Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 6, 1981, 89—94

EXTENSION OF QUASISYMMETRIC AND LIPSCHITZ EMBEDDINGS OF THE REAL LINE INTO THE PLANE

PEKKA TUKIA

1. For the definitions of the terms *Lipschitz embedding, Lipschitz homeomorphism* and *L-embedding,* we refer to [7, Section 1].

If $f\colon X\to R^2$, $X\subset R^2$, is an embedding, we say that f is a *quasisymmetric* embedding if there is $H\geqq 1$ such that

$$|f(b)-f(x)|\leqq H|f(a)-f(x)|$$

provided $a, b, x\in X$ and $|b-x|\leqq|a-x|$; cf. [8], where f was said to be weakly quasisymmetric if this is true. We say also that f is *H-quasisymmetric* if we wish to emphasize H. If $X=R=f(X)$, this definition coincides with the usual definition of a quasisymmetric (and H-quasisymmetric) mapping except that f may be also decreasing. Note that an L-embedding is always L^2-quasisymmetric.

2. We prove in this paper the following theorem whose quasiconformal part is more or less known [4, 6], although we have not found it in an exactly equivalent form.

Theorem. *Let $f\colon R\to R^2$ be an H-quasisymmetric embedding. Then there is an extension F of f to a K-quasiconformal homeomorphism of R^2 which is continuously differentiable outside R and where K depends only on H. If f is an L-embedding, then F is an L'-homeomorphism where L' depends only on L.*

We proved a variant of the Lipschitz part of the theorem in [7], where the proof was an explicit geometric construction, based on a certain compactness property of Lipschitz embeddings. The proof we offer here is analytic in character, the main tools being Riemann's mapping theorem and the Beurling—Ahlfors extension of a quasisymmetric mapping. This proof is also much shorter than the proof of [7]. However, it is questionable whether it is fundamentally simpler, so powerful are the theorems on which it is based.

We remark that the construction of [7] would, with minor modifications and with some results of [8], have given also the quasiconformal part of the theorem. Then an easy argument (1° of Section 12 of [7]) gives also the Lipschitz part. How-

doi:10.5186/aasfm.1981.0624

ever, we were not aware of this possibility when writing [7]. If this had been done, we would have had as a corollary a new, geometric proof of the following theorem, proved by Ahlfors [1] using analytic methods. (For the definition of the term "bounded turning", see Section 4.)

Let $S\subset R^2$ *and assume that* $S\cup\{\infty\}$ *is a topological circle. Then* S *is the image* $g(R)$ *of the real line under a quasiconformal map* g *of* R^2 *if and only if* S *is of bounded turning.*

This would follow since by [8, 4.9], combined with a normal family argument using [8, Section 3], there is a quasisymmetric embedding $f\colon R\to R^2$ such that $f(R)=S$.

We now proceed to the proof. For further discussion, see Section 7.

3. *The Beurling—Ahlfors extension.* Let $g\colon R\to R$ be an increasing homeomorphism and let $U=\{z\in C\colon \operatorname{im} z>0\}$. One can extend g to a homeomorphism B_g of $\operatorname{cl} U$ by setting $B_g=g$ in R and for $(x,y)\in U$

$$B_g(x,y)=\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta)\,i, \tag{1}$$

where

$$\alpha(x,y)=\int_0^1 g(x+ty)\,dt,\quad \beta(x,y)=\int_0^1 g(x-ty)\,dt. \tag{2}$$

This is the Beurling—Ahlfors extension [2, Section 6] of g. It is always a homeomorphism of $\operatorname{cl} U$ and it is continuously differentiable in U. In addition, if g is H-quasisymmetric, B_g is $8H$-quasiconformal (Reed [3]). We need the following result. Let $\varphi\colon R^2\to R^2$ be affine, $\varphi(z)=az+b$, where $a,b\in R$, $a>0$. Then $\varphi(R)=R$ and we have

$$B_{g\circ\varphi}=B_g\circ\varphi,\quad\text{and}\quad B_{\varphi\circ g}=\varphi\circ B_g, \tag{3}$$

where we have denoted $\varphi|R$ also by φ. Equations (3) are an immediate consequence of (1) and (2).

If $A\colon R^2\to R^2$ is a linear mapping, we denote

$$|A|=\max_{|x|=1}|A(x)|$$

$$l(A)=\min_{|x|=1}|A(x)|.$$

A mapping is *normalized* if it fixes 0, 1 and ∞. (If a map g is undefined at ∞, we set $g(\infty)=\infty$.) The derivative of a map h is Dh. Now we can state

Lemma 1. *Let* $g\colon R\to R$ *be* H*-quasisymmetric. Then*

(a) *there is* $L_1=L_1(H)\geqq 1$ *such that* $B_g|U$ *is an* L_1*-homeomorphism of* U *in the hyperbolic metric of* U, *and*

(b) *there is* $L_2=L_2(H)\geqq 1$ *such that, if* g *is in addition normalized,*

$$1/L_2 \leqq l\big(DB_g(1/2, 1/2)\big) \leqq |DB_g(1/2, 1/2)| \leqq L_2.$$

Proof. For (a) see Ahlfors [1, p. 293]. Let $(x, y)\in U$ and let $(x', y')=B_g(x, y)$. Then (a) implies

$$1/L_1 \leqq yl\big(DB_g(x, y)\big)/y' \leqq y|DB_g(x, y)|/y' \leqq L_1. \tag{4}$$

Now, if g is H-quasisymmetric and normalized, B_g is $8H$-quasiconformal and normalized. Then the compactness properties of quasiconformal mappings imply that there is $M=M(H)\geqq 1$ such that $\operatorname{im} B_g(1/2, 1/2)\in[1/M, M]$ whenever g is H-quasisymmetric and normalized. It follows by (4) that (b) is true with $L_2=2ML_1$.

4. Let $c\geqq 1$. We say that an (open or closed) arc $J\subset R^2$ is of c-bounded turning if, whenever $a, b\in J$,

$$\operatorname{diam}(J') \leqq c|a-b|,$$

where $J'\subset J$ is the subarc with endpoints a and b. We let $\mathscr{F}_c$ be the family of normalized embeddings g: cl $U\to R^2$ such that $g|U$ is conformal and $g(R)$ is of c-bounded turning. There is $K=K(c)$ such that every $g\in\mathscr{F}_c$ can be extended to a K-quasiconformal homeomorphism of R^2 (cf. [1]). It follows that $\mathscr{F}_c$ is compact: if $g_1, g_2, \ldots\in\mathscr{F}_c$, there is a subsequence $g_{n(1)}, g_{n(2)}, \ldots$ such that there is $\lim g_{n(i)}\in\mathscr{F}_c$.

Lemma 2. *Let* $X\subset U$ *be compact and let* $c\geqq 1$. *Then there is* $L_3=L_3(X, c)\geqq 1$ *such that the derivative* $g'(z)$ *satisfies*

$$1/L_3 \leqq |g'(z)| \leqq L_3 \tag{5}$$

for every $g\in\mathscr{F}_c$ *and* $z\in X$.

Proof. We choose a closed path γ: $[0, 1]\to U$ such that $\gamma([0, 1])\cap X=\emptyset$ and that for every $z\in X$ the index of z with respect to γ is 1. Then

$$g'(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)\,d\zeta}{(\zeta-z)^2} \tag{6}$$

for every $z\in X$ and $g\in\mathscr{F}_c$. Now (6) and the compactness of $\mathscr{F}_c$ imply immediately that there is L_3 for which the right inequality of (5) is valid. If there is no such L_3 for which the left inequality is also valid, we can find functions $g, g_1, g_2, \ldots\in\mathscr{F}_c$ and points $z, z_1, z_2, \ldots\in X$ such that $\lim g_i=g$, $\lim z_i=z$ and $\lim g_i'(z_i)=0$. However, $g'(z)\neq 0$. This and (6) now imply a contradiction.

5. Let g: $R\to R^2$ be a normalized embedding such that if we set $g(\infty)=\infty$, g is continuous at ∞. Let C_1 and C_2 be the components of $R^2\setminus g(R)$. We choose the notation in such a way that there is an orientation preserving homeomorphism G: cl $U\to$ cl C_1 extending g. Now we construct a canonical extension

F_g of g to a homeomorphism of R^2. We give the definition of F_g only in the upper half-plane U; the case of the lower half-plane is analogous.

There is a well-defined homeomorphism A_g: cl $U \to$ cl C_1 which is normalized and conformal in U. Consider the map $\bar{g}=A_g^{-1}\circ g$ which is an increasing homeomorphism of R; thus the Beurling—Ahlfors extension $B_{\bar{g}}$ is a homeomorphism of cl U extending $\bar{g}$. We set now

$$F_g|\text{cl } U = A_g \circ B_{\bar{g}}, \tag{7}$$

F_g being defined similarly in $R^2\setminus U$. We have $F_g|R=A_g\circ\bar{g}=A_g\circ A_g^{-1}\circ g=g$; thus F_g is indeed an extension of g.

Let then φ: $R\to R$ be an increasing affine map and ψ: $R^2\to R^2$ be a conformal affine map, and assume that $\psi g\varphi$ is normalized. We show

$$F_{\psi g\varphi} = \psi\circ F_g\circ\varphi \tag{8}$$

(where on the right side φ is extended to the unique conformal affine map extending φ). This is the fundamental property of F_g which makes the proof of our theorem possible. We observe first that, if $h=\psi g\varphi$,

$$A_h = \psi\circ A_g\circ\bar{\varphi} \tag{9}$$

where $\bar{\varphi}=A_g^{-1}\circ\psi^{-1}\circ A_h$ is a conformal affine map of cl U. Let $\bar{h}=A_h^{-1}\circ h$. Then

$$\bar{h} = \bar{\varphi}^{-1}\circ A_g^{-1}\circ\psi^{-1}\circ\psi\circ g\circ\varphi = \bar{\varphi}^{-1}\circ A_g^{-1}\circ g\circ\varphi = \bar{\varphi}^{-1}\circ\bar{g}\circ\varphi.$$

Consequently, by (3), $B_{\bar{h}}=\bar{\varphi}^{-1}\circ B_{\bar{g}}\circ\varphi|\text{cl } U$. Thus, by (9),

$$F_h|\text{cl } U = A_h\circ B_{\bar{h}} = \psi\circ A_g\circ\bar{\varphi}\circ\bar{\varphi}^{-1}\circ B_{\bar{g}}\circ\varphi|\text{cl } U = \psi\circ F_g\circ\varphi|\text{cl } U.$$

In the same manner one sees that (8) is valid also in $R^2\setminus U$.

If now g is a quasisymmetric embedding of R into R^2, necessarily $\lim_{|t|\to\infty}|g(t)|=\infty$ ([8, 2.1 and 2.16]) and thus the above discussion is valid for all normalized quasisymmetric maps. Note that, if g is H-quasisymmetric, then $g(R)$ is of $2H$-bounded turning.

Lemma 3. *Let $H\geqq 1$. There is a number $L_4=L_4(H)\geqq 1$ such that if f: $R\to R^2$ is a normalized H-quasisymmetric embedding, then the extension F_f is continuously differentiable in $R^2\setminus R$ and satisfies*

$$\frac{|f(x+y)-f(x-y)|}{|2yL_4|} \leqq l\big(DF_f(x, y)\big) \leqq |DF_f(x, y)| \leqq \frac{|f(x+y)-f(x-y)|}{|2y|}L_4$$

for $(x, y)\in R^2\setminus R$.

Proof. Choose $(x, y)\in R^2\setminus R$. We can assume that $y>0$. Let φ be the increasing affine map of R such that $\varphi(0)=x-y$ and $\varphi(1)=x+y$; we denote by φ also the extension of φ to a conformal affine map of R^2. Let ψ be the conformal affine map of R^2 such that $\psi(f(x-y))=0$ and $\psi(f(x+y))=1$. Then $g=\psi f\varphi$ is normalized and by (8)

$$DF_f(x, y) = D\psi^{-1}\circ DF_g(1/2, 1/2)\circ D\varphi^{-1}.$$

Now $D\varphi$ and $D\psi$ (which are constants) are similarities, and $D\varphi$ multiplies distances by the factor $2y$ and $D\psi$ multiplies them by the factor $1/|f(x+y)-f(x-y)|$. Thus the lemma is true if we can show that there is $L_4=L_4(H)$ such that

$$(10)\qquad 1/L_4 \leqq l(DF_g(1/2, 1/2)) \leqq |DF_g(1/2, 1/2)| \leqq L_4.$$

Observe that g is normalized and H-quasisymmetric. We show that, in fact, there is $L_4=L_4(H)$ such that (10) is true for all normalized and H-quasisymmetric g. By (7),

$$(11)\qquad DF_g(1/2, 1/2) = DA_g(B_{\bar{g}}(1/2, 1/2))\circ DB_{\bar{g}}(1/2, 1/2).$$

Here $\bar{g}=A_g^{-1}\circ g$ is normalized. It is also H'-quasisymmetric for some $H'=H'(H)$. To see this, observe first that $A_g\in\mathscr{F}_{2H}$ since $g(R)$ is of $2H$-bounded turning. Thus A_g, and hence also A_g^{-1}, can be extended to a $K(H)$-quasiconformal homeomorphism of R^2. But a $K(H)$-quasiconformal homeomorphism of R^2 is H_1-quasisymmetric for some $H_1=H_1(K(H))=H_1(H)$. This follows by a normal family argument or by [9, 2.4]. But then $A_g^{-1}\circ g$ is H'-quasisymmetric for some $H'=H'(H_1, H)=H'(H)$ by [8, 2.16 and 2.2].

Let $X=\{B_h(1/2, 1/2)\colon h\colon R\to R$ is normalized and H'-quasisymmetric$\}$. Then a normal family argument shows that $X=X(H)\subset U$ is compact (observe that B_g depends continuously on g). Now (11) and Lemmas 1 and 2 imply that there are $L_2=L_2(H')=L_2(H)$ and $L_3=L_3(X, 2H)=L_3(H)$ such that (11) is true with $L_4=L_2L_3=L_4(H)$.

6. *The proof* of the main theorem is now easy. We can assume that f is normalized, possibly by increasing L' for non-normalized f. Obviously, F_f is continuously differentiable outside R. Thus $F_f|R^2\setminus R$ is L_4^2-quasiconformal, which implies that also F_f is L_4^2-quasiconformal since R is a removable singularity for quasiconformal maps of R^2. Assume then that f is an L-embedding. We observe that f is L^2-quasisymmetric and thus we can apply Lemma 3, which implies that $1/LL_4\leqq l(DF_f(x, y))\leqq|DF_f(x, y)|\leqq LL_4$ if $(x, y)\in R^2\setminus R$. It follows that F_f is an L'-embedding where $L'=LL_4=L'(L)$.

7. Actually, Theorem A of [7] and the present theorem consider a slightly different situation since in [7] we considered a Lipschitz embedding $f\colon S=\partial I^2\to R^2$. However, these two cases can be fairly easily reduced to each other by means of the following observation, due to J. Väisälä. *For every $L\geqq 1$ there is $K=K(L)\geqq 1$ such that if $f\colon R\to R^2$ or $f\colon R^2\to R^2$ is an L-embedding in the euclidean metric with $f(0)=0$, then f is a K-embedding in the spherical metric, and vice versa.* This

follows easily by the expression $|dz|/(1+|z|^2)$ for the spherical metric. This observation would have simplified the discussion in Section 12 of [7], where we extended $f\colon S\to R^2$ outside S. We refer to this discussion for the details of how to reduce the theorems to each other (apart from the requirement that the extensions are PL or continuously differentiable outside S or R).

Presumably one would get by this method also the extension for Lipschitz or quasisymmetric embeddings of arcs into R^2 (cf. Theorem B of [7]) since quasiconformal arcs can be characterized in a manner similar to quasiconformal circles (Rickman [5]).

Finally, I am indebted to J. Luukkainen and J. Väisälä for reading the manuscript and for valuable remarks.

Added in proof. After this paper was completed, I received the paper "Hardy spaces, A_∞, and singular integrals on chord-arc domains" by D.S,Jerison and C,E.Kenig whose Proposition 1.13. is equivalent to the Lipschitz part of the theorem of this paper.

References

[1] AHLFORS, L. V.: Quasiconformal reflections. - Acta Math. 109, 1963, 291—301.

[2] BEURLING, A., and L. AHLFORS: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125—142.

[3] REED, T. J.: Quasiconformal mappings with given boundary values. - Duke Math. J. 33, 1966, 459—464.

[4] REED, T. J.: On the boundary correspondence of quasiconformal mappings of domains bounded by quasi circles. - Pacific J. Math. 28, 1969, 653—661.

[5] RICKMAN, S.: Characterization of quasiconformal arcs. - Ann. Acad. Sci. Fenn. Ser. A I Math. 395, 1966, 1—30.

[6] RICKMAN, S.: Quasiconformally equivalent curves. - Duke Math. J. 36, 1969, 387—400.

[7] TUKIA, P.: The planar Schönflies theorem for Lipschitz maps. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 49—72.

[8] TUKIA, P., and J. VÄISÄLÄ: Quasisymmetric embeddings of metric spaces. - Ibid. 5, 1980, 97—114.

[9] VÄISÄLÄ, J.: Quasisymmetric embeddings in euclidean spaces. - Trans. Amer. Math. Soc. 264, 1981, 191-204.

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 9 May 1980