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ON THE EXTREMALITY OF AFFINE MAPPINGS WITH SMALL DILATATION

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1. Introduction. In a paper [1] of 1975, Ahlfors introduced a new dilatation for quasiconformal mappings in space. This dilatation is apparently due to Clifford Earle, and has a number of theoretically desirable properties, but (perhaps) suffers from computational awkwardness. In his paper, Ahlfors notes that a crucial test for the dilatation would be whether the affine mapping is extremal for "the classical rectangular box problem of Grötzsch".

In this paper I will present an affirmative answer to this question for the special case of three dimensional mappings with small dilatation. I believe that the method promises to extend to higher dimensions, but that the three dimensional presentation is desirable for clarity of exposition. There is precedent for special treatment of dimension three, e.g. [2], and indeed the pioneering work of Gehring and Väisälä ([4], [5], [7]) was all carried out in this context.

Because the Earle—Ahlfors dilatation is somewhat difficult to explain, I shall defer its introduction until Section 5, where the main result will be proved. In the final Section 6, I will prove as an application of the method, that the dilatation, if small, is not increased under uniform convergence. This result and its proof are somewhat similar to the results of Bühlmann [3] for the more usual dilatations. Bühlmann's article is also confined to the case of three dimensions.

The essential feature of the present method is the finding of a convex function with various special additional properties, which are summarized in Section 4. Once this function is in hand, the applications follow by Jensen's inequality and the usual length-area estimates common in the field. The method cannot overcome the requirement for small dilatations, but may prove valuable at some later time for the more general case.

2. Notation and preliminaries. Some of the time I will be working with the vector space structure of the Euclidean space $\boldsymbol{R}^n$, and then I will usually think

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of points as column vectors, using letters like u, a. The usual norm will be $\|u\| = (\sum u_i^2)^{1/2}$. The symbol $\boldsymbol{S}$ will be used to identify the sphere $\{u\colon \|u\|=1\}$.

However, I prefer to use traditional notation as far as possible. For example, $\boldsymbol{R}^{3+}$ will denote the first octant, that is

$$\{p = (x, y, z)\colon\ x > 0,\ y > 0,\ z > 0\}.$$

For any real $\beta > 0$, and $p=(x, y, z)\in\boldsymbol{R}^{3+}$, I shall denote the point $(x^\beta, y^\beta, z^\beta)$ by p^β.

As regards functions, C^2 will mean twice continuously differentiable, and M will mean monotone. I will use the relation $p \leqq q$ to mean that each coordinate of q is not less than the corresponding coordinate of p, so that monotonicity will have the usual formal appearance:

$$f(p) \leqq f(q) \quad \text{whenever} \quad p \leqq q.$$

I shall consider extensively the hyperboloid

$$H = \{q = (\xi, \eta, \zeta)\in\boldsymbol{R}^{3+}\colon\ \xi\eta\zeta = 1\},$$

which bounds the convex set $H^+ = \{(x, y, z)\colon\ xyz \geqq 1\}$. We note that the map $p \mapsto p^\beta$ always maps H^+ onto itself, and also H onto itself.

For $q=(\xi, \eta, \zeta)\in H$, I will denote by $\delta(q)$ the matrix

$$\begin{bmatrix} \xi & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \zeta \end{bmatrix}.$$

A point $q\in H$ will be termed a *representative* of the matrix $A\in GL_0(3, \boldsymbol{R})$*) if there exists an orthogonal matrix $U\in SO(3, \boldsymbol{R})$ with

$$\frac{U^T A^T A U}{\det^{2/3} A} = \delta(q).$$

In particular, then, q^2 is a representative of $\delta(q)$.

An important notion for $q\in H$ will be a set which I denote by $S(q)$, defined as follows: if $p=(x, y, z)\in\boldsymbol{R}^{3+}$, then $p\in S(q)$ if and only if there exists an orthogonal matrix $V\in SO(3, \boldsymbol{R})$ with (x, y, z) the respective diagonal entries of the matrix

$$V^T \delta(q) V.$$

The motivation for considering this set is that if A is any matrix in $GL_0(3, \boldsymbol{R})$ whose representative is q, and whose columns are respectively $a_1, a_2, a_3\in\boldsymbol{R}^3$, then

$$p_A = \left(\frac{\|a_1\|^2}{\det^{2/3} A}, \frac{\|a_2\|^2}{\det^{2/3} A}, \frac{\|a_3\|^2}{\det^{2/3} A}\right) \tag{2.1}$$

belongs to $S(q)$. Evidently $S(q) \subseteqq H^+$.

*) $GL_0(3, \boldsymbol{R})$ will be those matrices in $GL(3, \boldsymbol{R})$ with positive determinant.

We have the important map $T\colon H^+\to \boldsymbol{R}$, defined for $p=(x,y,z)\in H^+$ by $T(p)=x+y+z$. This is motivated by the fact that the trace of a matrix is invariant under orthogonal conjugation. Therefore, if $p\in S(q)$, we must have $T(p)=T(q)$.

We have the inner and outer dilatations K_I, K_O defined for any matrix $A\in GL_0(3,\boldsymbol{R})$ by

$$\frac{1}{K_I[A]}=\inf_{u\in \boldsymbol{S}}\frac{\|Au\|^2}{\det^{2/3}A},\quad K_O[A]=\sup_{u\in \boldsymbol{S}}\frac{\|Au\|^2}{\det^{2/3}A},$$

and functions $M, m\colon H^+\to\boldsymbol{R}$, defined for $p=(x,y,z)$ by

$$M(p)=\max\{x,y,z\},\quad m(p)=\min\{x,y,z\}.$$

The connections here are that if $q\in H$ is a representative of a matrix $A\in GL_0(3,\boldsymbol{R})$, and if $p\in S(q)$, then

$$\frac{1}{K_I[A]}=m(q)\leqq m(p)\leqq M(p)\leqq M(q)=K_O[A].$$

The following lemma is a consequence of the preceding considerations:

Lemma 2.1. *For $q\in H$, the set $S(q)$ is a subset of the convex hull of the permutations of q, which is in turn a subset of the plane $x+y+z=T(q)$.*

Proof. Since the result depends only on the permutations of q, we may suppose that $q=(\xi,\eta,\zeta)$, with $\xi\leqq\eta\leqq\zeta$. In this case then, $S(q)$ is evidently a subset of the intersection of the plane

$$x+y+z=\xi+\eta+\zeta$$

with the cube

$$\{(x,y,z)\colon\ \xi\leqq x\leqq\zeta,\ \xi\leqq y\leqq\zeta,\ \xi\leqq z\leqq\zeta\}.$$

The vertices of this intersection are precisely the permutations of q. □

3. Convexity. A set E in $\boldsymbol{R}^n$ is convex if it contains the entire line segment connecting any pair of its points. A real valued function φ defined in a convex set E, is said to be convex if, for $x,y\in E, 0\leqq\lambda\leqq 1$, we have

$$\varphi(\lambda x+(1-\lambda)y)\leqq\lambda\varphi(x)+(1-\lambda)\varphi(y).$$

For $\varphi\in C^2$, it suffices that for each $x\in E$, the second differential

$$\varphi''(x)\colon \boldsymbol{R}^n\times\boldsymbol{R}^n\to\boldsymbol{R}$$

be positive semidefinite. This form may be identified with the matrix whose (i,j) entry is $\partial^2\varphi/\partial x_i\partial x_j$, and the requirement then is that for all $u\in\boldsymbol{S}$, we have

$$\langle\varphi''(x);\,u,u\rangle=\sum\frac{\partial^2\varphi}{\partial x_i\partial x_j}u_iu_j\geqq 0.$$

Some functions are not convex, but may be made convex by composition with a suitable nondecreasing function of a real variable. Thus, suppose $0 \leqq \psi \in C^2$, and $\alpha \in C^2 \cap M$. What are the requirements that $\varphi = \alpha \circ \psi$ be convex?

Routine calculation shows that

$$\frac{\partial \varphi}{\partial x_i} = \alpha'(\psi) \frac{\partial \psi}{\partial x_i}$$

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \alpha'(\psi) \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \alpha''(\psi) \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j},$$

or, in the language of differentials,

$$\langle \varphi'(x); u \rangle = \alpha'(\psi(x)) \langle \psi'(x); u \rangle,$$

$$\langle \varphi''(x); u, v \rangle = \alpha'(\psi(x)) \langle \psi''(x); u, v \rangle + \alpha''(\psi(x)) \langle \psi'(x); u \rangle \langle \psi'(x); v \rangle.$$

Consequently, if a function α of a real variable, belonging to $M \cap C^2$, and defined on the interval $(0, \sup \psi]$ has the property that

$$\frac{-\alpha''(c)}{\alpha'(c)} \leqq \frac{\langle \psi''(x); u, u \rangle}{\langle \psi'(x); u \rangle^2} \tag{3.1}$$

whenever $0 < \psi(x) = c$ and $u \in \boldsymbol{S}$, then it is apparent that $\alpha \circ \psi$ is convex. The behavior at $c = 0$ will not be a problem, because that is the absolute minimum.

For a function $\psi(x, y)$ defined in $\boldsymbol{R}^2$, we adopt the temporary notation $\nabla^\perp \psi$ for the vector $(-\partial\psi/\partial y, \partial\psi/\partial x)$, which is perpendicular to the gradient $\nabla \psi$. The following result is probably well known, but is in any case provable by elementary calculus:

Theorem. *Let ψ be of class C^2 in $D \subseteqq \boldsymbol{R}^2$, and let p be a point in D. Suppose the number*

$$E = E_\psi(p) = \langle \psi''(p); \nabla^\perp \psi(p), \nabla^\perp \psi(p) \rangle$$

is positive. Then as u ranges over $\boldsymbol{S}$, the ratio

$$\frac{\langle \psi''(p); u, u \rangle}{\langle \psi'(p); u \rangle^2}$$

has a finite minimum, given by

$$m_\psi(p) = \frac{\det \psi''(p)}{E_\psi(p)}.$$

We shall study extensively the specific function $\psi: \boldsymbol{R}^{2+} \to \boldsymbol{R}$, defined by

$$\psi(x, y) = \log^2 x + \log x \log y + \log^2 y. \tag{3.2}$$

Accordingly, we set $\alpha=\log x^2y$, $\beta=\log y^2x$, and calculate for $p=(x, y)$

$$\nabla\psi(p)=\left(\frac{\alpha}{x}, \frac{\beta}{y}\right), \quad \nabla^{\perp}\psi(p)=\left(-\frac{\beta}{y}, \frac{\alpha}{x}\right).$$

As a matrix, $\psi''(p)$ has the form

$$\begin{bmatrix} \dfrac{2-\alpha}{x^2} & \dfrac{1}{xy} \\ \dfrac{1}{xy} & \dfrac{2-\beta}{y^2} \end{bmatrix}$$

and therefore,

$$E_\psi(p)=\frac{1}{x^2y^2}[(2-\alpha)\beta^2-2\alpha\beta+(2-\beta)\alpha^2]$$

$$=\frac{1}{x^2y^2}[2(\beta+\alpha)^2-6\alpha\beta-\alpha\beta(\alpha+\beta)],$$

$$\det\psi''(p)=\frac{(2-\alpha)(2-\beta)-1}{x^2y^2}$$

$$=\frac{3-2(\alpha+\beta)+\alpha\beta}{x^2y^2},$$

and assuming $E_\psi(p)>0$,

$$m_\psi(p)=\frac{3-2(\alpha+\beta)+\alpha\beta}{2(\beta+\alpha)^2-6\alpha\beta-\alpha\beta(\alpha+\beta)}. \tag{3.3}$$

In fact, the mapping $(x, y)\mapsto(\alpha, \beta)$ is a univalent transformation of $\boldsymbol{R}^{2+}$ onto $\boldsymbol{R}^2$, but it is advisable to move on further, via the transformation $(\alpha, \beta)\mapsto(\xi, \eta)$, where

$$\xi=\alpha+\beta, \quad \eta=\alpha\beta.$$

Because (α, β) are real roots of the equation $0=t^2-\xi t+\eta$, the image domain is precisely $\{(\xi, \eta)\colon \xi^2-4\eta\geqq 0\}$. The composed mapping $F\colon (x, y)\mapsto(\xi, \eta)$ is univalent in $\{x\geqq y\}$, and in general $F(x, y)=F(y, x)$.

The expression $E_\psi(p)$ has the same sign as

$$2\xi^2-6\eta-\xi\eta,$$

and the ratio $m_\psi(p)$ becomes

$$m_\psi(x, y)=\frac{3-2\xi+\eta}{2\xi^2-6\eta-\xi\eta} \quad ((\xi, \eta)=F(x, y)). \tag{3.4}$$

Finally, define $\zeta=\zeta(\eta)$ by the expression

$$\zeta=\left(\frac{3}{2}(\eta^2+5\eta)\right)^{1/2},$$

and define r_0 by the expression

$$\sup_{0<\eta<1} \frac{(-27\eta^2+22\eta\zeta+15\zeta-84\eta)\sqrt{6}}{(9+6\eta-5\zeta)(9+5\eta-4\zeta)^{1/2}3\sqrt{\eta}} \tag{3.5}$$

Lemma 3.1. *As defined by* (3.5), *the number* r_0 *is finite, and for any* $r \geqq r_0$, *the function*

$$\varphi_r = \exp\{r\sqrt{\psi}\}$$

is convex in the convex domain $D=\{(x, y)\colon \psi(x, y) \leqq 1/3\}$.

Proof. We shall employ the previous remarks, by showing that for $0<c<1/3$, the expression $E_\psi(p)$ is positive on the entire level curve $\{\psi(x, y)=c\}$. Then it will remain to show that for the function $\alpha_r(c)=\exp\{r\sqrt{c}\}$, we have

$$\frac{1-r\sqrt{c}}{2c} = -\frac{\alpha_r''(c)}{\alpha_r'(c)} \leqq m_\psi(x, y)$$

whenever $\psi(x, y)=c$. The first equality is a routine calculation. For the inequality, in view of (3.4), we are to show

$$\frac{1-r\sqrt{c}}{2c} \leqq \frac{3-2\xi+\eta}{2\xi^2-6\eta-\xi\eta} \tag{3.6}$$

whenever

$$\begin{aligned} c = \psi(x, y) &= \log^2 x+\log x \log y+\log^2 y \\ &= \frac{1}{3}(\alpha^2-\alpha\beta+\beta^2) \\ &= \frac{1}{3}(\xi^2-3\eta). \end{aligned}$$

In view of this constraint, the relation (3.6) reduces to showing

$$\frac{1-r\sqrt{c}}{2c} \leqq \frac{3-2\xi+\eta}{6c-\xi\eta}, \tag{3.7}$$

whenever we have the four conditions

$$r \geqq r_0, \quad \xi^2 \geqq 4\eta, \quad \xi^2-3\eta = 3c, \quad 0 < c < \frac{1}{3}.$$

When all else is established, the convexity of the domain D will follow as the closure of the increasing union of the convex sub-level sets

$$D_c = \{(x, y)\colon \varphi_{r_0}(x, y) \leqq \exp\{r_0\sqrt{c}\}\} = \{\psi(x, y) \leqq c\}$$

as $c \nearrow 1/3$.

Turning to the details, it seems at first best to fix c, and use ξ as the parameter for the parabola

$$P_c = \{(\xi, \eta)\colon \xi^2-3c = 3\eta\}.$$

The constraint $\xi^2 \geqq 4\eta$ has the effect of limiting the range for the parameter ξ to the interval

$$I_c = [-2\sqrt{3c},\ 2\sqrt{3c}].$$

The quantity $\xi\eta$ becomes $(\xi^3 - 3c\xi\}/3$, and has its maximum value $6c\sqrt{3c}$ at $\xi = 2\sqrt{3c}$, which maximum reaches $6c$ for the first time when $\sqrt{3c} = 1$, hence $c = 1/3$. Therefore, for $0 < c < 1/3$, $E_\psi(x, y)$ is indeed positive whenever $\psi(x, y) = c$.

We shall then turn to the problem of minimizing

$$g_c(\xi) = \frac{3 - 2\xi + \eta}{6c - \xi\eta}; \quad \xi \in I_c.$$

One readily calculates

$$\frac{d}{d\xi} g_c(\xi) = \frac{h_c(\xi)}{3(6c - \xi\eta)^2},$$

where

$$h_c(\xi) = 12c\xi - 36c + 9\eta + 3\eta^2 + 6\xi^2 - 4\xi^3.$$

This is evidently a fourth degree polynomial in ξ. However, we do not wish to express it as such.

Values for ξ of special interest will be the endpoints of the interval I_c, and two relevant roots of h_c, which I shall denote by $\xi_1, \xi_0 \in I_c$, $\xi_1 \leqq \xi_0$. Denoting the corresponding η values by η_1, η_0, the points (ξ_i, η_i) may be shown, by elimination of the parameter c, to lie on the intersection of P_c with the hyperbola

$$R = \{(\xi, \eta)\colon \eta^2 + 15\eta - 4\xi\eta - 2\xi^2 = 0\}.$$

One routinely shows that both $h_c(-2\sqrt{3c})$ and $h_c(2\sqrt{3c})$ are positive, and therefore the serious contenders for the minimum are $-2\sqrt{3c}$ and ξ_0. Since the value $g_c(-2\sqrt{3c}) = (3 + \sqrt{3c})/6c$ is comfortably larger than $(1 - r\sqrt{c})/2c$ whenever $r > 0$, this poses no problem.

In order to study the inequality (3.7) at (ξ_0, η_0) it becomes far more efficient to take η_0 as parameter for both ξ_0 and c, through the relations $(\xi_0, \eta_0) \in P_c \cap R$. Dropping the subscript from η_0, we find easily that the explicit formulae are

$$\left.\begin{aligned} \zeta &= \left(\frac{3}{2}(\eta^2 + 5\eta)\right)^{1/2} \\ \xi_0 &= -\eta + \zeta \\ c &= \frac{1}{3}\xi_0^2 - \eta \end{aligned}\right\} \quad 0 < \eta < 1,$$

and finally, our task is to show that for $0 < \eta < 1$, $r \geqq r_0$, we have

$$\frac{3(1 - r\sqrt{c})}{\eta(5\eta - 4\zeta + 9)} = \frac{(1 - r\sqrt{c})}{2c}$$

not more than

$$g_c(\xi_0) = \frac{3-2\xi_0+\eta}{6c-\xi_0\eta} = \frac{3+3\eta-2\zeta}{\eta(6\eta-5\zeta+9)}.$$

Since $0<\eta<1$, the factor of $1/\eta$ may be cancelled from the inequality, and after some algebraic manipulation, our task is to prove

$$3r\sqrt{c} \geqq \frac{-27\eta^2+22\eta\zeta+15\zeta-84\eta}{(9+6\eta-5\zeta)}$$

or

$$r \geqq \frac{(-27\eta^2+22\eta\zeta+15\zeta-84\eta)\sqrt{6}}{(9+5\eta-4\zeta)^{1/2}(9+6\eta-5\zeta)3\sqrt{\eta}}.$$

This of course is guaranteed, since r_0 is the supremum of the right hand quotient $q(\eta)$. However, it remains to verify that r_0 is finite, and for this we must examine the behavior of $q(\eta)$ as $\eta\searrow 0$ and as $\eta\nearrow 1$.

For the case of $\eta\searrow 0$, and in view of $\zeta=(15\eta/2)^{1/2}(1+O(\eta))$, we see that

$$\lim_{\eta\searrow 0} q(\eta) = \frac{5\sqrt{5}}{9} < \infty.$$

On the other hand, as $\eta\nearrow 1$, we see that $\zeta\to 3$, and it is necessary to write $\eta=1-\varepsilon$, and find that

$$\zeta = 3-\frac{7}{4}\varepsilon+O(\varepsilon^2),$$

$$9+6\eta-5\zeta = \frac{11}{4}\varepsilon+O(\varepsilon^2),$$

$$-27\eta^2+22\eta\zeta+15\zeta-84\eta = \frac{29}{4}\varepsilon+O(\varepsilon^2),$$

and finally

$$\lim_{\eta\nearrow 1} q(\eta) = \frac{29\sqrt{3}}{33} < \infty. \quad \square$$

We have one final result for this section, which is a consequence of the convexity of the domains D_c and the earlier Lemma 2.1. Here, ψ is still defined by (3.2), and we make the permanent definition for $q=(\xi, \eta, \zeta)\in H$.

$$\Psi(q) = \frac{1}{2}(\log^2\xi+\log^2\eta+\log^2\zeta). \tag{3.11}$$

Corollary 3.2. *For* $q\in H, 0\leqq\Psi(q)\leqq 1/3$, *and for* $p=(x, y, z)\in S(q)$, *we have*

$$\psi(x, y) \leqq \Psi(q).$$

Proof. Lemma 2.1 tells us that $S(q)$ lies in the convex hull $\mathscr{K}$ of the permutations $\{q'\}$ of q. Let π be projection on the (x, y)-plane. The six (or three) projections $\{\pi(q')\}$ lie on the level curve $\{\psi=c=\Psi(q)\}$, which bounds the set D_c, and therefore

$$\pi(p)\in\pi(S(q))\subseteqq\pi(\mathscr{K})\subseteqq D_c,$$

hence

$$\psi(x, y)=\psi(\pi(p))\leqq c=\Psi(q). \quad \square$$

4. A special function. We shall now extend the function $\Psi\colon H\to\boldsymbol{R}$ to a new function which we will call $\hat{\Psi}\colon H^+\to\boldsymbol{R}$, and defined for $p=(x, y, z)\in H^+$ by

$$\hat{\Psi}(p)=\max\{\psi(x, y),\ \psi(y, z),\ \psi(z, x)\}.$$

The formula is an extension, because if $p\in H$, then all three competitors coincide with $\Psi(p)$.

Let us now set up the subdomain $\mathscr{E}\subseteqq H^+$, defined by

$$\mathscr{E}=\{(x, y, z)\in H^+\colon\ (x, y)\in D,\ (y, z)\in D,\ (z, x)\in D\}.$$

$\mathscr{E}$ is convex, as the intersection of the three convex cylinders exemplified by $\{(x, y, z)\colon (x, y)\in D\}$ with H^+. Moreover, by Corollary 3.2 we have the basic property

(4.1) $$\hat{\Psi}(p)\leqq\Psi(q)\quad\text{whenever}\quad p\in S(q)\quad\text{and}\quad q\in\mathscr{E}\cap H,$$

and from the definition, the obvious:

(4.2) $$\hat{\Psi}(x, y, z)\geqq\psi(x, y).$$

We define the functions $\Phi_r\colon\mathscr{E}\to\boldsymbol{R}$ by

$$\Phi_r(p)=\exp\left\{r\sqrt{\hat{\Psi}(p)}\right\}.$$

Direct consequences of (4.1) and (4.2) are the properties

(4.3) $$\Phi_r(p)\leqq\Phi_r(q)\quad\text{whenever}\quad p\in S(q),$$

(4.4) $$\Phi_r(x, y, z)\geqq\varphi_r(x, y),$$

and we also note that for any $\beta>0$,

(4.5) $$\Phi_r(p^\beta)=\Phi_{\beta r}(p)=\Phi_r^\beta(p).$$

Since it is evident that for $p=(x, y, z)\in\mathscr{E}$,

$$\Phi_r(p)=\max\{\varphi_r(x, y),\ \varphi_r(y, z),\ \varphi_r(z, x)\},$$

and the competing functions are convex, it follows that Φ_r is *convex* in $\mathscr{E}$ whenever $r\geqq r_0$. This being so, it now follows that Φ_r is *monotone*. For, indeed, the relation (4.4) shows that for $(x, y, z)\in\mathscr{E}$, we have

$$\Phi_r(x, y, z)\geqq\varphi_r(x, y)=\Phi_r\left(x, y, \frac{1}{xy}\right).$$

But a convex function of one variable (z) which assumes its minimum at the left endpoint $(1/(xy))$ of its interval of definition is necessarily monotone. From the relation $\hat{\Psi}=(\log^2 \Phi_r)/r^2$ it follows that $\hat{\Psi}$ itself is monotone, and in turn that Φ_r is monotone for all $r>0$.

5. Applications to extremal problems. The Earle—Ahlfors dilatation for a matrix $A\in GL_0(3, \boldsymbol{R})$ is defined by setting $s=\sqrt{3}/2$, and

$$(5.0)\quad K_E[A]=\Phi_s(q)=\exp\left\{\left(\frac{3}{4}\Psi(q)\right)^{1/2}\right\}=\exp\left\{\left(\frac{3}{8}(\log^2\xi+\log^2\eta+\log^2\zeta)\right)^{1/2}\right\}$$

whenever $q=(\xi, \eta, \zeta)\in H$ is a representative of A.

Apparently, [1], the motivation for the choice $s=\sqrt{3}/2$ is to have, whenever $\mu\geqq 1$,

$$K_E[\delta(\mu, 1, 1)]=\mu.$$

This is a useful mnemonic convention. We also note that by this convention, the requirement $q\in\mathscr{E}\cap H$ (i.e. $\Psi(q)\leqq 1/3$) corresponds to

$$K_E[A]\leqq\exp\left\{\frac{1}{2}\right\}=\sqrt{e}.$$

It is a routine consequence of the definitions and of (4.1) that

$$(5.1)\qquad p\in\mathscr{E}\quad\text{whenever}\quad q\in\mathscr{E}\cap H\quad\text{and}\quad p^{1/3}\in S(q).$$

For a quasiconformal mapping f of a domain G into $\boldsymbol{R}^3$, we have at almost every $u\in G$, the total differential

$$f'(u)\colon \boldsymbol{R}^3\to\boldsymbol{R}^3$$

which is a linear map with the property

$$f(u+h)=f(u)+\langle f'(u); h\rangle+o(\|h\|),$$

and nonzero Jacobian $J_f(u)$, which is defined as the Lebesgue derivative of the measure function $E\mapsto V(f(E))$. Here, V is Lebesgue measure in $\boldsymbol{R}^3$.

Now suppose $\{e_1, e_2, e_3\}$ are orthogonal unit vectors in $\boldsymbol{R}^3$, and let (u_1, u_2, u_3) be coordinates of $u\in G$ in the sense that

$$u=\sum u_i e_i,$$

and let $\{e_1', e_2', e_3'\}$ be appropriately oriented orthogonal unit vectors in $\boldsymbol{R}^3$ such that one may write

$$f(u)=\sum f_j(u_1, u_2, u_3)e_j'.$$

If a matrix $Df(u)$ is defined by having entry (j, i) equal to $\partial f_j/\partial u_i$ (the i-th column will be known as $\partial_i f$), then as is well known, $f'(u)$ has matrix $Df(u)$ with respect to the bases $\{e_i\}$, $\{e_j'\}$, and $J_f(u)=\det Df(u)$. The quantity $K_E[Df(u)]$ is invariant with respect to orthogonal changes of coordinates, and we denote it by $K_f(u)$.

The entries of Df are Borel functions, and K_f is Borel measurable because of the relations

$$K_f(u) = \sup_U \Phi_s(p_U(u)),$$

where $p_U = (x_U, y_U, z_U)$ are the diagonal entries of

$$\frac{U^T Df^T(u)\, Df(u)\, U}{J_f^{2/3}(u)},$$

and U ranges over a countable dense subset of $SO(3, \boldsymbol{R})$. It is natural to set $K_E[f] = \text{ess sup}_{u \in G} K_f(u)$.

When E is a measurable set in G, with $V(E) < \infty$, we shall denote by $⨍_E g$ the normalized integral

$$\frac{1}{V(E)} \int_E g(u)\, dV(u).$$

Jensen's inequality will assure us that if $R = R(u)$ is a measurable vector valued function with values in $\mathscr{E}$, and if $r \geqq r_0$, then

$$\Phi_r\left(⨍_E R\right) \leqq ⨍_E \Phi_r \circ R. \tag{5.2}$$

We shall apply this inequality to the case

$$R = R_f(u) = \left(\frac{\|\partial_1 f\|^3}{J_f}, \frac{\|\partial_2 f\|^3}{J_f}, \frac{\|\partial_3 f\|^3}{J_f}\right)^{1/2}, \tag{5.3}$$

where we assume that $K_E[f] \leqq \sqrt{e}$. Because of (5.3) and (2.1), we see that $R(u)^{4/3} \in S(q)$ for each representative q of $Df(u)$, and therefore by (5.1), we are assured that $R(u) \in \mathscr{E}$.

To obtain an upper bound for the right side of (5.2), we recall (4.5), (4.3), and (5.0) to find:

$$\begin{aligned} \Phi_r(R(u)) &= \Phi_{\frac{3}{4}r}(R(u)^{4/3}) \leqq \Phi_{\frac{3}{4}r}(q) \\ &= \Phi_s^{sr}(q) = K_f(u)^{sr}, \end{aligned} \tag{5.4}$$

hence setting $t = sr = r\sqrt{3}/2$, we come to

$$⨍_E \Phi_r \circ R \leqq ⨍_E K_f^t. \tag{5.5}$$

On the other side, we shall employ the monotone property. First, however, we note that by Hölder's inequality,

$$⨍_E \|\partial_i f\| \leqq \left\{⨍_E \frac{\|\partial_i f\|^{3/2}}{J_f^{1/2}}\right\}^{2/3} \left\{⨍_E J_f\right\}^{1/3},$$

and since

$$\left\| \underset{E}{\int} \partial_i f \right\| \leqq \underset{E}{\int} \|\partial_i f\|$$

we have

$$\frac{\|\int_E \partial_i f\|^{3/2}}{[\int_E J_f]^{1/2}} \leqq \underset{E}{\int} R_i .$$

When the left side is taken as the i-th component of a point $p=p_f(E)$, and *provided* $p\in\mathscr{E}$, then we have by the monotone property of Φ_r,

$$\Phi_r(p) \leqq \Phi_r\left(\underset{E}{\int} R\right). \tag{5.6}$$

Now let $x=(\xi_1, \xi_2, \xi_3)\in \boldsymbol{R}^{3+}$, and let $Q(x)$ be the set of quasiconformal mappings $f\colon \boldsymbol{R}^3\to\boldsymbol{R}^3$, with the properties:

(5.7)
i) $K_E[f] \leqq \sqrt{e}$

ii) $f(u+e_i) = f(u)+\xi_i e_i \quad (i = 1, 2, 3;\ u\in\boldsymbol{R}^3)$.

This class includes the linear mapping f_0 with matrix $A_0=\delta(x)$ if we assume, *as we do,* that $K_E[A_0]\leqq\sqrt{e}$. We take for E the unit cube C_0:

$$C_0 = \{u\in\boldsymbol{R}^3\colon\ 0 \leqq u_i \leqq 1,\ i = 1, 2, 3\}.$$

Theorem 5.1. *For each* $t\geqq sr_0$, *the affine mapping* f_0 *is mean* t*-extremal over* C_0 *for* K_f *in the class* $Q(x)$, *which is to say that*

$$\underset{C_0}{\int} K_f^t \geqq \underset{C_0}{\int} K_{f_0}^t = K_E[A_0]^t$$

for all $f\in Q(x)$.

The proof is immediate when we note that since C_0 has volume 1, the average integrals can be replaced by ordinary integrals. Furthermore, by Fubini's theorem,

$$\underset{C_0}{\int} \partial_1 f = \int_0^1\int_0^1 f(1, u_2, u_3)-f(0, u_2, u_3)\, du_2\, du_3$$

$$= \int_0^1\int_0^1 (\xi_1, 0, 0)\, du_2\, du_3 = (\xi_1, 0, 0)$$

and at the same time,

$$\underset{C_0}{\int} J_f = V(f(C_0)) = \xi_1\xi_2\xi_3 .$$

Therefore, in this case,

$$p_1 = \xi_1^{3/2}/(\xi_1\xi_2\xi_3)^{1/2},$$

or, more generally,

$$p_i^{4/3} = \xi_i^2/(\xi_1\xi_2\xi_3)^{2/3} \quad (i = 1, 2, 3).$$

Hence, $p^{4/3}\in H\cap\mathscr{E}$ is a representative for A_0. Therefore, using (4.5), and reasoning as in (5.4) with $t=sr$, and using (5.6), (5.2), and (5.5), we find

$$K_E[A_0]^t = \Phi_s^t(p^{4/3}) = \Phi_{\frac{3}{4}r}(p^{4/3}) = \Phi_r(p)$$

$$\leqq \Phi_r\left(\int_{C_0} R\right) \leqq \int_{C_0} \Phi_r \circ R \leqq \int_{C_0} K_f^t. \quad \square$$

Corollary 5.2. *In the class* $Q_0(x)$ *with only property* (5.7) (ii), *the affine mapping* f_0 *is extremal for* K_E.

Proof. Fix $f\in Q_0(x)$ and t sufficiently large. If K_f does not satisfy condition (5.7) (i), then

$$K_E[f] \geqq \sqrt{e} \geqq K_E[f_0].$$

Otherwise, we have $f\in Q(x)$, and

$$K_E[f]^t \geqq \int_{C_0} K_f^t \geqq K_E[f_0]^t. \quad \square$$

We are now ready for the formulation of the Grötzsch rectangular box problem.

Corollary 5.3. *Assume that* $x=(\xi_1, \xi_2, \xi_3)\in \boldsymbol{R}^{3+}$, *with* $K_E[\delta(x)]\leqq\sqrt{e}$. *Suppose a quasiconformal mapping* $w=f(u)$ *maps the unit cube* C_0 *onto the rectangular box*

$$R_0 = \{w\colon\ 0 \leqq w_i \leqq \xi_i,\ i = 1, 2, 3\},$$

in such a manner that the face-correspondence is as follows:

$$\left.\begin{array}{l}\{u_i = 0\} \mapsto \{w_i = 0\} \\ \{u_i = 1\} \mapsto \{w_i = \xi_i\}\end{array}\right\} \quad (i = 1, 2, 3).$$

Let f_0 *be the affine mapping with this property. Then* $K_E[f]\geqq K_E[f_0]$.

Proof. We may, by repeated reflections in the faces of C_0 and R_0, extend f to a quasiconformal mapping of $2C_0$ onto $2R_0$, with the property that the map $g\colon u\mapsto(1/2)f(2u)$ has an extension belonging to the class $Q_0(x)$. Therefore by Corollary 5.2,

$$K_E[f] = K_E[g] \geqq K_E[f_0]. \quad \square$$

6. Convergent sequences. Turning to a different situation, we have the following result for convergent sequences. In the hypothesis it is only necessary to assume that f_0 is non-constant, but for simplicity, we state it as follows:

Theorem 6.1. *Suppose that* f_0, f_n $(n=1, 2, 3, \ldots)$ *are quasiconformal mappings of a domain* $G\subseteqq\boldsymbol{R}^3$. *Suppose that* $K_E[f_n]\leqq\sqrt{e}$, *and suppose that* $f_n\to f_0$ *uniformly on compact subsets of* G. *Then*

$$K_E[f_0] \leqq \liminf_{n\to\infty} K_E[f_n]$$

and, more precisely, for almost every $u \in G$,

(6.1) $$K_{f_0}(u) \leqq \limsup_{n\to\infty} K_{f_n}(u).$$

Proof. Fix t sufficiently large. Since the first conclusion follows from the second, we shall prove that (6.1) holds at every point u_0 for which

(a) f_0 is totally differentiable at u_0, with $J_{f_0}(u_0)>0$,

(b) the integral of the function $g=\limsup K^t_{f_n}$ has strong Lebesgue derivative at u_0.

The condition (b) means that

$$\lim_{E\to\{u_0\}} \int_E g = g(u_0)$$

for a wide range of sets E containing u_0, and in particular including all cubes of arbitrary orientation. The fact that condition (b) holds almost everywhere follows because the integrand g is measurable and *bounded.* See, for example, [6].

Now fix such a point u_0 and choose coordinates $u=(u_1, u_2, u_3)$ in G, and coordinates in $f(G)$, such that $u_0=(0, 0, 0)$, and with respect to which $f'(u_0)$ has matrix $\delta(x)$, where $x=(\xi_1, \xi_2, \xi_3)\in \boldsymbol{R}^{3+}$.

Fix $\varepsilon>0$, and take $E=C_\varepsilon$ as the cube

$$\{u\colon 0\leqq u_i \leqq \varepsilon,\ i=1, 2, 3\}.$$

The argument procedes as in Section 5 to the point just before (5.6), where $p=p_{n,\varepsilon}$ and $R=R_n$ depend on f_n and C_ε. At this point, we let $n\to\infty$, and make use of Fatou's lemma in (5.2) and (5.5) to come to

$$\limsup_{n\to\infty} \Phi_r\Big(\int_{C_\varepsilon} R_n\Big) \leqq \int_{C_\varepsilon} \limsup_{n\to\infty} K^t_{f_n} \quad (t=sr),$$

and therefore by condition (b),

(6.3) $$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \Phi_r\Big(\int_{C_\varepsilon} R_n\Big) \leqq \limsup_{n\to\infty} K^t_{f_n}(u_0).$$

As before, we have the lower estimates $p_{n,\varepsilon}$ for $\int_{C_\varepsilon} R_n$, and in fact, as $n\to\infty$, we can show that

$$p_{n,\varepsilon} \to p_{0,\varepsilon}.$$

Indeed, it only depends on the uniform convergence. For example,

$$\int_{C_\varepsilon} \partial_1 f_n = \frac{1}{\varepsilon^3}\int^{\varepsilon}\int^{\varepsilon} f_n(\varepsilon, u_2, u_3) - f_n(0, u_2, u_3)\, du_2\, du_3$$

and, as $n \to \infty$, the quantity on the right approaches

$$\frac{1}{\varepsilon^3} \int_0^{\varepsilon} \int_0^{\varepsilon} f_0(\varepsilon, u_2, u_3) - f_0(0, u_2, u_3)\, du_2\, du_3 = \int_{C_\varepsilon} \partial_1 f_0. \tag{6.4}$$

Similarly, as $n \to \infty$,

$$\int_{C_\varepsilon} J_{f_n} = \frac{1}{\varepsilon^3} V(f_n(C_\varepsilon)) \to \frac{1}{\varepsilon^3} V(f_0(C_\varepsilon)) = \int_{C_\varepsilon} J_{f_n}.$$

This takes care of n. Now using the differentiability of f_0 at u_0, and for example (6.4), it follows that as $\varepsilon \to 0$,

$$p_{0,\varepsilon}^{4/3} \to (\xi_1 \xi_2 \xi_3)^{-2/3} (\xi_1^2, \xi_2^2, \xi_3^2) = x_0 \in H \cap \mathscr{E}. \tag{6.5}$$

To see this, we may by our assumption, write

$$f_0(u_1, u_2, u_3) = f_0(0) + \xi_1 u_1 a_1 + \xi_2 u_2 a_2 + \xi_3 u_3 a_3 + o(\|u\|),$$

where $\{a_1, a_2, a_3\}$ are orthogonal unit vectors in $\boldsymbol{R}^3$. Therefore the integrand in the left side of (6.4) is $\varepsilon \xi_1 a_1 + o(\varepsilon)$, the integral is $\varepsilon^3 \xi_1 a_1 + o(\varepsilon^3)$, and on taking norms,

$$\left\| \int_{C_\varepsilon} \partial_1 f_0 \right\| = \xi_1 + o(1).$$

In like manner, we have the estimate from the differential

$$\int_{C_\varepsilon} J_{f_0} = \frac{1}{\varepsilon^3} V(f_0(C_\varepsilon)) = \xi_1 \xi_2 \xi_3 + o(1),$$

and (6.5) follows as claimed. Note that we do not appeal to the Lebesgue differentiation here. This is because the orientation of the cubes C_ε is not known in advance, while on the other hand, the functions $\partial_i f_0$ and J_{f_0} are not necessarily bounded, and their integrals are not known to possess strong derivatives.

To resume the argument, the problem now presenting itself is that we are not absolutely sure that the quantities $p_{n,\varepsilon}$ or $p_{0,\varepsilon}$ lie in H^+. However, this is not a serious difficulty. Indeed, let us take any sequence $\varepsilon_m \to 0$, and by diagonalization and pruning, we may assume that for each $m = 1, 2, 3, \ldots,$ we have

$$\lim_{n \to \infty} \int_{C_{\varepsilon_m}} R_n = q_m \in \mathscr{E}$$

and as $m \to \infty$,

$$q_m \to q_0 \in \mathscr{E}.$$

For the inclusions, one need only note that since the integrands R_n are in $\mathscr{E}$, the integrals are surely in $\mathscr{E}$ and furthermore that $\mathscr{E}$ is compact. Then in view of (6.3), we have

$$\Phi_r(q_0) \leqq \limsup_{n\to\infty} K_{f_n}(u_0)^t.$$

Now it follows from (6.5) that

$$x_0^{3/4} = \lim_{m\to\infty} p_{0,\varepsilon_m} = \lim_{m\to\infty}\lim_{n\to\infty} p_{n,\varepsilon_m} \leqq \lim_{m\to\infty} q_m = q_0,$$

and we may finally apply the monotone property of Φ_r to conclude

$$K_{f_0}(u_0)^t = \Phi_r(x_0^{3/4}) \leqq \Phi_r(q_0) \leqq \limsup_{n\to\infty} K_{f_n}(u_0)^t. \quad \square$$

I will conclude this note with a slight generalization of Corollary 5.2. This particular result has an interpretation relevant to the Teichmüller spaces of real 3-tori, which I hope to expand upon in a subsequent article.

Theorem 6.2. *Fix* $X, Y \in GL_0(3, \mathbf{R})$, *with respective columns*

$$x_1, x_2, x_3;\ y_1, y_2, y_3 \in \mathbf{R}^3,$$

and assume that $K_E[YX^{-1}] \leqq \sqrt{e}$. *Then, in the class* $Q(X, Y)$ *of quasiconformal mappings* $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ *with*

$$f(u+x_i) = f(u)+y_i \quad (i = 1, 2, 3;\ u\in \mathbf{R}^3),$$

the affine mapping f_0 *with matrix* YX^{-1} *is extremal for* K_E.

Proof. Fix $f\in Q(X, Y)$. Since $f-f(0)$ also belongs to $Q(X, Y)$ there is no loss in generality in assuming that $f(0)=0$. Let $\mu=\sup\{\|f(u)-f_0(u)\|: u\in R\}$, where R is the parallelepiped spanned by the columns of X. Then because of the reproducing nature of f, it is clear that

$$\mu = \sup\{\|f(u)-f_0(u)\|: u\in \mathbf{R}^3\}.$$

Next, for each integer n, consider the map $f_n\colon \mathbf{R}^3\to\mathbf{R}^3$ in which

$$u \mapsto \frac{1}{n} f(nu).$$

One sees at once that $K_E[f_n]=K_E[f]$ for any n, and that

$$\sup\{\|f_n(u)-f_0(u)\|: u\in \mathbf{R}^3\} = \frac{\mu}{n} \to 0.$$

Therefore $f_n\to f_0$ uniformly in $\mathbf{R}^3$, and by Theorem 6.1,

$$K_E[f_0] \leqq \liminf K_E[f_n] = K_E[f]. \quad \square$$

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