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A NOTE ON MAXIMUM MODULUS ALGEBRAS

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1. Let X be a locally compact Hausdorff space and let A be an algebra of complex-valued continuous functions on X. Then A is called a (local) *maximum modulus algebra* on X, provided that for each compact subset K of X with topological boundary ∂K, and for each f in A, we have

$$(*) \qquad |f(p_0)| \leqq \max\{|f(p)| \mid p\in\partial K\} \quad \text{for} \quad p_0\in K$$

(see [9], [11]). One of the main results of [9] reads as follows ([9, Theorem 2]).

Theorem A. *If A is a maximum modulus algebra on a plane domain G, and if A contains a function which is analytic and not constant in G, then every member of A is analytic in G.*

Recently, Bear and Hile ([2, Theorem 4]) gave the following extension of Theorem A.

Theorem B. *Let G be a plane domain and A a maximum modulus algebra on G. If for each $z\in G$ there is an open neighborhood U_z of z and a function $f_z\in A$ such that f_z is an interior (i.e., light and open) mapping on U_z, then there is a homeomorphism φ of G onto a plane domain G' such that $g\circ\varphi^{-1}$ is analytic in G' for each $g\in A$.*

In this note we show that the requirement that f_z be open is not needed to establish the conclusion of Theorem B; in other words, openness turns out to be a consequence of the remaining assumptions. Actually, our first theorem states that members of any maximum modulus algebra are quasiopen mappings in the sense of Whyburn ([12], [13]); moreover, it is a simple matter to verify that a quasiopen and light mapping is open. Relying on Theorem 1, we will also revise some results of Bear and Hile ([2, Theorem 3]), W. C. Fox ([3]) and Kra ([5, Theorem III]). The note is concluded with some simple observations on algebras of quasiconformal functions (cf. [1]).

2. Let X and Y be topological spaces. A continuous mapping $f\colon X\to Y$ is said to be *quasiopen*, provided that for any $y\in f(X)$ and any open set U in X containing a compact component of $f^{-1}(y)$, y is an interior point of $f(U)$. The following characterization is due to Whyburn [13, p. 112].

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Lemma. *If X and Y are locally compact Hausdorff spaces, a mapping $f\colon X\to Y$ is quasiopen if and only if for each relatively compact open set U in X $\partial f(U)\subset f(\partial U)$.*

In fact, Whyburn limited himself to locally compact separable metric spaces, but the validity of the Lemma is readily seen even in the setting given above.

Theorem 1. *Let X be a locally compact Hausdorff space and let A be a maximum modulus algebra on X. Then every member of A is a quasiopen mapping $X\to\mathbf{C}$.*

Proof. By [9, Lemma 1], we may assume that A contains the constants. Let $f\in A$ and let U be a relatively compact open set in X. Suppose that $\partial f(U)\not\subset f(\partial U)$ and take a point $z_0\in\partial f(U)$ such that $z_0\notin f(\partial U)$. Since $f(\partial U)$ is compact (note that ∂U is nonempty by $(*)$), we find a point $z_1\in\mathbf{C}\setminus f(\overline{U})$ such that $|z_1-z_0|<\min\{|z_1-z|\,|\,z\in f(\partial U)\}$. Then pick out a point $z_2\in f(\overline{U})$ such that $|z_1-z_2|=\min\{|z_1-z|\,|\,z\in f(\overline{U})\}$. Clearly, $z_2\notin f(\partial U)$.

Set $z_3=(z_1+z_2)/2$ and

$$d=\min\left\{\frac{1}{4}|z_1-z_2|,\ \frac{1}{2}\min\{|z-z_2|\,|\,z\in f(\partial U)\}\right\}.$$

Further, let $D(z_2,d)$ stand for the set $\{z\in\mathbf{C}\,|\,|z-z_2|<d\}$. Now choose a point $q\in U$ such that $f(q)=z_2$ and denote by C the nonempty compact set $f^{-1}(\overline{D(z_2,d)})\cap\overline{U}$. It is clear that $C\subset U$ and $z_2\notin f(\partial C)$. Thus $|z_3-z_2|<\min\{|z-z_3|\,|\,z\in f(\partial C)\}$.

Denote by g the mapping $p\mapsto(z_3-f(p))^{-1}$, $p\in C$. By the previous inequality,

$$(**)\qquad |g(q)|>\max\{|g(p)|\,|\,p\in\partial C\}.$$

Consider the identity

$$(z_3-f(p))^{-1}=(z_3-z_2)^{-1}\cdot\sum_{i=0}^{n}\left(\frac{f(p)-z_2}{z_3-z_2}\right)^i+\left(\frac{f(p)-z_2}{z_3-z_2}\right)^{n+1}\cdot(z_3-f(p))^{-1},$$

$p\in C$, $n\in\mathbf{N}$. Clearly, the function

$$f_n\colon\ p\mapsto(z_3-z_2)^{-1}\cdot\sum_{i=0}^{n}\left(\frac{f(p)-z_2}{z_3-z_2}\right)^i$$

is a member of A for each n. On the other hand,

$$\left|(z_3-f(p))^{-1}\cdot\left(\frac{f(p)-z_2}{z_3-z_2}\right)^n\right|\leqq\frac{1}{|z_3-z_2|}\cdot\left(\frac{1}{2}\right)^n$$

in C for each n. It follows that $f_n\to g$ uniformly on C. But this implies that g also attains its maximum modulus on ∂C, a contradiction to $(**)$.

We conclude that $\partial f(U)\subset f(\partial U)$ for each relatively compact open set U in X. The assertion now follows from the preceding Lemma. □

Corollary. *Let X be a locally compact Hausdorff space and let A be a maximum modulus algebra on X. Then $f\in A$ is an open mapping $X\to\mathbf{C}$ whenever f is light.*

Remark. It is clear that for an *individual* function, in general, validity of the maximum principle does not imply quasiopenness (see also [3]).

3. Our first application provides the generalization of Theorem B mentioned before. Although it is an immediate consequence of Theorem B, in view of Corollary to Theorem 1, we prefer to base the proof on Theorem A and hence reproduce some arguments from [2].

Theorem 2. *Let G be a domain in $\hat{C}$, the extended plane, and let A be a maximum modulus algebra on G. If for each $z \in G$ there is an open neighborhood U_z of z and a function $f_z \in A$ such that f_z is light on U_z, then there is a homeomorphism Φ of G onto a plane domain G' such that $g \circ \Phi^{-1}$ is analytic in G' for each $g \in A$. Accordingly, the conclusion holds whenever A contains a mapping light on G.*

Proof. Let $z \in G$, and choose an open neighborhood $U_z \subset G$ of z and $f_z \in A$ such that $f_z | U_z$ is light. By Corollary to Theorem 1, $f_z | U_z$ is interior. By Stoïlow's theorem ([10, p. 121]), there is a homeomorphism φ_z on U_z such that $f_z \circ \varphi_z^{-1}$ is analytic on $\varphi_z(U_z)$. Let A_z stand for $\{g \circ \varphi_z^{-1} | g \in A\}$. Then A_z is a maximum modulus algebra on $\varphi_z(U_z)$ which contains the nonconstant analytic function $f_z \circ \varphi_z^{-1}$. By Theorem A, $g \circ \varphi_z^{-1}$ is analytic on $\varphi_z(U_z)$ for each $g \in A$.

It is now readily verified that G together with the local parameters (U_z, φ_z), $z \in G$, constitutes a Riemann surface $\tilde{G}$; moreover, the members of A are analytic on $\tilde{G}$. Since $\tilde{G}$ is planar, there is a conformal mapping Φ of $\tilde{G}$ onto a plane domain G'. Clearly, $g \circ \Phi^{-1}$ is analytic in G' for each $g \in A$. □

Example. Define $\varphi \colon \boldsymbol{C} \to \boldsymbol{R}$, $\varphi(z) = \operatorname{Re} z$, and denote by $C(\boldsymbol{R})$ the algebra of all continuous complex-valued functions on $\boldsymbol{R}$. Then $\{g \circ \varphi | g \in C(\boldsymbol{R})\}$ is a maximum modulus algebra on $\boldsymbol{C}$. This simple example shows that lightness, or something like that, is really needed to guarantee some sort of analyticity.

Next consider the situation of [2, Theorem 3]. In other words, let $G \subset \hat{C}$ be a domain and A a uniform algebra on $\bar{G}$ such that the maximal ideal space of A is $\bar{G}$ and the Shilov boundary Γ is a proper subset of $\bar{G}$ (for the terminology, we refer to [4]).

Let $z \in G \setminus \Gamma$ and let $U_z \subset G \setminus \Gamma$ be a connected open neighborhood of z. Suppose that there is a function $f \in A$ such that $f | U_z$ is light. Since A_z, the restriction of A to U_z, is a maximum modulus algebra on U_z by Rossi's theorem [4, p. 92], there is, by Theorem 2, a homeomorphism φ_z on U_z such that $g \circ \varphi_z^{-1}$ is analytic on $\varphi_z(U_z)$ for each $g \in A$.

As before, the local parameters (U_z, φ_z) are compatible in an obvious way. Consequently, for each component D of $G \setminus \Gamma$ there is a homeomorphism Φ of D onto a plane domain such that $g \circ \Phi^{-1}$ is analytic for each $g \in A$. Thus the property of being countable-to-one in the version of Bear and Hile is replaced by lightness in

Theorem 3. *Let* $G \subset \hat{C}$ *be a domain and A a function algebra on* $\bar{G}$. *Assume that the maximal ideal space of A is* $\bar{G}$ *and the Shilov boundary* Γ *is a proper subset of* $\bar{G}$. *If for each* $z \in G \setminus \Gamma$ *there is a neighborhood* U_z *of z and a function* $f_z \in A$ *such that* $f_z | U_z$ *is light, then there is on each component of* $G \setminus \Gamma$ *a homeomorphism* Φ *onto a plane domain such that* $g \circ \Phi^{-1}$ *is analytic for each* $g \in A$.

Remarks. (1) The assumptions of [2, Theorem 1] admit a similar relaxation. This follows immediately from Corollary to Theorem 1, in view of Rossi's theorem.

(2) Apparently, a result analogous to Theorem 3 can be obtained whenever G is a relatively compact domain in any Riemann surface. Similarly, in Theorem 2 G could be taken as an arbitrary Riemann surface.

In a similar fashion, we can establish the following extension of a result of W. C. Fox (see [3]).

Theorem 4. *Let X be a topological manifold of dimension two, and let f and g be functions, not both constants, sending X into* $\boldsymbol{C}$. *There exists a conformal structure for X relative to which both f and g are analytic if and only if the algebra generated by f and g is a maximum modulus algebra on X and at least one member in this algebra is also light.*

The next theorem generalizes a striking result of Kra ([5], [7]).

Theorem 5. *Let X be a connected, locally compact, Hausdorff space, and let A be a maximum modulus algebra on X which separates points and contains the constants. Suppose that, for every* $p \in X$, *the ideal* $M(p) = \{f \in A | f(p) = 0\}$ *is principal. Then X can be given a unique conformal structure which respects the topology such that every* $f \in A$ *becomes an analytic function on X. In particular, X is an open Riemann surface.*

Remark. In Kra's version, the nonconstant functions in A were assumed to be open mappings. Cf. also Remarks (2) and (3) in [5, p. 239].

Proof (as in [7]). Let $p \in X$ and let $t \in A$ be a function which generates $M(p)$. Since A separates points, $t(q) \neq 0$ for each $q \neq p$ in X. Let V be an open neighborhood of p with compact closure $\bar{V}$, and denote $\delta = \min \{|t(q)| \,|\, q \in \partial V\}$ (again, $\partial V \neq \emptyset$ by $(*)$).

Given any $f \in M(p)$, $|f|$ and $|f/t|$ attain their maxima for $\bar{V}$ at points on ∂V. Hence

$$\|f/t\| \leqq (1/\delta) \cdot \|f\|, \quad f \in M(p),$$

where $\| \ \|$ refers to the sup norm on $\bar{V}$.

Thus the assumptions of the lemma of Porcelli and Connell (see [7, pp. 318—319]) are satisfied. Consequently, on the open set $U = \{q \in V \,|\, |t(q)| < \delta/2\}$, every function $f \in A$ is equal to a convergent power series in t; i.e., $f | U = g \circ (t | U)$, where g is analytic on $\{z \in \boldsymbol{C} \,|\, |z| < \delta/2\}$. Since A separates points, t must be injective on U.

It follows from Corollary to Theorem 1 that t is also open on U. Accordingly, $t|U$ is a homeomorphism of U onto $t(U)=\{z\in C\big||z|<\delta/2\}$.

Now, clearly, the pairs $(U, t|U)$ constitute a unique conformal structure on X in such a way that each member of A becomes an analytic function on X. □

4. For the sake of illustration, we will add some observations on $QC(G)$, the class of quasiconformal functions on a plane domain G. Recall that a quasiconformal function on G can be defined as a function f which admits a representation $f=g\circ\varphi$, where φ is a quasiconformal homeomorphism $G\to\varphi(G)$ and g is an analytic function on $\varphi(G)$; thus we include the constants but exclude functions with "poles" (cf. [6, p. 250]).

Assume that $A\subset QC(G)$ is a nontrivial algebra with the usual operations. By Theorem B, we find a homeomorphism $\varphi\colon G\to\varphi(G)$ and an algebra, say B, of analytic functions on $\varphi(G)$ such that $A=B\circ\varphi=\{g\circ\varphi|g\in B\}$. Plainly, φ is quasiconformal on G. Thus $A\subset QC(G)$ constitutes an algebra if and only if there is a quasiconformal homeomorphism φ on G and an algebra B of analytic functions on $\varphi(G)$ such that $A=B\circ\varphi$. In particular, the complex dilatations (see [6, pp.191—192]) of any two nonconstant members of an algebra coincide (as elements of $L^\infty(G)$, of course).

Assume now that $A\subset QC(G)$ is a maximal algebra, i.e., $A=A'$ whenever A' is an algebra such that $A\subset A'\subset QC(G)$. Then clearly $A=H(G')\circ\varphi$, where $H(G')$ stands for the algebra of all analytic functions on $G'=\varphi(G)$. Obviously, there is a one-to-one correspondence between the class of maximal algebras in $QC(G)$ and the set $\{\mu\in L^\infty(G)\big|\|\mu\|<1\}$ (see [6, p. 204]).

Assume then that $A_i\subset QC(G)$ is a maximal algebra and φ_i a corresponding quasiconformal homeomorphism, $i=1, 2$. Suppose that $T\colon A_1\to A_2$ is an algebraic homomorphism. Let φ_i^* denote the homomorphism $g\mapsto g\circ\varphi_i$, $H(\varphi_i(G))\to A_i$, $i=1, 2$. Then $T'=\varphi_2^{*-1}\circ T\circ\varphi_1^*$ is an algebraic homomorphism $H(\varphi_1(G))\to H(\varphi_2(G))$. By [8, Theorem 1], there is a unique analytic mapping ψ of $\varphi_2(G)$ into $\varphi_1(G)$ such that $T'g=g\circ\psi$ for each $g\in H(\varphi_1(G))$. Consequently, there is a one-to-one correspondence between the class of homomorphisms $T\colon A_1\to A_2$ and the class of analytic mappings $\psi\colon \varphi_2(G)\to\varphi_1(G)$. In particular, A_1 and A_2 are algebraically isomorphic if and only if $\varphi_1(G)$ and $\varphi_2(G)$ are conformally equivalent.

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