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ON INTEGRABLE AUTOMORPHIC FORMS AND P-SEQUENCES OF ADDITIVE AUTOMORPHIC FUNCTIONS

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1. Let ∂D and D be the unit circle and the open unit disk, respectively. Denote the hyperbolic distance by $d(z_1, z_2)(z_1, z_2 \in D)$ and the hyperbolic disk $\{z|d(z, z_0)<r\}$ by $U(z_0, r)$. A sequence of points (z_n) in D tending to ∂D is said to be a sequence of P-points for a meromorphic function f if for each $r>0$, and every subsequence of points (z_k) of (z_n), f assumes every value, except perhaps two, infinitely often in the set $\bigcup_{k=1}^{\infty} U(z_k, r)$.

Let W' be an automorphic form with respect to a Fuchsian group Γ acting in D. We suppose that all residues of W' vanish. Then W' has

$$W(z) = \int_0^z W'(t)\,dt, \quad z\in D, \tag{1.1}$$

as an integral function and it satisfies the equality

$$W(T(z)) = \int_0^{T(z)} W'(t)\,dt = W(z)+A_T, \quad T\in\Gamma, \tag{1.2}$$

where A_T is called the period of W with respect to T. The integral function W will be called an additive automorphic function with respect to Γ.

For Γ the fundamental domain F_0 is fixed to be the set of points in D each of which is exterior to the isometric disks of all transformations of Γ. The fundamental domain F_0 is called thick if there exist positive constants Γ, Γ' such that for each sequence of points $(z_n)\subset F_0$ there is a sequence of points (z_n') for which $d(z_n, z_n')\leqq r$ and $U(z_n', r')\subset F_0$ for each $n=1, 2, \ldots$.

An additive automorphic function W is said to be of the second kind if there exists a sequence of points (z_n) in the closure $\bar{F}_0$ such that the sequence of functions

$$g_n(\zeta) = W\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right) \tag{1.3}$$

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tends uniformly to a constant limit in some neighbourhood of $\zeta=0$. An additive automorphic function W is said to be of the first kind if it is not of the second kind [1].

An automorphic form W' with respect to Γ is called integrable if

$$\iint_{F_0} \frac{1}{1-|z|^2} |W'(z)|\, d\sigma_z < \infty, \tag{1.4}$$

where $d\sigma_z$ is the euclidean area element.

In a previous paper [2], the author showed the relationship between the integral functions of integrable automorphic forms and Bloch functions in the case where the fundamental domain is thick, as stated in the following theorem.

1.1 Theorem. *Let W' be an integrable automorphic form with respect to Γ and the fundamental domain F_0 thick. Then W is a Bloch function in D.*

1.2. Remark. Concerning Theorem 1.1 one can allow parabolic vertices to appear in F_0 and demand F_0 to be thick outside parabolic vertices. This is due to the fact that W cannot have singularities at parabolic vertices because of the finiteness of the integral $\iint_{F_0} (1/(1-|z|^2))|W'(z)|d\sigma_z$.

1.3. Remark. Niebur and Sheingorn in [5, Theorem 2] have proved a result which corresponds to Theorem 1.1 supposing that the lower bound for the traces of the hyperbolic transformations in Γ is greater than 2 (cf. also [4, Theorem 1]).

2. In this section we continue to consider an additive automorphic function W satisfying the assumption (1.4) of the above theorem. In the case where the fundamental domain F_0 is thick we show W to be of the second kind. Further, we investigate by this theorem the behaviour of W' near the boundary ∂D and, in a special case, angular limits on ∂D.

2.1. Theorem. *If W' is an integrable automorphic form with respect to Γ and the fundamental domain F_0 thick, then W is of the second kind.*

Proof. Let (z_n) be a sequence of points converging to ∂D for which there exists a positive real number r such that $U(z_n, r)\subset F_0$ for each $n=1, 2, \ldots$. We form the functions

$$g_n(\zeta) = W\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right) - W(z_n). \tag{2.1}$$

By [2, Theorem 3.1] $\{g_n\}$ forms a normal family in D. Hence we can find a subsequence (g_k) of (g_n) such that $\lim_{k\to\infty} g_k(\zeta)=g(\zeta)$ uniformly on every compact part of D. We show first that $g'(\zeta_0)=0$ for each $\zeta_0\in U(0, r/2)$. Denote

$$z_k' = (\zeta_0+z_k)/(1+\bar{z}_k\zeta_0).$$

Since $d(z_k, z_k')=d(0, \zeta_0)<r/2$, we have $U(z_k', r/2)\subset F_0$. By [2, (3.2)]

$$|g_k'(\zeta_0)| = \frac{1}{1-\delta(z_k, z_k')^2}\cdot(1-|z_k'|^2)|W'(z_k')| \tag{2.2}$$

$$\leqq \frac{1}{1-\delta(z_k, z_k')^2}\cdot C \iint\limits_{U(z_k', r/2)} \frac{1}{1-|z|^2}|W'(z)|\, d\sigma_z,$$

where $\delta(z_k, z_k')=|z_k-z_k'|/|1-\bar{z}_k z_k'|$ and C is independent of k. Since $d(z_k, z_k')<r/2$, the expression $1/(1-\delta(z_k, z_k')^2)$ is bounded for each $k=1, 2, \ldots$. The boundedness of the integral $\iint_{F_0}(1/(1-|z|^2))|W'(z)|d\sigma_z$ implies that

$$\iint\limits_{U(z_k', r/2)} (1/(1-|z|^2))|W'(z)|d\sigma_z \to 0 \quad \text{for} \quad k\to\infty.$$

Hence $|g'(\zeta_0)|=\lim_{k\to\infty}|g_k'(\zeta_0)|=0$. Thus $g'(\zeta)\equiv 0$ in $U(0, r/2)$ and as an analytic function in the whole unit disk D. This implies, since $g(0)=\lim_{k\to\infty} g_k(0)=0$, that $g(\zeta)\equiv 0$ in D. Therefore W is of the second kind.

Using Theorem 2.1 we obtain knowledge of the boundary behaviour of the automorphic form W'.

2.2. Theorem. *Let W' be an integrable automorphic form with respect to Γ and the fundamental domain F_0 thick. Then for each sequence of points $(z_n)\subset G_R=\{z|d(z, F_0)<R\}$ converging to the boundary ∂D it holds*

$$\lim_{n\to\infty}(1-|z_n|^2)|W'(z_n)| = 0. \tag{2.3}$$

Proof. Suppose, on the contrary, that there is a sequence of points (z_n) in G_R converging to ∂D such that $\inf(1-|z_n|^2)|W'(z_n)|=a>0$. Choose a sequence of points (w_n) in F_0 such that for some positive constants R', r it holds $d(z_n, w_n)<R'$ and $U(w_n, r)\subset F_0$ for each $n=1, 2, \ldots$. We form the functions

$$g_n(\zeta) = W\left(\frac{\zeta+w_n}{1+\bar{w}_n\zeta}\right)-W(w_n) \tag{2.4}$$

and let (g_k) be a subsequence of (g_n) such that $\lim_{k\to\infty} g_k(\zeta)=g(\zeta)$ uniformly on every compact part of D. By the proof of Theorem 2.1 $g'(\zeta)\equiv 0$ in D. Let $\zeta_k=(z_k-w_k)/(1-\bar{w}_k z_k)$. Since $d(0, \zeta_k)=d(w_k, z_k)<R'$, we may suppose that there exists $\lim_{k\to\infty}\zeta_k=\zeta_0\in D$. Hence

$$\lim_{k\to\infty}(1-|z_k|^2)|W'(z_k)| \tag{2.5}$$

$$= \lim_{k\to\infty}(1-|\zeta_k|^2)|g_k'(\zeta_k)| = 0.$$

This is a contradiction, and thus the theorem is proved.

Under certain circumstances we can show the existence of an angular limit for W at a point for which the corresponding radius belongs to a fundamental domain.

2.3. Theorem. *Let W' be an integrable automorphic form with respect to Γ, the fundamental domain F_0 thick and (z_n) a sequence of points converging in a Stolz angle to the boundary point ξ for which the radius $0\xi \subset F_0 \cup \{\xi\}$. If $d(z_n, z_{n+1}) \leqq M < \infty$ and $\lim_{n\to\infty} W(z_n) = c$, then W has the angular limit c at ξ.*

Proof. Suppose, on the contrary, that W does not have the angular limit c at ξ. Then there is a sequence of points $(w_n) \subset \alpha$ (a Stolz angle at ξ) converging to ξ such that $\lim_{n\to\infty} W(w_n) = d \neq c$. By the assumptions we can choose subsequences of (z_n) and (w_n) and endow them with new indexes in such a way that there are points $(z_k') \subset F_0$ for which $\max\{d(z_k', z_k), d(z_k', w_k)\} \leqq M' < \infty$ and form the functions

$$g_k(\zeta) = W\left(\frac{\zeta + z_k'}{1 + \bar{z}_k'\zeta}\right). \tag{2.6}$$

Let (g_m) be a subsequence of (g_k) such that $\lim_{m\to\infty} g_m(\zeta) = g(\zeta)$ uniformly on every compact part of D. By the proof of Theorem 2.1 g is constant in D. Let $\zeta_m^z = (z_m - z_m')/(1 - \bar{z}_m' z_m)$ and $\zeta_m^w = (w_m - z_m')/(1 - \bar{z}_m' w_m)$. We can suppose, without loss of generality, that the limits $\lim_{m\to\infty} \zeta_m^z = \zeta^z$ and $\lim_{m\to\infty} \zeta_m^w = \zeta^w$ exist in D. Consequently,

$$g(\zeta^z) = \lim_{m\to\infty} g_m(\zeta_m^z) = \lim_{m\to\infty} W(z_m) = c$$

and

$$g(\zeta^w) = \lim_{m\to\infty} g_m(\zeta_m^w) = \lim_{m\to\infty} W(w_m) = d,$$

which is a contradiction. The theorem follows.

3. In what follows we shall consider the images of sequences of P-points for an additive automorphic function W under the transformations of Γ in different cases.

3.1. Let W be a non-normal additive automorphic function with respect to Γ and (z_n) a sequence of P-points for W. Choose the transformations $T_n \in \Gamma$ such that $T_n(z_n) = z_n' \in \bar{F}_0$ for each $n = 1, 2, \ldots$. If the period set $\{A_{T_n}\} = \{W(z_n') - W(z_n)\}$ is bounded, we show that the image sequence (z_n') of (z_n) is also a sequence of P-points for W.

By the properties of sequences of P-points we can find a sequence of points (w_n) such that $\lim_{n\to\infty} d(z_n, w_n) = 0$ and $\lim_{n\to\infty} (1 - |w_n|^2)|W'(w_n)|/(1 + |W(w_n)|^2) = \infty$. Let $T_n(w_n) = w_n'$. Now

$$(1 - |w_n'|^2)\frac{|W'(w_n')|}{1 + |W(w_n')|^2} \tag{3.1}$$

$$= (1 - |w_n|^2)\frac{|W'(w_n)|}{1 + |W(w_n) + A_{T_n}|^2} \to \infty$$

for $n \to \infty$. This implies that (w_n') is a sequence of P-points for W. Since $d(z_n', w_n') = d(z_n, w_n) \to 0$ for $n \to \infty$, $(z_n') = (T_n(z_n))$ is also a sequence of P-points for W.

3.2. Remark. Since (z_n') is a sequence of P-points for a meromorphic function W in D, $\lim_{n\to\infty} |z_n'| = 1$.

3.3. If the period set $\{A_{T_n}\} = \{W(z_n') - W(z_n)\}$ is arbitrary, then Remark 3.2 holds provided W is an analytic function in D. In order to prove this we suppose, on the contrary, that $|z_k'| < r < 1$ for some subsequence (z_k') of (z_n'). We may assume that $\lim_{k\to\infty} z_k' = z_0' \in D$. Form the functions $g_k(\zeta) = W(S_k(\zeta)) = W((\zeta + z_k')/(1 + \bar{z}_k'\zeta))$ defined in D. Then $\lim_{k\to\infty} g_k(\zeta) = W((\zeta + z_0')/(1 + \bar{z}_0'\zeta)) = g(\zeta)$ uniformly in $U(0, r)$ for any positive number r. Here g is a bounded analytic function in $U(0, r)$. Let $h_k(\zeta) = W(T_k^{-1}(S_k(\zeta))) = g_k(\zeta) - A_{T_k}$. We can find a subsequence (h_m) of (h_k) such that either 1) $\lim_{m\to\infty} h_m(\zeta)$ is a bounded analytic function or 2) $\lim_{m\to\infty} h_m(\zeta) = \infty$ in $U(0, r)$. Both results contradict the definition of P-points for W.

3.4. Remark. The assertion in 3.3 does not hold if we allow W to be meromorphic in D. We give a simple example. Let W be an additive automorphic function with respect to Γ for which the fundamental domain F_0 is compact in D. Suppose that W has poles in D and at least one non-zero period. Then, by [3, Theorem 2.1], W is not normal and one can find a sequence of P-points (z_n) for W. If $T_n(z_n) = z_n' \in \bar{F}_0$, $T_n \in \Gamma$, then $\lim_{n\to\infty} |z_n'| \neq 1$.

3.5. Remark. In the cases of Remark 3.2 and 3.3 it follows that the fundamental domain F_0 cannot be compact in D. Thus if, in 3.3, F_0 is compact, then (z_n) is not a sequence of P-points. Hence W is normal in D. On the other hand, it is known that W is even a Bloch function in D if $W(F_0)$ is bounded in the complex plane (cf. [6, Theorem 3]).

3.6. Remark. The conclusion of 3.1 does not hold even for an analytic function W if the period set $\{A_{T_n}\}$ is unbounded. This we can see by the following example.

Consider the function f: $f(w) = \log w + i \log (w-1)$ on the universal covering surface of the domain $C - \{0, 1\}$. Since the corresponding additive automorphic function W in D has the periods $2\pi i$, 2π for which $\operatorname{Im}(2\pi i/2\pi) = 1 \neq 0$, W is not normal (cf. [3, 6.1′]). Let (z_n) be any sequence of P-points for W in D. Suppose that $(T_n(z_n)) = (z_n') \subset \bar{F}_0$. By 3.3 at least some subsequence (z_k') of (z_n') converges to a parabolic vertex p of F_0. On the other hand, W has the angular limit ∞ at p. Thus (z_n') is not a sequence of P-points for W and the assertion is proved.

An additive automorphic function W is called F_0-normal if for each sequence of points (z_n) in the closure $\bar{F}_0$ the sequence of functions

$$g_n(\zeta) = W\left(\frac{\zeta + z_n}{1 + \bar{z}_n \zeta}\right) \tag{3.2}$$

forms a normal family in D.

Concluding the consideration of sequences of P-points we suppose W to be F_0-normal, of the first kind and to possess a sequence of P-points. By these assumptions, in the next theorem, one obtains a condition between the sequence of P-points and poles of W.

3.7. Theorem. *Let W be an F_0-normal but non-normal additive automorphic function of the first kind with respect to Γ. If (z_n) is a sequence of P-points for W and (w_n) the sequence of poles of W in D, then* $d((z_n), (w_n)) = \inf_{n,m} d(z_n, w_m) = 0$.

Proof. Let $T_n(z_n) = z_n' \in \bar{F}_0$, $T_n \in \Gamma$. Note that $d((z_n), (w_n)) = d((z_n'), (w_n))$. Suppose, on the contrary, that $d((z_n'), (w_n)) = r > 0$. Form the functions

$$g_n(\zeta) = W\left(\frac{\zeta + z_n'}{1 + \bar{z}_n' \zeta}\right). \tag{3.3}$$

Since W is F_0-normal and of the first kind, one can find a subsequence (g_k) of (g_n) converging uniformly to a non-constant meromorphic function g on every compact part of D. By the Hurwitz theorem g is a bounded analytic function in $U(0, r/2)$. If we consider the functions

$$\begin{aligned} h_k(\zeta) &= W\left(T_k^{-1}\left(\frac{\zeta + z_k'}{1 + \bar{z}_k' \zeta}\right)\right) \\ &= W\left(\frac{\zeta + z_k'}{1 + \bar{z}_k' \zeta}\right) - A_{T_k} = g_k(\zeta) - A_{T_k}, \end{aligned} \tag{3.4}$$

some subsequence denoted by (h_m) converges uniformly either to an analytic function or the constant ∞ on every compact part of $U(0, r/2)$. This contradicts the definition of a sequence of P-points for W. The assertion follows.

In [2] we proved the following theorem, where $d^*(w_1, w_2)$ denotes the spherical distance.

Theorem. *Let W be an analytic F_0-normal additive automorphic function of the first kind and r any positive number. Then there exists a positive number m_r such that*

$$d^*(W(z), w) > m_r \quad \text{in} \quad D - \bigcup_{n=1}^{\infty} U(z_n(w), r),$$

where $z_n(w)$, $n = 1, 2, \ldots$, denote the w-points of W.

The above theorem does not hold if we suppose W to be only an F_0-normal additive automorphic function of the first kind, that is, we omit the condition of analyticity. This we show by a simple example.

Let W be an additive automorphic function with respect to Γ for which the fundamental domain F_0 is compact in D. We denote $\{z_m(\infty) | m = 1, 2, \ldots\}$ for the set of poles for W and suppose that the period $A_T \neq 0$ for some hyperbolic transformation $T \in \Gamma$. It follows trivially that W is F_0-normal and of the first kind. Since F_0 is compact, $\inf_{m,k} d(z_m(\infty), z_k(\infty)) = s > 0$. Let r be any positive number such that $r < s/2$. Denote $\inf \{d^*(W(z), \infty) | z \in D - \bigcup_{m=1}^{\infty} U(z_m(\infty), r)\} = M$. Choose a point $z_0 \in \partial U(z_1(\infty), r)$. Then $W(z_0) = w_0 \neq \infty$. Now

$$M \leqq d^*(W(T^n(z_0)), \infty) = \frac{1}{(1+|w_0+nA_T|^2)^{1/2}} \to 0 \tag{3.5}$$

as $n \to \infty$. As a non-negative constant $M=0$ and thus the assertion is proved.

However, the theorem holds in the following modified form where $G_R = \{z | d(z, F_0) < R\}$.

3.8. Theorem. *Let W be an F_0-normal additive automorphic function of the first kind, $\{z_n(w)\}$ the set of w-points for W in D and R, r any positive numbers. Then there exists $m_r > 0$ such that*

$$d^*(W(z), w) > m_r \quad in \quad G_R - \bigcup_{n=1}^{\infty} U(z_n(w), r). \tag{3.6}$$

Proof. Suppose, on the contrary, that there exist positive numbers R, r and a sequence of points $(z_n) \subset G_R - \bigcup_{n=1}^{\infty} U(z_n(w), r)$ such that $\lim_{n\to\infty} W(z_n) = w$. Form the functions

$$g_n(\zeta) = W\left(\frac{\zeta + z_n}{1 + \bar{z}_n \zeta}\right). \tag{3.7}$$

By the assumption there is a subsequence (g_k) of (g_n) such that $\lim_{k\to\infty} g_k(\zeta) = g(\zeta)$ uniformly on every compact part of D. Further, g is a non-constant meromorphic function in D. Now $g(0) = \lim_{k\to\infty} g_k(0) = \lim_{k\to\infty} W(z_k) = w$. By the Hurwitz theorem there is a sequence of points (ζ_k) in $U(0, r/2)$ such that $g_k(\zeta_k) = w$ for $k \geqq k_0$. Denote $z_k' = (\zeta_k + z_k)/(1 + \bar{z}_k \zeta_k)$. Then $W(z_k') = g_k(\zeta_k)$ and W has thus the w-point at z_k' for $k \geqq k_0$. However, it holds

$$\inf_n d(z_k', z_n(w)) \geqq r/2. \tag{3.8}$$

This is a contradiction and the theorem is proved.

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