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MEASURES OF NONCOMPACTNESS FOR ELEMENTS OF C^*-ALGEBRAS

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1. Introduction and notation

For Banach spaces E and F, let $L(E, F)$ denote the Banach space of bounded linear operators from E to F (and let $L(E)$ stand for $L(E, E)$). Several measures of noncompactness for $T\in L(E, F)$ have been considered in the literature. The ball measure of noncompactness, $\|T\|_q$, of T is defined by

$$\|T\|_q = \inf\left\{r > 0 \mid T(B_E) \subset \bigcup_{k=1}^{n} B(x_k, r),\ x_k\in F,\ n\in \mathbf{N}\right\}$$

(see e.g. [3], [5], [6], [1]). Here $B_E=\{x\in E \mid \|x\|\leqq 1\}$ and $B(x_k, r)=\{x\in F \mid \|x-x_k\|<r\}$. Another measure of noncompactness, $\|T\|_m$, is defined in [6, p. 7] to be the greatest lower bound of those numbers $\eta>0$ for which there exists a subspace $M\subset E$ with finite codimension and such that $\|Tx\|\leqq\eta\|x\|$ whenever $x\in M$. These two measures are equivalent seminorms on $L(E, F)$ [6, p. 7]. In case F has what [6] calls the compact approximation property (which is weaker than the metric approximation property), they are equivalent to the seminorm $T\mapsto\|T\|_{\mathscr{K}}$,

$$\|T\|_{\mathscr{K}} = \inf\{\|T-K\| \mid K\in L(E, F) \text{ is a compact operator}\}$$

[6, pp. 7, 11—12]. Let H be a complex Hilbert space, and $T\in L(H)$. We show that $\|T\|_m=\|T\|_{\mathscr{K}}$ (Theorem 1); the same technique yields the equation $\|T\|_q=\|T\|_{\mathscr{K}}$ proved in [10, p. 340]. Motivated by these results we define below for an element of an arbitrary C^*-algebra a measure of noncompactness modelled on $\|T\|_{\mathscr{K}}$; specializing in $L(H)$ we thus return to any one of the three measures discussed above.

Let A be a C^*-algebra. Following Vala [9] we call an element $u\in A$ compact if the mapping $x\mapsto uxu$ is a compact operator on A. We denote by $C(A)$ the set of the compact elements of A. As the compact elements of the C^*-algebra $L(H)$ are the same as the compact operators on the Hilbert space H [8], the following definition generalizes that of $\|T\|_{\mathscr{K}}$ for $T\in L(H)$.

Definition. If $u\in A$, we denote $k(u)=\inf\{\|u-x\| \mid x\in C(A)\}$ and call $k(u)$ the *(quotient) measure of noncompactness* of u.

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Even a measure of weak noncompactness has a simple connection with the present situation. Following the notation of [1] we write

$$\gamma_{\mathscr{W}}(T) = \inf\{r > 0 | T(B_E) \subset W + rB_F,\ W \subset F \text{ weakly compact}\}$$

for each $T \in L(E, F)$ (see Example 3.2 (b) in [1, p. 12]). In Theorem 2 we show that $k(u) = \gamma_{\mathscr{W}}(L_u) = \gamma_{\mathscr{W}}(R_u)$ for all $u \in A$, where L_u (resp. R_u) is the image of u under the left (resp. right) regular representation of A.

2. The equality of measures of noncompactness

The two theorems mentioned above are based on the following observation.

Lemma. *Let A be a C^*-algebra and I a closed two-sided ideal of A. Denote $q(x) = \inf\{\|x-y\| \,|\, y \in I\}$ for $x \in A$. Let p: $A \to \mathbf{R}$ be a seminorm such that $p(x) \leqq q(x)$ and $p(xy) \leqq p(x)p(y)$ for all $x, y \in A$, and $\{x \in A | p(x) = 0\} = I$. Then $p = q$.*

Proof. Let B denote the quotient algebra A/I and π: $A \to A/I$ the quotient map. Equipped with the involution $\pi(x) \mapsto \pi(x)^* = \pi(x^*)$ and the quotient norm $\pi(x) \mapsto \|\pi(x)\| = q(x)$, B is a C^*-algebra (see [4], Proposition 1.8.2). From our assumptions it follows that by setting $\|\pi(x)\|_1 = p(x)$ for $x \in A$ we get a well-defined norm $\|\cdot\|_1$ on B satisfying $\|u\|_1 \leqq \|u\|$ and $\|uv\|_1 \leqq \|u\|_1 \|v\|_1$ for all $u, v \in B$. Thus Corollary 4.8.4 in [7] (or the proof of Proposition 1.8.1 in [4]) shows that for any $u = \pi(x)$, $x \in A$, we get $\|u^*\|_1 \|u\|_1 \geqq \|u\|^2 = \|u^*\| \, \|u\|$, and since $\|u^*\|_1 \leqq \|u^*\|$ and $\|u\|_1 \leqq \|u\|$, we must have $p(x) = \|u\|_1 = \|u\| = q(x)$.

Theorem 1. *If H is a complex Hilbert space and $T \in L(H)$, then $\|T\|_q = \|T\|_m = \|T\|_{\mathscr{K}}$.*

Proof. Both $\|\cdot\|_q$ and $\|\cdot\|_m$ are submultiplicative seminorms on the C^*-algebra $L(H)$, they are majorized by $\|\cdot\|_{\mathscr{K}}$ and vanish precisely on the ideal of the compact operators on H (see [6, pp. 7, 9]). Thus the preceding Lemma implies the assertion.

Theorem 2. *Let A be a C^*-algebra and $u \in A$. Define L_u: $A \to A$ by $L_u x = ux$ and R_u: $A \to A$ by $R_u x = xu$. Then $k(u) = \gamma_{\mathscr{W}}(L_u) = \gamma_{\mathscr{W}}(R_u)$.*

Proof. Define $p(x) = \gamma_{\mathscr{W}}(L_x)$ for $x \in A$. From Proposition 3.7 in [1, p. 14] it follows that p is a seminorm on A, and applying (1) in [1, p. 17] we get $p(xy) = \gamma_{\mathscr{W}}(L_x L_y) \leqq \gamma_{\mathscr{W}}(L_x)\gamma_{\mathscr{W}}(L_y) = p(x)p(y)$. Theorem 3.1 in [12] states that an element x of A belongs to $C(A)$ if and only if L_x is a weakly compact operator. Since $\{T \in L(A) | \gamma_{\mathscr{W}}(T) = 0\}$ is the set of the weakly compact operators T: $A \to A$ (see Lemma 1 in [2, p. 259] or Theorem 3.11 in [1, p. 16]), it therefore follows that $p(x) = 0$ if and only if $x \in C(A)$. Furthermore, $p(x) \leqq \inf\{\|L_x - T\| \,|\, T \in L(A)$ weakly compact$\} \leqq \inf \{\|L_x - L_y\| \,|\, y \in C(A)\} = k(x)$ (see Corollary 3.9 or the proof of Theorem 3.8 in [1], and 1.3.5 in [4]). From the Lemma it now follows that $k=p$, since $C(A)$ is a closed two-sided ideal in A (see Theorem 3.10 in [11, p. 26]). A similar argument shows that $k(x)=\gamma_{\mathscr{W}}(R_x)$ for all $x \in A$.

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