6, Satz 16]. This result was extended to Brelot spaces by Ikegami in [11, Theorem 7].

In this paper we prove two more general theorems concerning harmonic morphisms between harmonic spaces. The spaces we consider satisfy the axioms of Constantinescu and Cornea in [8] with some additional assumptions. The notion of a co-fine filter we use was introduced by Sieveking in [15, p. 21].

1. Assumptions and notations

Let X be a noncompact harmonic space in the sense of [8, p. 30]. *We assume throughout this paper that X satisfies the following additional conditions:*

(A1) X is $\mathscr{P}$-harmonic.

(A2) X has a countable base.

(A3) The sheaf of harmonic functions on X has the property of nuclearity.

(A4) There exists an extremal superharmonic function on X which is harmonic.

(A5) There exists a superharmonic function s_0 on X with $\inf s_0(X)>0$.

(A6) 1 is a Wiener function on X.

The conditions (A1)—(A3) make possible the integral representation of positive superharmonic functions ([8, p. 330]). The conditions (A5)—(A6) are related to the theory of Wiener functions presented in [10]. Together with (A1) they imply the existence and harmonicity of the function h_1 ([10, p. 12]). The condition (A4) follows from (A1), (A5) and (A6), unless h_1 is identically zero (cf. [8, Corollary 11.5.3]).

We denote the Martin space of X by M_X and the Riesz space (resp. the Poisson space) of X by R_X (resp. P_X). Then $M_X=R_X\cup P_X$ ([8, p. 312]). By [8, Theorem

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11.5.1] there exists a semi-regular Riesz—Martin kernel $(x, \xi)\mapsto k_\xi(x)$ on M_X. *In what follows we shall keep* k_ξ *fixed.*

For any positive superharmonic function u on X there exists a unique measure μ_u on M_X (in the sense of [8, p. 301]) such that

$$u = \int^{\bullet} k_\xi \, d\mu_u(\xi)$$

(cf. [8, Theorem 11.5.1]). We call μ_u *the canonical measure* of u *In this paper we always assume* u *to be harmonic.* For any $A\subset M_X$ let χ_A be the characteristic function of A. Then

$$\mu_u(R_X) = \int \chi_{R_X} \, d\mu_u = 0$$

(cf. [8, Proposition 11.4.12.c]), and μ_u is a measure on P_X. By (A4) we know that $P_X\neq\emptyset$.

2. The co-fine filters

Let $\mathcal{H}_e$ be the set of extremal positive superharmonic functions on X which are harmonic. Two elements of $\mathcal{H}_e$ are called equivalent if they are proportional. Let $\psi\colon \mathcal{H}_e\to P_X$ be the canonical mapping with respect to this equivalence relation ([8, p. 311]).

Definition 2.1. Let $\xi\in P_X$. The co-fine filter of ξ is

$$\mathcal{F}_\xi = \{E\subset X \,|\, \hat{R}_u^{X\setminus E} \neq u, \text{ for } u\in\psi^{-1}(\{\xi\})\}$$

(cf. [15, p. 21], [12, p. 185]).

By [8, Exercise 11.4.4] $\mathcal{F}_\xi$ is a filter on X. Obviously

$$\mathcal{F}_\xi = \{E\subset X \,|\, \hat{R}_{k_\xi}^{X\setminus E} \neq k_\xi\}.$$

Proposition 2.2. *For every* $\xi\in P_X$ *the filter* $\mathcal{F}_\xi$ *has no cluster points in* X.

Proof. Since k_ξ does not vanish identically, $\mathcal{F}_\xi$ is finer than the filter of the complements of relatively compact subsets of X ([8, Proposition 5.3.5]). □

Let X^* be a resolutive compactification of X (cf. [10, p. 16]) and $\Delta=X^*\setminus X$. We denote by $\mu(A)$ the harmonic measure of a μ-measurable set A of Δ ([14, p. 41]). The function $\mu(A)$ is positive and harmonic.

Lemma 2.3. *Let* X^* *be a resolutive compactification of* X *and* U^* *an open set of* X^*. *If* $\mu(U^*\cap\Delta)$ *does not vanish identically, there exists a* $\xi\in P_X$ *with* $U^*\cap X\in\mathcal{F}_\xi$.

Proof. Let $u=\mu(U^*\cap\Delta)$;

$$\hat{R}_u^{X\setminus U^*}\neq u$$

by [14, Lemma 8.3]. Then

$$u=\int^{\bullet} k_\xi\, d\mu_u(\xi),$$

where μ_u is the canonical measure of u. By [8, Proposition 11.4.12.e]

$$\hat{R}_u^{X\setminus U^*}=\int^{\bullet}\hat{R}_{k_\xi}^{X\setminus U^*}\,d\mu_u(\xi).$$

If $\hat{R}_{k_\xi}^{X\setminus U^*}=k_\xi$ for all $\xi\in P_X$, then $\hat{R}_u^{X\setminus U^*}=u$, which is a contradiction. □

The following theorem is related to e.g. [9, Theorem IV.4].

Theorem 2.4. *Let X^* be a resolutive compactification of X. There exists a set $E\subset\Delta$ with $\mu(E)=0$ such that for every $x\in\Delta\setminus E$ and every neighbourhood U_x^* of x, $U_x^*\cap X\in\mathcal{F}_\xi$ for some $\xi\in P_X$.*

Proof. Let E be the set of all $x\in\Delta$ for which there exists a neighbourhood U_x^* with $\mu(U_x^*\cap\Delta)=0$. Then $E\subset\Delta\setminus\operatorname{supp}\mu_z$ for every $z\in X$. Hence

$$\mu_z(E)\leqq\mu_z(\Delta\setminus\operatorname{supp}\mu_z)=0$$

for every $z\in X$, and $\mu(E)=0$. The assertion of the theorem holds for every $x\in\Delta\setminus E$ by Lemma 2.3. □

Definition 2.5. Let $E\subset X$. We define

$$\mathcal{E}_E=\{\xi\in P_X|X\setminus E\in\mathcal{F}_\xi\}=\{\xi\in P_X|\hat{R}_u^E\neq u,\ u\in\psi^{-1}(\{\xi\})\}.$$

Remark 2.6. Let E and F be subsets of X with $E\subset F$. Then $\mathcal{E}_F\subset\mathcal{E}_E$. This is obvious since $\xi\in\mathcal{E}_F$ implies $\hat{R}_u^F(x)<u(x)$, $u\in\psi^{-1}(\{\xi\})$, for some $x\in X$. As $\hat{R}_u^E\leqq\hat{R}_u^F$, $\xi\in\mathcal{E}_E$.

Remark 2.7. We recall [8, Exercise 11.4.5]. The outlines of the proof are also given in [8].

Let F be a closed set of X and K a compact set of P_X. By Definition 2.5

$$\mathcal{E}_F\cap K=\{\xi\in K|\hat{R}_u^F\neq u,\ u\in\psi^{-1}(\{\xi\})\}.$$

There exists a countable set $B\subset X$ with

$$\mathcal{E}_F\cap K=\{\xi\in K|(\exists x)(x\in B,\hat{R}_u^F(x)<u(x),\ u\in\psi^{-1}(\{\xi\}))\},$$

and $\mathcal{E}_F\cap K$ is a K_σ-set on K. Since $k_\xi\in\psi^{-1}(\{\xi\})$,

$$\mathcal{E}_F\cap K=\bigcup_{x\in B}\{\xi\in K|\hat{R}_{k_\xi}^F(x)\neq k_\xi(x)\}.$$

3. Some lemmas

Let $\mathscr{K}$ be the set of compact sets of P_X ordered by the inclusion relation. Let u be a positive harmonic function on X. Then a family of measures $(\mu_{u_K})_{K\in\mathscr{K}}$ defines the canonical measure μ_u of u ([8, p. 301]).

Lemma 3.1. *Let $A\subset P_X$ and $K\in\mathscr{K}$ such that $A\cap K$ is a Borel set on K. Then the function on X, defined by*

$$x\mapsto \int^* \chi_{A\cap K}(\xi)(k_\xi(x)|K)\,d\mu_{u_K}(\xi),$$

is harmonic.

Proof. Let $\nu_K=\chi_{A\cap K}\mu_{u_K}$. The integral $\int f\,d\nu_K$ exists for every $f\in C(K)$, defined by

$$\int f\,d\nu_K=\int f\chi_{A\cap K}\,d\mu_{u_K}$$

(cf. [3, IV, § 5, 4, Corollaire 3] and [3, IV, § 5, 6, Corollaire 3]). Then $f\mapsto\int f\,d\nu_K$ is a continuous linear functional on $C(K)$, and ν_K is a measure on K. By [8, Proposition 11.4.12.c]

$$\int^* (k_\xi|K)\,d\nu_K(\xi)=\int^* \chi_{A\cap K}(\xi)(k_\xi|K)\,d\mu_{u_K}(\xi)$$

is harmonic. □

Let $A\subset P_X$. We define on X a function $\omega_u(A)$ by

$$x\mapsto\int^* \chi_A(\xi)\,k_\xi(x)\,d\mu_u(\xi).$$

Lemma 3.2. *Let $A\subset P_X$. The function $\omega_u(A)$ is harmonic.*

Proof. Let $K\in\mathscr{K}$, $x\in X$ and

$$\tau_K^x=(k_\xi(x)|K)\mu_{u_K}.$$

The function $k_\xi(x)|K$ is μ_{u_K}-measurable. Then the integral

$$\int f(\xi)(k_\xi(x)|K)\,d\mu_{u_K}(\xi)=\int f(\xi)\,d\tau_K^x(\xi)$$

exists for every $f\in C(K)$ (cf. [3, IV, § 5, 6, Théorème 5] and [3, IV, § 5, 3, Corollaire 5]). Hence τ_K^x is a measure on K, and the family $(\tau_K^x)_{K\in\mathscr{K}}$ defines a measure on P_X for every $x\in X$.

For every $K\in\mathscr{K}$ and $x\in X$

$$\int^* \chi_{A\cap K}(\xi)\,d\tau_K^x(\xi)=\inf_{U\supset A\cap K}\int^* \chi_U(\xi)\,d\tau_K^x(\xi),$$

where U is an open set of K ([3, IV, § 1, 4, Proposition 19]). By Lemma 3.1

$$x\mapsto\int^* \chi_U(\xi)\,d\tau_K^x(\xi)$$

is harmonic. Thus

$$x \mapsto \int^{*} \chi_{A\cap K}(\xi)\, d\tau_K^x(\xi)$$

is harmonic ([8, Proposition 1.1.2]). Further,

$$\omega_u(A) = \sup_{K\in\mathcal{K}} \int^{*} \chi_{A\cap K}(\xi)\, d\tau_K^x(\xi)$$

is dominated by u and hence harmonic. □

We say that a property holds μ_u-a.e. on P_X, if it holds for every $\xi\in P_X$ except for a set A with

$$\mu_u^{\bullet}(A) = \int^{\bullet} \chi_A\, d\mu_u = 0.$$

Lemma 3.3. *Let f, g and h be positive numerical functions on P_X.*

a) *If $f=g$ μ_u-a.e., then*

$$\int^{\bullet} f\, d\mu_u = \int^{\bullet} g\, d\mu_u.$$

b) *If $f\leqq g$, then*

$$\int^{\bullet} f\, d\mu_u \leqq \int^{\bullet} g\, d\mu_u.$$

c)

$$\int^{\bullet} (f+g)\, d\mu_u \leqq \int^{\bullet} f\, d\mu_u + \int^{\bullet} g\, d\mu_u.$$

d) *If g and h are μ_u-measurable, then*

$$\int^{\bullet} f(g+h)\, d\mu_u = \int^{\bullet} fg\, d\mu_u + \int^{\bullet} fh\, d\mu_u.$$

Proof. The proof in [4, V, § 1, 1, Propositions 1 and 2] carries over to our case. □

Lemma 3.4. *Let E be a closed set of X. Then $\hat{R}_u^E = u$ if and only if*

$$\mu_u^{\bullet}(\mathcal{E}_E) = \int^{\bullet} \chi_{\mathcal{E}_E}\, d\mu_u = 0$$

(cf. [9, Theorem II.2]).

Proof. Let $\mu_u^{\bullet}(\mathcal{E}_E)=0$. For any $x\in X$,

$$\hat{R}_u^E(x) = \int^{\bullet} \hat{R}_{k_\xi}^E(x)\, d\mu_u(\xi)$$

(cf. [8, Proposition 11.4.12.e]). By Definition 2.5 and Lemma 3.3.a

$$\hat{R}_u^E = \int^{\bullet} \hat{R}_{k_\xi}^E\, d\mu_u(\xi) = \int^{\bullet} \chi_{P_X\setminus\mathcal{E}_E} \hat{R}_{k_\xi}^E\, d\mu_u(\xi)$$

$$= \int^{\bullet} \chi_{P_X\setminus\mathcal{E}_E} k_\xi\, d\mu_u(\xi) = \int^{\bullet} k_\xi\, du_u(\xi) = u.$$

Secondly, let $\hat{R}_u^E = u$. Then

$$\int^{\bullet} \hat{R}_{k_\xi}^E\, d\mu_u(\xi) = \int^{\bullet} k_\xi\, d\mu_u(\xi).$$

The functions $\hat{R}^E_{k_\xi}$ and $k_\xi - \hat{R}^E_{k_\xi}$ are μ_u-measurable (cf. [8, Proposition 11.4.12.e] and [3, IV, § 5, 3, Corollaire 3]). By Lemma 3.3.d

$$\int^{\bullet} k_\xi \, d\mu_u(\xi) = \int^{\bullet} (k_\xi - \hat{R}^E_{k_\xi}) \, d\mu_u(\xi) + \int^{\bullet} \hat{R}^E_{k_\xi} \, d\mu_u(\xi).$$

Hence

$$\int^{\bullet} (k_\xi - \hat{R}^E_{k_\xi}) \, d\mu_u(\xi) = 0.$$

Then for every $x \in X$ and for every $K \in \mathscr{K}$

$$\int^{*} (k_\xi(x) - \hat{R}^E_{k_\xi}(x)) \, d\mu_{u_K}(\xi) = 0.$$

We obtain

$$k_\xi(x) = \hat{R}^E_{k_\xi}(x)$$

for all $\xi \in K$, except for a set of μ_{u_K}-measure zero ([3, IV, § 2, 3, Théorème 1]). By Remark 2.7

$$\mathscr{E}_E \cap K = \bigcup_{x \in B} \{\xi \in K \mid \hat{R}^E_{k_\xi}(x) \neq k_\xi(x)\},$$

where B is a countable subset of X. Hence

$$\mu_u^{\bullet}(\mathscr{E}_E) = \sup_{K \in \mathscr{K}} \mu^*_{u_K}(\mathscr{E}_E \cap K) = 0. \quad \square$$

Lemma 3.5. *Let E be a closed set of X. Then $\omega_u(P_X \setminus \mathscr{E}_E)$ is the greatest positive harmonic minorant of $\hat{R}^E_u$* (cf. [9, Corollary on p. 327]).

Proof. By Lemma 3.2, $\omega_u(P_X \setminus \mathscr{E}_E)$ is positive and harmonic. By Lemma 3.3.b

$$\hat{R}^E_u = \int^{\bullet} \hat{R}^E_{k_\xi} \, d\mu_u(\xi) \geqq \int^{\bullet} \chi_{P_X \setminus \mathscr{E}_E} \hat{R}^E_{k_\xi} \, d\mu_u(\xi)$$
$$= \int^{\bullet} \chi_{P_X \setminus \mathscr{E}_E} k_\xi \, d\mu_u(\xi) = \omega_u(P_X \setminus \mathscr{E}_E).$$

Hence $\omega_u(P_X \setminus \mathscr{E}_E)$ is a minorant of $\hat{R}^E_u$.

Let u' be the greatest harmonic minorant of $\hat{R}^E_u$. Then

$$u' = \hat{R}^E_{u'}$$

by [8, Exercise 5.3.2]. Hence $\mu^{\bullet}_{u'}(\mathscr{E}_E) = 0$ by Lemma 3.4. We have

$$u' \leqq \hat{R}^E_u \leqq u.$$

For positive harmonic functions v the mapping $\mu_v \mapsto v$ is an additive injection by [8, Corollary 11.4.4.c]. Then

$$\mu_{u'} \leqslant \mu_u.$$

Thus

$$u' = \int^{\bullet} \chi_{P_X \setminus \mathscr{E}_E} k_\xi \, d\mu_{u'}(\xi) \leqq \int^{\bullet} \chi_{P_X \setminus \mathscr{E}_E} k_\xi \, d\mu_u(\xi)$$
$$= \omega_u(P_X \setminus \mathscr{E}_E).$$

So, $u' = \omega_u(P_X \setminus \mathscr{E}_E)$. $\square$

Theorem 3.6. *Let* E *be a closed set of* X. *Then* $\hat{R}^E_{\omega_u(\mathscr{E}_E)}$ *is a potential.*

Proof. For every $K \in \mathscr{K}$ the functions $\chi_{\mathscr{E}_E}|K$ and $\chi_{P_X \setminus \mathscr{E}_E}|K$ are μ_{u_K}-measurable by Remark 2.7 and [3, IV, § 5, 4, Corollaire 3]. Thus $\chi_{\mathscr{E}_E}$ and $\chi_{P_X \setminus \mathscr{E}_E}$ are μ_u-measurable. Then by Lemma 3.3.d

$$u = \int^{\bullet} \chi_{P_X \setminus \mathscr{E}_E} k_\xi \, d\mu_u(\xi) + \int^{\bullet} \chi_{\mathscr{E}_E} k_\xi \, d\mu_u(\xi).$$

Hence

$$\hat{R}^E_u = \hat{R}^E_{\omega_u(P_X \setminus \mathscr{E}_E) + \omega_u(\mathscr{E}_E)} = \hat{R}^E_{\omega_u(P_X \setminus \mathscr{E}_E)} + \hat{R}^E_{\omega_u(\mathscr{E}_E)}$$

([8, Theorem 4.2.1]). By Lemma 3.5

$$\omega_u(P_X \setminus \mathscr{E}_E) = \hat{R}^E_{\omega_u(P_X \setminus \mathscr{E}_E)}$$

is the greatest positive harmonic minorant of $\hat{R}^E_u$. Hence $\hat{R}^E_{\omega_u(\mathscr{E}_E)}$ is a potential. □

Remark 3.7. The assumptions (A5) and (A6) were not used in this section.

4. Definitions

Let X and X' be two noncompact harmonic spaces. Let $\varphi \colon X \to X'$ be a continuous mapping. We denote by X'^* an arbitrary compactification of X'. For $\xi \in P_X$, let $\mathscr{F}_\xi$ be the co-fine filter of ξ.

Definition 4.1. The co-fine cluster set of φ at ξ is

$$\varphi^{\wedge}(\xi) = \bigcap_{U \in \mathscr{F}_\xi} \overline{\varphi(U)},$$

where the closure is taken in X'^* (cf. [7, p. 146], [11, Theorem 7]).

Lemma 4.2. *Let* $\varphi \colon X \to X'$ *be a continuous mapping and* $\xi \in P_X$. *If* U'^* *is an open set of* X'^* *with* $\varphi^{\wedge}(\xi) \subset U'^*$, *then* $\varphi^{-1}(U'^* \cap X') \in \mathscr{F}_\xi$.

Proof. Cf. [7, Hilfssatz 14.1]. □

The assumptions about X imply the existence of the positive harmonic function h_1 on X. For any set $A \subset P_X$ we denote by $\omega(A)$ the function $\omega_{h_1}(A)$ on X, i.e.

$$x \mapsto \int^{\bullet} \chi_A(\xi) k_\xi(x) \, d\mu_1(\xi),$$

where $\mu_1 = \mu_{h_1}$. By Lemma 3.2 $\omega(A)$ is harmonic.

Remark 4.3. For every $x \in X$ we can regard $k_\xi(x)\mu_1$ as a measure on P_X (cf. the proof of Lemma 3.2).

We say that A is *of harmonic measure zero* if $\omega(A)$ equals zero.

Remark 4.4. For the case of a Brelot space X with some additional assumptions a resolutive Martin compactification exists for X. Let ω_x, $x \in X$, denote the harmonic (Radon) measure defined on the Martin boundary by the solution of the Dirichlet problem. Then for every boundary set A

$$\int^{*} \chi_A(\xi)\, d\omega_x(\xi) = \int^{*} \chi_A(\xi)\, k_\xi(x)\, d\mu_1(\xi)$$

([7, p. 140], [11, p. 262]). This motivates our definition.

Remark 4.5. If $\{A_n\}_{n\in N}$ is a sequence of sets with $A_n \subset P_X$ and $A = \bigcup_{n\in N} A_n$,

$$\omega(A) \leqq \sum_{n\in N} \omega(A_n).$$

This follows from [3, IV, § 1, 4, Proposition 18].

5. Theorems of the Riesz type

In this section we consider a harmonic morphism $\varphi: X \to X'$. The target space X' *is supposed to satisfy* (A5) *and* (A6). We also assume that X' *is an MP-set.* By [10, p. 21] X' has resolutive compactifications.

The concept of a harmonic morphism (earlier also called a harmonic mapping, e.g. in [13]) is defined as in [13, Definition 2.2]. The definition of a polar set in a resolutive compactification of X' can be found in [14, Definitions 6.1 and 6.7]. For the notion of a locally polarly nonconstant mapping we refer to [14, Definition 2.1].

Lemma 5.1. *Let B be a semi-polar set of X and u a hyperharmonic function on X with $u \geqq 0$ on $X \setminus B$. Then $u \geqq 0$.*

Proof. X endowed with the fine topology is a Baire space ([8, Corollary 5.1.1]). By [2, p. 193] and [8, Corollary 6.3.3] $X \setminus B$ is finely dense in X. The fine continuity of u then implies $u \geqq 0$ on X. □

We proceed to the proofs of our two main theorems. Theorem 5.3 is similar to [7, Satz 14.1] and [11, Theorem 7]. In the axiomatics of [8] corresponding results for neighbourhood filters of an ideal boundary point in a resolutive compactification of X were proved in [14, Theorems 8.5 and 8.8].

Theorem 5.2. *Let $\varphi: X \to X'$ be a locally polarly nonconstant harmonic morphism. Let $A \subset P_X$ and let*

$$A' = \bigcup_{\xi \in A} \varphi^{\wedge}(\xi)$$

be polar in a resolutive compactification X'^* *of* X'. *We suppose that there exists an open set* W'^* *of* X'^* *satisfying the following three conditions:*

(a) $W'=W'^*\cap X'$ *is a* $\mathcal{P}$*-set.*

(b) $A'\subset W'^*$.

(c) *Either* X' *is elliptic or* $A'\cap X'$ *is contained in a* σ*-compact set of* W'.
Then A *is of harmonic measure zero.*

Proof. Let

$$W=\varphi^{-1}(W'),\quad F=X\setminus W.$$

We choose an $x\in W$ and a positive hyperharmonic function u' on W' with $(u'\circ\varphi)(x)<\infty$ and

$$\lim_{W'\ni y'\to A'\cap\overline{W'}} u'(y')=\infty.$$

By [14, Theorem 2.4 and Lemma 6.9] such a function exists for every $x\in W$, outside a polar set of W. Let $\alpha>0$ be arbitrary and define

$$W_\alpha'^*=\{x'\in W'^*|\liminf_{W'\ni y'\to x'} u'(y')>\alpha\}.$$

Then $W_\alpha'^*$ is open in X'^*. Let

$$F_\alpha=X\setminus\varphi^{-1}(W_\alpha'^*\cap X').$$

Hence $F_\alpha\supset F$. Since $A'\subset W_\alpha'^*$, by Lemma 4.2 $\varphi^{-1}(W_\alpha'^*\cap X')\in\mathcal{F}_\xi$ and

$$\hat{R}_{k_\xi}^{F_\alpha}\neq k_\xi$$

for every $\xi\in A$. Denoting $\mathcal{E}_\alpha=\mathcal{E}_{F_\alpha}$ we obtain

$$A\subset\mathcal{E}_\alpha.$$

Let

$$\omega_\alpha=\omega(\mathcal{E}_\alpha)=\dot{\int}\chi_{\mathcal{E}_\alpha}(\xi)k_\xi\,d\mu_1(\xi).$$

Then $0\leqq\omega_\alpha\leqq h_1$, and ω_α is harmonic. Theorem 3.6 implies that

$$p_\alpha=\hat{R}_{\omega_\alpha}^{F_\alpha}$$

is a potential.

Let $\{U_n\}_{n\in N}$ be an exhaustion of X by relatively compact open sets. Then

$$\lim_{n\to\infty} R_{p_\alpha}^{X\setminus\bar{U}_n}=0$$

by [1, Korollar 2.4.5] (which carries over to our case). Hence we can assume that the positive hyperharmonic function

$$s=\sum_{n\in N} R_{p_\alpha}^{X\setminus\bar{U}_n}$$

is finite at x. There exists a positive hyperharmonic function u on W such that $u(x)<\infty$ and $\overline{H}^W_{\omega_\alpha}+\varepsilon u\in\overline{\mathscr{U}}^W_{\omega_\alpha}$ for every $\varepsilon>0$ (cf. [8, Exercise 2.4.8]). We define for every $\varepsilon>0$ a hyperharmonic function u_ε on W by

$$u_\varepsilon = \overline{H}^W_{\omega_\alpha}+\varepsilon u-\omega_\alpha+\varepsilon s+\alpha^{-1}(u'\circ\varphi).$$

By the $\mathscr{P}$-harmonicity of X there exists a potential p on X with

$$0 \leqq h_1 \leqq |h_1-1|+1 \leqq p+1$$

([10, Proposition 1.4.5]). Since $\omega_\alpha\leqq h_1$,

$$u_\varepsilon+p \geqq 0 \tag{1}$$

on $\varphi^{-1}(W'^*_\alpha\cap X')$. For every $n\in N$

$$s \geqq np_\alpha \tag{2}$$

on $X\setminus\overline{U}_n$. There exists a semi-polar set B of X such that on $F_\alpha\setminus B$

$$p_\alpha = \hat{R}^{F_\alpha}_{\omega_\alpha} = \omega_\alpha \tag{3}$$

(cf. [8, Corollary 6.3.6]). Let $n_\varepsilon>1/\varepsilon$. Then by (2) and (3)

$$\varepsilon s \geqq \omega_\alpha \tag{4}$$

on $((X\setminus\overline{U}_{n_\varepsilon})\cap F_\alpha)\setminus B$. From (1) and (4) we deduce the existence of a compact set K_ε of X such that (1) holds on $W\setminus B$ outside K_ε. Since $B\cap(W\setminus K_\varepsilon)$ is semi-polar in $W\setminus K_\varepsilon$, (1) is valid on $W\setminus K_\varepsilon$ (Lemma 5.1).

If $\partial W\neq\emptyset$, then

$$\liminf_{W\ni z\to y}\left(\overline{H}^W_{\omega_\alpha}(z)+\varepsilon u(z)\right) \geqq \omega_\alpha(y)$$

for every $y\in\partial W$. Since W is an MP-set, (1) holds on W.

We recall that $u(x)$ and $s(x)$ are finite. Further, $\overline{H}^W_{\omega_\alpha}=\hat{R}^F_{\omega_\alpha}$ on W ([8, Proposition 5.3.3]). As ε was arbitrary,

$$\omega_\alpha(x) \leqq \hat{R}^F_{\omega_\alpha}(x)+p(x)+\alpha^{-1}(u'\circ\varphi)(x).$$

Since $A\subset\mathscr{E}_\alpha\subset\mathscr{E}_F$ (Remark 2.6)

$$\omega(A)(x) \leqq \hat{R}^F_{\omega(\mathscr{E}_F)}(x)+p(x)+\alpha^{-1}(u'\circ\varphi)(x).$$

Observing that $(u'\circ\varphi)(x)<\infty$ and letting $\alpha\to\infty$ we obtain

$$\omega(A)(x) \leqq \hat{R}^F_{\omega(\mathscr{E}_F)}(x)+p(x). \tag{5}$$

The relation (5) holds for every $x\in W$, outside a polar set of W. By Lemma 5.1, (5) holds everywhere on W. On $F=X\setminus W$ we obtain $\omega(\mathscr{E}_F)=\hat{R}^F_{\omega(\mathscr{E}_F)}$, outside a semi-polar set of X. Hence (5) holds on X, outside a semi-polar set of X. Since $\omega(A)$ is harmonic and $\hat{R}^F_{\omega(\mathscr{E}_F)}+p$ is a potential by Theorem 3.6, we obtain $\omega(A)=0$ by Lemma 5.1. □

In the following theorem let P' be the set of points of X' where all potentials on X' vanish. This set was introduced in [5, p. 901]. All three possibilities $P'=\emptyset$ (in which case X' is $\mathcal{P}$-harmonic), $\emptyset\neq P'\neq X'$, and $P'=X'$ may occur in the theorem.

Theorem 5.3. *Let* $\varphi\colon X\to X'$ *be a locally polarly nonconstant harmonic morphism. Let* $A\subset P_X$ *and let*

$$A'=\bigcup_{\xi\in A}\varphi\hat{\ }(\xi)$$

be polar in a resolutive compactification X'^* *of* X'. *Then* A *is of harmonic measure zero if one of the following two conditions holds:*

(a) X' *is elliptic and connected.*

(b) X' *has a countable base and* P' *has a finite number of components.*

Proof. If $P'=\emptyset$, the conditions of Theorem 5.2 are valid.

If $P'\neq\emptyset$, the proof of [14, Theorem 8.8] carries over to the present situation. This follows by Remark 4.5, since $A\subset P_X$, $A=\cup_{n\in N}A_n$ implies

$$\omega(A)\leqq\sum_{n\in N}\omega(A_n).\qquad\square$$

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