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ON PSEUDO-MONOTONE OPERATORS AND NONLINEAR PARABOLIC INITIAL-BOUNDARY VALUE PROBLEMS ON UNBOUNDED DOMAINS

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1. Introduction

Let Ω be an arbitrary domain in R^N $(N\geqq 1)$ and let Q be the cylinder $\Omega\times(0, T)$ with a given $T>0$. We shall consider on Q the quasilinear parabolic partial differential operator of order $2m$ $(m\geqq 1)$ of the form

$$\frac{\partial u(x, t)}{\partial t}+Au(x, t), \tag{1}$$

where A is an elliptic operator given in the divergence form

$$Au(x, t)=\sum_{|\alpha|\leqq m}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x, t, u, Du, \ldots, D^m u). \tag{2}$$

The coefficients A_α are regarded as real-valued functions of the point (x, t) in Q, of $\eta=\{\eta_\beta\colon |\beta|\leqq m-1\}$ in R^{N_1} and of $\zeta=\{\zeta_\beta\colon |\beta|=m\}$ in R^{N_2}, where $\alpha=(\alpha_1, \ldots, \alpha_N)$ and $\beta=(\beta_1, \ldots, \beta_N)$ are N-tuples of nonnegative integers, $|\beta|=\beta_1+\ldots+\beta_N$ and $D^\alpha=\prod_{i=1}^N(\partial/\partial x_i)^{\alpha_i}$.

If we assume that the functions A_α satisfy the familiar condition

(A_1) *Each* $A_\alpha(x, t, \eta, \zeta)$ *is measurable in* (x, t) *for fixed* $\xi=(\eta, \zeta)$ *and continuous in* ξ *for fixed* (x, t). *For a given* $p>1$ *there exists a constant* $c_1>0$ *and a function* $k_1\in L^{p'}(Q)$ *with* $p'=p/(p-1)^{-1}$ *such that*

$$|A_\alpha(x, t, \eta, \zeta)|\leqq c_1\big(|\zeta|^{p-1}+|\eta|^{p-1}+k_1(x, t)\big)$$

for all $|\alpha|\leqq m$, *all* $(x, t)\in Q$ *and all* $\xi=(\eta, \zeta)\in R^{N_1+N_2}=R^{N_0}$,

then the operator A gives rise to a bounded map S from the space $\mathscr{V}=L^p(0, T; V)$ to its dual space $\mathscr{V}^*$, V being a closed subspace of the Sobolev space $W^{m,p}(\Omega)$.

When Ω is a bounded domain, the operator $\partial/\partial t$ induces a maximal monotone map L from the subset $D(L)=\{v\in\mathscr{V}\colon \partial v/\partial t\in\mathscr{V}^*,\ v(x, 0)=0 \text{ in } \Omega\}$ to $\mathscr{V}^*$, and

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from a simple set of additional hypotheses of the Leray—Lions type it can be derived that S is pseudo-monotone on $D(L)$. This result is then applicable to the existence of weak solutions for the parabolic initial-boundary value problems for the operator (1).

When Ω is unbounded, the situation is different while the compactness part of the Sobolev embedding theorem and Relich's selection theorem are no more available and the above definition of the set $D(L)$ does not make sense.

The purpose of the present note is to show, for arbitrary domains Ω, that the map S induced by the elliptic operator A is pseudo-monotone as a map from the space $\mathcal{W}=\mathcal{V}\cap L^2(Q)$ to $\mathcal{W}^*$ on the set $D(L)=\{v\in\mathcal{W}:\ \partial v/\partial t\in\mathcal{W}^*,\ v(x,0)=0 \text{ in } \Omega\}$ whenever the coefficients A_α satisfy the following conditions (cf. [5] p. 323, [6]) in addition to (A_1):

(A_2) *For each* $(x,t)\in Q$, *each* $\eta\in R^{N_1}$ *and any pair of distinct elements* ζ *and* ζ^* *in* R^{N_2},

$$\sum_{|\alpha|=m}\{A_\alpha(x,t,\eta,\zeta)-A_\alpha(x,t,\eta,\zeta^*)\}(\zeta_\alpha-\zeta_\alpha^*)>0.$$

(A_3) *There exist a constant* $c_2>0$ *and functions* $k_2\in L^1(Q)$, $h_\alpha\in L^{p'}(Q)$ *for all* $|\alpha|\leqq m$, *such that*

$$\sum_{|\alpha|\leqq m}A_\alpha(x,t,\xi)\xi_\alpha\geqq-\sum_{|\alpha|\leqq m}h_\alpha(x,t)\xi_\alpha-k_2(x,t)$$

for all $(x,t)\in Q$ *and all* $\xi\in R^{N_0}$.

This result is analogous to the elliptic case studied by F. E. Browder [1]. In fact, our method here is a modification of the method introduced by R. Landes and V. Mustonen [4], which makes it possible to relax one of the classical conditions imposed on the coefficients A_α.

The result of pseudo-monotonicity can be applied to the variational problems for the operator (1) involving a domain which is not necessarily bounded. As an example we shall show that the partial differential equation

$$\frac{\partial u}{\partial t}+Au=f \quad \text{in} \quad Q \tag{3}$$

with the initial-boundary conditions

$$\begin{cases} u(x,0)=0 & \text{in } \Omega\\ D^\alpha u=0 & \text{on } \partial\Omega\times(0,T) \quad \text{for } |\alpha|\leqq m-1\end{cases} \tag{4}$$

admits a solution u for any given f in $L^{p'}(Q)$. Under similar conditions (a condition stronger than our (A_3) was needed) this existence theorem was also proved by G. Mahler [6] by an ad hoc approximation method which was originally introduced by P. Hess [2] for elliptic Dirichlet problems.

2. Prerequisites

Let Ω be an open subset of R^N. The Sobolev space of functions u such that u and its distributional derivatives $D^\alpha u$ lie in $L^p(\Omega)$ for all $|\alpha|\leqq m$ is denoted by $W^{m,p}(\Omega)$. By $W_0^{m,p}(\Omega)$ we mean the closure in $W^{m,p}(\Omega)$ of $C_0^\infty(\Omega)$, the space of test functions with compact support in Ω. If $u\in W^{m,p}(\Omega)$, we shall write $\eta(u)=\{D^\alpha u\colon |\alpha|\leqq m-1\}$, $\zeta(u)=\{D^\alpha u\colon |\alpha|=m\}$ and $\xi(u)=\{D^\alpha u\colon |\alpha|\leqq m\}$. When $T>0$ is given and V is a closed subspace of the Sobolev space $W^{m,p}(\Omega)$, we denote $\mathscr{V}=L^p(0, T; V)$, a Banach space equipped with the norm

$$\|u\|_{\mathscr{V}}=\Big\{\int_0^T \|u(t)\|_V^p\,dt\Big\}^{1/p}.$$

We let further $\mathscr{W}$ stand for the Banach space $\mathscr{V}\cap L^2(Q)$ with $Q=\Omega\times(0, T)$ and with the norm $\|\cdot\|_{\mathscr{W}}=\|\cdot\|_{\mathscr{V}}+\|\cdot\|_{L^2(Q)}$.

The duality pairing between the elements u in a Banach space X and f in X^* is denoted by $(f, u)_X$, where the subscript X will be omitted when no confusion is possible. If $1<p<\infty$, $\mathscr{W}$ is reflexive and its dual space is $\mathscr{W}^*=\mathscr{V}^*+L^2(Q)$, where $\mathscr{V}^*=L^{p'}(0, T; V^*)$. Furthermore,

$$\mathscr{W}\subset L^2(Q)\subset\mathscr{W}^*\subset L^1(0, T; V^*+L^2(\Omega));$$

for each $u\in\mathscr{W}$ the distribution derivative $u'=\partial u/\partial t$ can be defined and the condition $u'\in\mathscr{W}^*$ makes sense. Each $u\in\mathscr{W}$ with $u'\in\mathscr{W}^*$ is (after a modification on a set of measure zero) a continuous function, $[0, T]\to L^2(\Omega)$ and the following integration formula holds (see [6], [7]) for all $u, v\in\mathscr{W}$ with $u', v'\in\mathscr{W}^*$:

(5) $$(u', v)_{\mathscr{W}}+(v', u)_{\mathscr{W}}=\big(u(T), v(T)\big)_{L^2(\Omega)}-\big(u(0), v(0)\big)_{L^2(\Omega)}.$$

Let L stand for the linear map from $\mathscr{W}$ to $\mathscr{W}^*$ which takes u to u' having the domain

$$D(L)=\{u\in\mathscr{W}\colon u'\in\mathscr{W}^*,\ u(x, 0)=0 \text{ in } \Omega\}.$$

It follows from (5) that $(Lu, u)\geqq 0$ for all $u\in D(L)$. Thus L is a monotone linear map.

We close this section by recalling the definition of a pseudo-monotone map and an abstract surjectivity result which we will employ in proving the existence theorem in Section 4. Indeed, Theorem 1.2 of [5] p. 319 can be stated as follows:

Proposition 1. *Let X be a reflexive Banach space with strictly convex norms in X and X^*. Let L be a linear maximal monotone map from $D(L)$ to X^* with $D(L)$ dense in X, let T be a bounded map from X to X^*, and suppose that T is $D(L)$-pseudo-monotone, i.e. for any sequence $(v_n)\subset D(L)$ with $v_n\rightharpoonup v$ (weak convergence) in X, $Lv_n\rightharpoonup Lv$ in X^* and* $\limsup\big(T(v_n), v_n-v\big)\leqq 0$, *it follows that $T(v_n)\rightharpoonup T(v)$ in X^* and $\big(T(v_n), v_n\big)\to\big(T(v), v\big)$. If T is coercive on X, i.e. $\big(T(u), u\big)\|u\|^{-1}\to\infty$ as $\|u\|\to\infty$ in X, then for any $f\in X^*$ there is $u\in D(L)$ such that $Lu+T(u)=f$.*

3. Theorem on pseudo-monotonicity

Let us assume that the coefficients A_α of the operator (2) satisfy the conditions (A_1), (A_2) and (A_3) in the given domain $Q=\Omega\times(0, T)$. On account of (A_1) the equation

$$a(u, v) = \sum_{|\alpha|\leqq m} \int_Q A_\alpha(x, t, \xi(u)) D^\alpha v \, dx \, dt \tag{6}$$

defines a bounded semilinear form on $\mathscr{V}\times\mathscr{V}$. Hence (6) gives rise to a bounded (nonlinear) map S from $\mathscr{V}$ to $\mathscr{V}^*$ by the rule

$$(S(u), v) = a(u, v), \quad u, v\in\mathscr{V}. \tag{7}$$

In view of (A_1) and (A_3) it is clear that $\mathscr{V}$ would be the natural space for the mapping S but, on the other hand, the map L is defined on the subset $D(L)\subset\mathscr{W}\subset\mathscr{V}$ only, with values in $\mathscr{W}^*$. Therefore we shall regard S as a map from $\mathscr{W}$ to $\mathscr{W}^*$ and prove

Theorem 1. *Let Ω be an arbitrary domain in R^N, $T>0$, $Q=\Omega\times(0, T)$ and let the functions A_α satisfy the conditions* (A_1), (A_2) *and* (A_3). *Then the map S from $\mathscr{W}$ to $\mathscr{W}^*$ defined by* (7) *is $D(L)$-pseudo-monotone.*

Proof. We can follow the lines of the proof of the elliptic case in [4]. Indeed, let $(v_n)\subset D(L)$ be a sequence such that $v_n\rightharpoonup v$ in $\mathscr{W}$, $Lv_n\rightharpoonup Lv$ in $\mathscr{W}^*$ and $\limsup (S(v_n), v_n-v)\leqq 0$. We must verify that $S(v_n)\rightharpoonup S(v)$ in $\mathscr{W}^*$ and that $(S(v_n), v_n)\to (S(v), v)$, at least for an infinite subsequence of (v_n). As $v_n\rightharpoonup v$ in $\mathscr{W}$, $D^\alpha v_n\rightharpoonup D^\alpha v$ in $L^p(Q)$ for all $|\alpha|\leqq m$ and $v_n\rightharpoonup v$ in $L^2(Q)$. Our aim is to show that $D^\alpha v_n(x, t)\to D^\alpha v(x, t)$ almost everywhere in Q for all $|\alpha|\leqq m$ for some subsequence. By (A_1) this implies that $A_\alpha(x, t, \xi(v_n))\to A_\alpha(x, t, \xi(v))$ a.e. in Q for all $|\alpha|\leqq m$. By (A_1) this also means that $A_\alpha(\cdot, \cdot, \xi(v_n))\rightharpoonup A_\alpha(\cdot, \cdot, \xi(v))$ in $L^{p'}(Q)$, and thus $S(v_n)\rightharpoonup S(v)$ in $\mathscr{W}^*$ follows. The a.e. convergence of $D^\alpha v_n(x, t)$ to $D^\alpha v(x, t)$ for all $|\alpha|\leqq m-1$ is established by Aubin's Lemma ([5] p. 57). Indeed, $W^{m,p}(\Omega)$ is compactly embedded in $W^{m-1,p}(\omega)$ for any subdomain ω with a compact closure in Ω. Thus $v_n\rightharpoonup v$ in $\mathscr{W}$ and $Lv_n\rightharpoonup Lv$ in $\mathscr{W}^*$ together imply (cf. [6] p. 205) that $v_n\to v$ (strongly) in $L^p(0, T; W^{m-1,p}(\omega))$, i.e. $D^\alpha v_n\to D^\alpha v$ in $L^p(\omega\times(0, T))$ for all $|\alpha|\leqq m-1$, and the a.e. convergence for a subsequence follows.

To verify that $D^\alpha v_n(x, t)\to D^\alpha v(x, t)$ a.e. in Q also for all $|\alpha|=m$ we denote

$$q_n(x, t) = \sum_{|\alpha|=m} \{A_\alpha(x, t, \eta(v_n), \zeta(v_n)) - A_\alpha(x, t, \eta(v_n), \zeta(v))\}(D^\alpha v_n - D^\alpha v),$$

$$p_n(x, t) = \sum_{|\alpha|\leqq m} A_\alpha(x, t, \xi(v_n))(D^\alpha v_n - D^\alpha v),$$

$$r_n(x, t) = \sum_{|\alpha|=m} A_\alpha(x, t, \eta(v_n), \zeta(v))(D^\alpha v - D^\alpha v_n),$$

$$s_n(x, t) = \sum_{|\alpha|\leqq m-1} A_\alpha(x, t, \xi(v_n))(D^\alpha v - D^\alpha v_n).$$

Then $q_n = p_n + r_n + s_n$ in Q. If we can show that $q_n(x, t) \to 0$ a.e. in Q, then the desired result follows from Lemma 6 due to R. Landes [3]. In fact, as $q_n(x, t) \geqq 0$ for almost all $(x, t) \in Q$ by (A_2), it suffices to show that

$$\limsup \int_{Q_k} q_n(x, t)\, dx\, dt \leqq \varepsilon_k, \tag{8}$$

where $Q_k = \Omega_k \times (0, T)$, (Ω_k) is a growing sequence of bounded subdomains of Ω such that $\mu(\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k) = 0$ and $\varepsilon_k \to 0$ as $k \to \infty$. For any fixed k we have

$$\int_{Q_k} q_n(x, t) = \int_{Q} p_n(x, t) - \int_{Q \setminus Q_k} p_n(x, t) + \int_{Q_k} (r_n(x, t) + s_n(x, t)),$$

where we know by assumption that $\limsup \int_Q p_n(x, t) \leqq 0$. Moreover, since $(D^\alpha v_n)$ is bounded in $L^p(Q)$ and $(A_\alpha(.,., \xi(v_n)))$ is bounded in $L^{p'}(Q)$, we get by (A_3),

$$\begin{aligned} -\int_{Q \setminus Q_k} p_n(x, t) &= -\sum_{|\alpha| \leqq m} \int_{Q \setminus Q_k} A_\alpha(x, t, \xi(v_n)) D^\alpha v_n \\ &\quad + \sum_{|\alpha| \leqq m} \int_{Q \setminus Q_k} A_\alpha(x, t, \xi(v_n)) D^\alpha v \\ &\leqq c \sum_{|\alpha| \leqq m} \Big\{ \int_{Q \setminus Q_k} |h_\alpha(x, t)|^{p'} \Big\}^{1/p'} + \int_{Q \setminus Q_k} k_2(x, t) \\ &\quad + c \sum_{|\alpha| \leqq m} \Big\{ \int_{Q \setminus Q_k} |D^\alpha v|^p \Big\}^{1/p} := \varepsilon_k, \end{aligned}$$

where obviously $\varepsilon_k \to 0$ as $k \to \infty$, c being some positive constant. Since $D^\alpha v_n \to D^\alpha v$ in $L^p(Q_k)$ for all $|\alpha| \leqq m-1$ and since $A_\alpha(.,., \eta(v_n), \zeta(v)) \to A_\alpha(.,., \eta(v), \zeta(v))$ in $L^{p'}(Q_k)$ for all $|\alpha| = m$ by (A_1) and the dominated convergence theorem, we can conclude that

$$\limsup \int_{Q_k} (r_n(x, t) + s_n(x, t)) \leqq 0$$

for any fixed k. Thus (8) has been verified.

We complete the proof by showing that $(S(v_n), v_n) \to (S(v), v)$. In view of the assumption, $\limsup (S(v_n), v_n) \leqq (S(v), v)$. Hence it suffices to prove that

$$\liminf (S(v_n), v_n) \geqq (S(v), v). \tag{9}$$

By (A_2) we have

$$\begin{aligned} \sum_{|\alpha| = m} A_\alpha(x, t, \xi(v_n)) D^\alpha v_n &\geqq \sum_{|\alpha| = m} A_\alpha(x, t, \eta(v_n), \zeta(v)) (D^\alpha v_n - D^\alpha v) \\ &\quad + \sum_{|\alpha| = m} A_\alpha(x, t, \xi(v_n)) D^\alpha v, \end{aligned}$$

and hence we get further for any fixed k,

$$\sum_{|\alpha|\leqq m}\int_Q A_\alpha(x,t,\xi(v_n))D^\alpha v_n \geqq \sum_{|\alpha|=m}\int_{Q_k} A_\alpha(x,t,\eta(v_n),\zeta(v))(D^\alpha v_n - D^\alpha v)$$

$$+\sum_{|\alpha|=m}\int_{Q_k} A_\alpha(x,t,\xi(v_n))D^\alpha v + \sum_{|\alpha|\leqq m-1}\int_{Q_k} A_\alpha(x,t,\xi(v_n))D^\alpha v_n$$

$$+\sum_{|\alpha|\leqq m}\int_{Q\setminus Q_k} A_\alpha(x,t,\xi(v_n))D^\alpha v_n.$$

By the same arguments as above in proving (8) we obtain

$$\liminf \sum_{|\alpha|\leqq m}\int_Q A_\alpha(x,t,\xi(v_n))D^\alpha v_n \geqq \sum_{|\alpha|\leqq m}\int_Q A_\alpha(x,t,\xi(v))D^\alpha v - \varepsilon_k,$$

where $\varepsilon_k \to 0$ as $k\to\infty$, and so the proof is complete.

4. Parabolic initial-boundary value problem

We shall employ Theorem 1 and Proposition 1 for obtaining an existence theorem for the parabolic equation (3) with initial-boundary conditions (4). Indeed, we can choose $V=W_0^{m,p}(\Omega)$ to obtain

Theorem 2. *Let Ω be an arbitrary domain in R^N, $T>0$, $Q=\Omega\times(0,T)$ and let the functions A_α satisfy the conditions* (A_1), (A_2) *and* (A_3). *If $a(u,u)\|u\|_{\mathcal{V}}^{-1}\to\infty$ for all $u\in\mathcal{V}$ with $\|u\|_{\mathcal{V}}\to\infty$, then for any $f\in L^{p'}(Q)+L^2(Q)$ the equation* (3) *admits a weak solution u in $D(L)$, i.e. there exists $u\in D(L)$ such that*

$$\left(\frac{\partial u}{\partial t}, w\right)_{\mathcal{W}} + a(u,w) = (f,w)_{\mathcal{W}} \quad \textit{for all} \quad w\in\mathcal{W}. \tag{10}$$

Proof. First we remark that by assumption the map S defined by (6) is coercive on $\mathcal{V}$, although not necessarily on $\mathcal{W}$. Therefore, to invoke Proposition 1 with $X=\mathcal{W}$ we perform the substitution $u=e^{kt}v$ with k a positive constant, as suggested in [6]. Then we obtain from (3) the equation

$$\frac{\partial v}{\partial t}+\tilde{A}v+kv=\tilde{f} \quad \text{in} \quad Q$$

with the initial-boundary conditions (4), where

$$\tilde{A}v(x,t)=\sum_{|\alpha|\leqq m}(-1)^{|\alpha|}D^\alpha\tilde{A}_\alpha(x,t,\xi(v)),$$

$$\tilde{A}_\alpha(x,t,\xi)=e^{-kt}A_\alpha(x,t,e^{kt}\xi) \quad \text{and} \quad \tilde{f}=e^{-kt}f.$$

It is obvious that the functions $\tilde{A}_\alpha$ also satisfy the conditions (A_1) to (A_3) with a new constant $\bar{c}_1$ and new functions $\bar{h}_\alpha\in L^{p'}(Q)$. Now we define a map

$$S_1=\tilde{S}+S_0\colon \mathscr{W}\to\mathscr{W}^*$$

by

$$(\tilde{S}(v), w)=\tilde{a}(v, w)=\sum_{|\alpha|\le m}\int_Q \tilde{A}_\alpha(x, t, \xi(v))D^\alpha w\, dx\, dt$$

and

$$(S_0(v), w)=k\int_Q vw\, dx\, dt.$$

On account of Theorem 1, $\tilde{S}$ is $D(L)$-pseudo-monotone. It is easily seen that also the sum $S_1=\tilde{S}+S_0$ is $D(L)$-pseudo-monotone. Moreover, S_1 is coercive on $\mathscr{W}$. Indeed, by the assumption of the theorem, $a(v, v)\|v\|_{\mathscr{V}}^{-1}\to\infty$ and therefore

$$\frac{(S_1(v), v)}{\|v\|_{\mathscr{W}}}=\frac{\tilde{a}(v, v)+k\|v\|_{L^2(Q)}}{\|v\|_{\mathscr{V}}+\|v\|_{L^2(Q)}}\to\infty \quad\text{as}\quad \|v\|_{\mathscr{W}}\to\infty.$$

For applying Proposition 1 we must finally verify that the map $L=\partial/\partial t\colon D(L)\to\mathscr{W}^*$ is maximal monotone and that $D(L)$ is dense in $\mathscr{W}$. By Lemma 1.2 of [5] p. 313 it suffices to show that for any $v\in\mathscr{W}$ and $w\in\mathscr{W}^*$ such that

(11) $$0\le(w-Lu, v-u)\quad\text{for all}\quad u\in D(L)$$

it follows that $v\in D(L)$ and $w=Lv$. To show this let $v\in\mathscr{W}$ and $w\in\mathscr{W}^*$ be given, let $\varphi\in C_0^\infty(0, T)$ and $\bar{u}\in V\cap L^2(\Omega)$ be arbitrary and let $u=\varphi\bar{u}$. Then $u\in\mathscr{W}$, $u(0)=u(T)=0$, $u'=\varphi'\bar{u}\in L^2(Q)\subset\mathscr{W}^*$ and hence $u\in D(L)$. Since $(Lu, u)=0$, we get from (11) by (5)

$$0\le(w, v)-(\varphi'\bar{u}, v)-(w, \varphi\bar{u}),$$

where

$$(\varphi'\bar{u}, v)=\Big(\int_0^T \varphi'(t)v(t)\, dt, \bar{u}\Big)_{V\cap L^2(\Omega)}$$

and

$$(w, \varphi\bar{u})=\Big(\int_0^T \varphi(t)w(t)\, dt, \bar{u}\Big)_{V\cap L^2(\Omega)},$$

while

$$\int_0^T \varphi'(t)v(t)\, dt\in L^2(\Omega)\subset V^*+L^2(\Omega)\quad\text{and}\quad\int_0^T \varphi(t)w(t)\, dt\in V^*+L^2(\Omega).$$

Consequently, for all $\bar{u}\in V\cap L^2(\Omega)$,

$$0\le(w, v)-\Big(\int_0^T \varphi'(t)v(t)\, dt+\int_0^T \varphi(t)w(t)\, dt, \bar{u}\Big)_{V\cap L^2(\Omega)}$$

implying that

$$\int_0^T \varphi'(t)v(t)\, dt=-\int_0^T \varphi(t)w(t)\, dt$$

for all $\varphi\in C_0^\infty(0, T)$, i.e. $w=v'=Lv$. The fact that $v(0)=0$ implying $v\in D(L)$ follows from (5) and (11) by standard argument (see e.g. [8] p. 176).

Now we are in a position to employ Proposition 1 to establish the existence of an element v in $D(L)$ such that

$$\left(\frac{\partial v}{\partial t}, w\right)_{\mathscr{W}} + \tilde{a}(v, w) + k(v, w)_{L^2(Q)} = (\tilde{f}, w)_{\mathscr{W}}$$

for all $w \in \mathscr{W}$, for any given $\tilde{f} \in \mathscr{W}^*$. Reversing the substitution we get (cf. [6])

$$\left(\frac{\partial u}{\partial t}, e^{-kt} w\right)_{\mathscr{W}} + a(u, e^{-kt} w) = (f, e^{-kt} w)_{\mathscr{W}}$$

for all $w \in \mathscr{W}$, which implies that (10) holds. Since also $u \in D(L)$, the proof is complete.

Remark. It is clear from the proof of Theorem 1 that the Dirichlet null boundary condition in (4) can be replaced by any boundary condition associated to V with $W_0^{m,p}(\Omega) \subset V \subset W^{m,p}(\Omega)$. On the other hand, it can be shown that (3) admits also a periodic solution $u(0) = u(T)$ if one chooses

$$D(L) = \{u \in \mathscr{W} \colon u' \in \mathscr{W}^*,\ u(0) = u(T)\}.$$

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