ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR STRONGLY NONLINEAR ELLIPTIC VARIATIONAL PROBLEMS

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1. Introduction

In this note we are concerned with the existence and uniqueness of solutions of nonlinear variational equations and inequalities for quasilinear elliptic partial differential operators of the form

$$(1.1) \qquad Au(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \big(g_\alpha (D^\alpha u(x)) \big), \quad x \in \Omega$$

which can be thought of as a simple nonlinear generalisation of the Laplace operator. Here Ω is an open bounded subset in R^N and the coefficients g_α are continuous functions obeying no growth restrictions from above.

The problems of this type have been extensively studied of late in the context of the theory of mappings of monotone type acting between Banach spaces which are determined by the behaviour of the coefficients. When the growth is at most of polynomial type, it is customary to work with reflexive Sobolev spaces (see [2], [4], [15], [16]), but when no such restriction is imposed, it is natural to tackle this problem in the framework of complementary systems of Sobolev—Orlicz spaces (see [6], [7], [8], [9], [10]). In this method the function space, where the problem is to be studied, is determined by the maximal growth of the coefficients. However, the abstract surjectivity results are available for the existence of solutions for the problem, provided an additional coercivity condition is fulfilled. That condition means a restriction for the growth from below. There have been several ways to widen the space between these two growth limitations. One may consider the sums of differential operators such that the principal part is coercive, obeying a polynomial growth condition, while the other terms may have liberal growth (see [3], [5], [11], [18], [19], [20], [21], [22]). The introduction of an anisotropic product of Sobolev spaces allows different growth for various coefficients of the same order ([7], [10]).

The growth limit from above for the lower order terms can be slightly lifted by the use of the full force of the Sobolev embedding theorem.

The purpose of the present note is to introduce a new rather simple regularisation method which allows more liberty for the coefficients g_α of the operator (1.1) than the methods described above and which can be applied to variational inequalities as well as to various boundary value problems. We shall make the operator coercive in some function space by imposing on the functions g_α a weak growth restriction from below which only means that $g_\alpha(t) \to \infty$ as $t \to \infty$ and $g_\alpha(t) \to -\infty$ as $t \to -\infty$. No growth limitations will be needed from above. To be more precise we introduce our conditions for the coefficients g_α.

(H$_1$) *For each* α, $|\alpha| \leqq m$, *the function* g_α: $\boldsymbol{R} \to \boldsymbol{R}$ *is continuous. There exists an Orlicz function M and a constant* $c_1 > 0$ *such that for each* $|\alpha| \leqq m$,

$$(1.2) \qquad\qquad g_\alpha(t)\,t \geqq M(t) - c_1 \quad \text{for all} \quad t \in \boldsymbol{R}.$$

(H$_2$) *For each α with $|\alpha| = m$ the function g_α is strictly increasing.*

(H$_3$) *For each α with $|\alpha| \leqq m - 1$ the function g_α has a representation $g_\alpha = \varphi_\alpha + r_\alpha$, where r_α is continuous and bounded and φ_α is continuous, satisfying the conditions $\varphi_\alpha(t)\,t \geqq 0$ for all $t \in \boldsymbol{R}$ and*

$$(1.3) \qquad\qquad \varphi_\alpha(t)\,s \leqq \varphi_\alpha(s)\,s + K_1 \varphi_\alpha(t)\,t + K_2$$

for all $s, t \in \boldsymbol{R}$, K_1 and K_2 being some positive constants.

We would like to mention that (1.2) guarantees the existence of the representation in (H$_3$) for all $|\alpha| \leqq m - 1$. On the other hand, it will be possible to relax (1.2) for $|\alpha| \leqq m - 1$ provided Poincaré's inequality holds in the Sobolev space involved. The inequality (1.3) is well known by the papers dealing with strongly nonlinear elliptic problems also in a more restrictive form (cf. [5], [18], [19], [20], [21]). It obviously holds for all nondecreasing functions (see [20] p. 11). Note also that we have made no oddness assumption for g_α (cf. [9], [10]).

Our existence theorems for variational equations and inequalities for the operator (1.1) satisfying the assumptions (H$_1$) to (H$_3$) will be proved in Section 3. For technical reasons we shall first treat the case where the function M appearing in (H$_1$) has a special form $M(t) = c_0|t|^p$ for some constants $c_0 > 0$ and $1 < p < \infty$. The general case will then be derived from this particular case. In Section 4 we give some examples of applications which can be handled by our existence theorems. The last section of this paper is devoted to studying the uniqueness of the solutions of the problems treated in Section 3 when all coefficients g_α are strictly increasing. A difficulty arises from the fact that $g_\alpha(D^\alpha u)$ for each $|\alpha| \leqq m$ lie in $L^1(\Omega)$, only. Therefore, for the uniqueness of the solutions of various interesting problems we shall assume the domain Ω to be starlike.

We finally mention that some similar existence results for the Dirichlet bound-ary value problem have been obtained by R. Landes ([13], [14]) by the Galerkin method. Our study can be easily extended for the more general operators

$$Au(x) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), Du(x), \dots, D^m u(x)), \quad x \in \Omega$$

by modification of the conditions (H$_1$) to (H$_3$) for the coefficients A_α accordingly.

2. Preliminaries

Let Ω be a bounded domain in R^N with $N \ge 2$, let $m \ge 1$ be an integer and let $1 \le p < \infty$. $W^{m,p}(\Omega)$ will stand for the *Sobolev space* (of equivalence classes) of func-tions u defined on Ω whose distributional derivatives up to the order m lie in $L^p(\Omega)$. The norm on $W^{m,p}(\Omega)$ is given by

$$(2.1) \qquad \|u\|_{m,p} = \Big\{ \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}^p \Big\}^{1/p},$$

where the summation extends over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ of non-negative integers with $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N \le m$ and $D^\alpha = \prod_{i=1}^N (\partial/\partial x_i)^{\alpha_i}$. $W_0^{m,p}(\Omega)$ will denote the closure in $W^{m,p}(\Omega)$ of $C_0^\infty(\Omega)$, the infinitely differentiable functions with compact support in Ω. In $W_0^{m,p}(\Omega)$ the expression

$$(2.2) \qquad |u|_{m,p} = \Big\{ \sum_{\alpha = m} \|D^\alpha u\|_{L^p}^p \Big\}^{1/p}$$

defines also a norm which is equivalent to (2.1).

An *Orlicz function* is any continuous map $M: R \to R$ which is even, convex and satisfies

$$\lim_{t \to 0} \frac{M(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{M(t)}{t} = \infty.$$

The *complementary Orlicz function* of M will be denoted by $\overline{M}$. The *Orlicz class* $K_M(\Omega)$ is defined as the set of measurable functions u such that

$$\int_\Omega M(u(x))\,dx < \infty,$$

and the *Orlicz space* $L_M(\Omega)$ as the linear hull of $K_M(\Omega)$ furnished with the Luxem-burg norm

$$\|u\|_M = \inf\Big\{ \lambda: \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big) dx \le 1 \Big\}.$$

The closure in $L_M(\Omega)$ of bounded functions with compact support in Ω is denoted by $E_M(\Omega)$. The *Sobolev—Orlicz space* of functions u such that u and its distributional

derivatives up to the order m lie in $L_M(\Omega)$ is denoted by $W^m L_M(\Omega)$. It becomes a Banach space with the norm

$$(2.3) \qquad \|u\|_{m,M} = \max\{\|D^\alpha u\|_M : |\alpha| \leq m\}.$$

A similar definition applies to the space $W^m E_M(\Omega)$. For a more detailed discussion on Orlicz spaces we refer to [12].

Let V be a closed subspace of $W^{m,p}(\Omega)$ containing $W_0^{m,p}(\Omega)$ and let $s > \max\{N, p\}$. Then the set $W = W^{m+1,s}(\Omega) \cap V$ is a closed subspace of $W^{m+1,s}(\Omega)$. If the domain Ω has the cone property, it follows from the Sobolev embedding theorem ([1] p. 97, for example) that $W^{m+1,s}(\Omega)$ is continuously embedded in $C_B^m(\Omega)$, a Banach space under the norm

$$\|u\|_{m,\infty} = \max\{\|D^\alpha u\|_{L^\infty} : |\alpha| \leq m\}.$$

For any $v \in W$ we introduce a partial differential operator

$$Bv(x) = \sum_{|\alpha| \leq m+1} (-1)^{|\alpha|} D^\alpha (|D^\alpha v(x)|^{s-2} D^\alpha v(x)),$$

which gives rise to a mapping S from W to W^* (a duality map with respect to the gauge $\Phi(r) = r^{s-1}$) by the rule

$$(2.4) \qquad (S(u), v) = \int_\Omega \sum_{|\alpha| \leq m+1} |D^\alpha u|^{s-2} D^\alpha u D^\alpha v \, dx,$$

where $u, v \in W$ and $(.,.)$ denotes the pairing between W and W^*, the dual space of W. The map S is *monotone*, i.e. $(S(u) - S(v), u - v) \geq 0$ for all $u, v \in W$ and it takes bounded sets of W onto bounded sets of W^*; indeed,

$$\left|(S(u), v)\right| \leq \|u\|_W^{s-1} \|v\|_W \quad \text{for all } u, v \in W.$$

Moreover, since $(S(u), u) = \|u\|_W^s$, S is *coercive* in W, i.e. $(S(u), u)\|u\|_W^{-1} \to \infty$ as $\|u\|_W \to \infty$. Observe finally that S is also continuous and *pseudo-monotone* (cf. [4]), that is, (PM) whenever (u_j) is a sequence in W which converges weakly to an element u in W (we write $u_j \rightharpoonup u$) and $\limsup (S(u_j), u_j - u) \leq 0$, it follows that $S(u_j) \rightharpoonup S(u)$ and $(S(u_j), u_j) \to (S(u), u)$.

Let us assume that the functions g_α satisfy the conditions (H_1) to (H_3) introduced in Section 1 for all $|\alpha| \leq m$. We shall denote

$$V_1 = \{v \in V : g_\alpha(D^\alpha v) \in L^1(\Omega) \text{ and } g_\alpha(D^\alpha v) D^\alpha v \in L^1(\Omega) \text{ for all } |\alpha| \leq m\}.$$

It is clear that $W \subset V_1 \subset V$. The semilinear form

$$(2.5) \qquad a(u, v) = \int_\Omega \sum_{|\alpha| \leq m} g_\alpha(D^\alpha u) D^\alpha v \, dx$$

is well-defined and bounded for all $u \in V_1$ and $v \in W$. Therefore we can define a (nonlinear) mapping T from V_1 to W^* by

$$(2.6) \qquad (T(u), v) = a(u, v) \quad \text{for all } u \in V_1, \quad v \in W.$$

We shall close this section by the following result which is an immediate consequence of the theory of mappings of monotone type acting between reflexive Banach spaces.

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the cone property, let K_1 be a closed convex subset of W containing the origin, let the functions g_α satisfy the conditions (H_1) to (H_3) and let the mappings S and T be defined by (2.4) and (2.6), respectively. Then for any $f \in W^*$ and any $\sigma > 0$ the variational inequality*

$$\sigma(S(u), v-u) + (T(u), v-u) \geq (f, v-u) \quad \text{for all} \quad v \in K_1$$

admits a solution u in K_1.

Proof. It suffices to verify that the mapping $\sigma S + T$ from W to W^* is continuous, pseudo-monotone and coercive (see [16] p. 247, for example). The continuity and pseudo-monotonicity of S are already clear and the respective properties for $\sigma S + T$ follow from the Sobolev embedding theorem (cf. [4]). By (H_1), $(T(v), v) \geq -c_1$ for all $v \in W$, implying that

$$\sigma(S(v), v) + (T(v), v)\|v\|_W^{-1} \geq \sigma\|v\|_W^{s-1} - c_1\|v\|_W^{-1}.$$

Thus $\sigma S + T$ is also coercive. □

3. Existence theorems

We shall apply Lemma 1 to produce existence theorems for variational inequalities and boundary value problems for the elliptic operator (1.1). To reduce the use of the rather complicated Sobolev—Orlicz spaces to the minimum we shall first deal with the particular case where the function M appearing in the condition (H_1) has the shape

$$M(t) = c_0|t|^p$$

for some p with $1 < p < \infty$ and $c_0 > 0$. This condition,

$(H_1)^*$ *For each α, $|\alpha| \leq m$, the function $g_\alpha: \mathbb{R} \to \mathbb{R}$ is continuous. There are constants $c_0, c_1 > 0$ and p with $1 < p < \infty$ such that for each $|\alpha| \leq m$,*

$$(1.2)^* \qquad\qquad g_\alpha(t)t \geq c_0|t|^p - c_1 \quad \text{for all} \quad t \in \mathbb{R}$$

enables us to stay within the frames of reflexive Sobolev spaces. The general case with the condition (H_1) we shall treat by making the necessary changes in the proof.

Let V be a closed subspace of $W^{m,p}(\Omega)$ containing $W_0^{m,p}(\Omega)$, where $W^{m,p}(\Omega)$ is determined through the condition $(H_1)^*$. By choosing $s > \max\{N, p\}$ we define $W = W^{m+1,s}(\Omega) \cap V$. Then we have

Theorem 1. *Let Ω be a bounded domain in R^N having the cone property, let the functions g_α satisfy the conditions $(H_1)^*$, (H_2) and (H_3), and let K be a closed convex subset of V containing the origin. Then for each $f \in V^*$ the variational inequality*

$$(3.1) \qquad a(u, w-u) \geqq (f, w-u) \quad \text{for all} \quad w \in W \cap K$$

admits a solution u in $K \cap V_1$.

Proof of Theorem 1. To invoke Lemma 1 we make two observations. Firstly, the natural injection map i of W to V is continuous. Thus $V^* \subset W^*$. For any $f \in V^*$ we shall write $f \in W^*$ instead of the more precise statement $i^* f \in W^*$. Secondly, when K is a closed convex set in V the set $K_1 = K \cap W$ inherits the same property as a subset of W. Consequently, by Lemma 1, for any $n \in N$ there exists $u_n \in K_1$ such that

$$(3.2) \qquad \frac{1}{n}\big(S(u_n), v-u_n\big) + \big(T(u_n), v-u_n\big) \geqq (f, v-u_n)$$

for all $v \in K_1$. Setting $v = 0 \in K_1$ and bearing in mind (2.5) and (2.6) we get

$$\frac{1}{n}\big(S(u_n), u_n\big) + \int_\Omega \sum_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n) D^\alpha u_n \, dx \leqq (f, u_n)$$

for all $n \in N$. By (2.4) and $(H_1)^*$ we have further

$$\frac{1}{n}\|u_n\|_W^s + c_0\|u_n\|_V^p - c_1 \leqq c_2\|u_n\|_V$$

with c_2 being some positive constant. Hence we can conclude that the real-valued sequences $\{\|u_n\|_V\}$, $\{(1/n)\|u_n\|_W^s\}$ and $\big\{\int_\Omega \sum_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n) D^\alpha u_n \, dx\big\}$ remain uniformly bounded. Since V is a reflexive Banach space, there exists a subsequence of $\{u_n\}$ such that (we do not change the notation, for simplicity) $u_n \rightharpoonup u$. As K is also weakly closed, $u \in K$. We shall show that u is a solution of (3.1). For showing this it is sufficient to establish the following four assertions for some further subsequence of $\{u_n\}$.

$$(1) \qquad \limsup\left\{\frac{1}{n}\big(S(u_n), w-u_n\big)\right\} \leqq 0 \quad \text{for any} \quad w \in W,$$

$$(2) \qquad \int_\Omega g_\alpha(D^\alpha u) D^\alpha u \, dx \leqq \liminf \int_\Omega g_\alpha(D^\alpha u_n) D^\alpha u_n \, dx \quad \text{for all} \quad |\alpha| \leqq m,$$

$$(3) \qquad \lim \int_\Omega g_\alpha(D^\alpha u_n) D^\alpha w \, dx = \int_\Omega g_\alpha(D^\alpha u) D^\alpha w \, dx \quad \text{for all} \quad w \in W,$$

$$(4) \qquad u \in V_1.$$

Indeed, provided that (1)—(4) has been verified, we obtain from (3.2) for any $w = v$ in $W \cap K$,

$$(f, w - u) = \lim (f, w - u_n) \leqq \limsup \left\{ \frac{1}{n} (S(u_n), w - u_n) \right\}$$

$$+ \limsup \left\{ \int\limits_\Omega \sum\limits_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n)(D^\alpha w - D^\alpha u_n) \, dx \right\}$$

$$\leqq \lim \int\limits_\Omega \sum\limits_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n) D^\alpha w \, dx - \sum\limits_{|\alpha| \leqq m} \left\{ \liminf \int\limits_\Omega g_\alpha(D^\alpha u_n) D^\alpha u_n \, dx \right\}$$

$$\leqq \int\limits_\Omega \sum\limits_{|\alpha| \leqq m} g_\alpha(D^\alpha u) D^\alpha w \, dx - \int\limits_\Omega \sum\limits_{|\alpha| \leqq m} g_\alpha(D^\alpha u) D^\alpha u \, dx = a(u; w - u),$$

which will prove (3.1), u being in $K \cap V_1$.

To show (1) we remark that $(1/n)\|u_n\|_W^{s-1} \to 0$ as $n \to \infty$, because $\{(1/n)\|u_n\|_W^s\}$ is bounded. Since $(S(u_n), u_n) \geqq 0$, we have

$$\frac{1}{n} (S(u_n), w - u_n) \leqq \frac{1}{n} (S(u_n), w) \leqq \frac{1}{n} \|u_n\|_W^{s-1} \|w\|_W,$$

which implies the assertion (1).

The next step is to show that (2) and (3) are true for all $|\alpha| \leqq m - 1$. Indeed, as

$$\int\limits_\Omega \sum\limits_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n) D^\alpha u_n \, dx \leqq c_3 \quad \text{for all} \quad n \in N$$

with c_3 a positive constant, we have, in view of (H$_3$), also

$$\int\limits_\Omega \sum\limits_{|\alpha| \leqq m} \varphi_\alpha(D^\alpha u_n) D^\alpha u_n \, dx \leqq c_4, \quad \int\limits_\Omega \sum\limits_{|\alpha| \leqq m} |g_\alpha(D^\alpha u_n) D^\alpha u_n| \, dx \leqq c_4$$

for all $n \in N$, c_4 being again some positive constant. On the other hand, by the Sobolev embedding theorem, $D^\alpha u_n(x) \to D^\alpha u(x)$ a.e. in Ω for all $|\alpha| \leqq m - 1$ for some subsequence. By Fatou's lemma

$$(3.4) \qquad \int\limits_\Omega \varphi_\alpha(D^\alpha u) D^\alpha u \, dx \leqq \liminf \int\limits_\Omega \varphi_\alpha(D^\alpha u_n) D^\alpha u_n \, dx.$$

By the dominated convergence theorem $r_\alpha(D^\alpha u_n) \to r_\alpha(D^\alpha u)$ in $L^{p'}(\Omega)$ and hence

$$(3.5) \qquad \lim \int\limits_\Omega r_\alpha(D^\alpha u_n) D^\alpha u_n \, dx = \int\limits_\Omega r_\alpha(D^\alpha u) D^\alpha u \, dx$$

for all $|\alpha| \leqq m - 1$. Now (3.4) and (3.5) imply that (2) holds true and that also $g_\alpha(D^\alpha u) D^\alpha u \in L^1(\Omega)$ for all $|\alpha| \leqq m - 1$.

For any $\delta > 0$,

$$\left| \varphi_\alpha(D^\alpha u_n(x)) \right| \leqq \max_{|t| \leqq \delta^{-1}} |\varphi_\alpha(t)| + \delta \varphi_\alpha(D^\alpha u_n(x)) D^\alpha u_n(x).$$

Applying Vitali's convergence theorem we obtain $\varphi_\alpha(D^\alpha u_n) \to \varphi_\alpha(D^\alpha u)$ and

$g_\alpha(D^\alpha u_n) \to g_\alpha(D^\alpha u)$ in $L^1(\Omega)$ for all $|\alpha| \leq m-1$. Finally, since $D^\alpha w \in L^\infty(\Omega)$ whenever $w \in W$ and $|\alpha| \leq m$, we obtain (3) for all $|\alpha| \leq m-1$.

To be able to establish the assertions (2) and (3) also for all $|\alpha| = m$ it is clearly sufficient, by exactly the same argument as above, to show that

$$(3.6) \qquad D^\alpha u_n(x) \to D^\alpha u(x) \quad \text{a.e. in } \Omega \text{ for all } |\alpha| = m.$$

Then it will also be plain that $g_\alpha(D^\alpha u)D^\alpha u \in L^1(\Omega)$ and $g_\alpha(D^\alpha u) \in L^1(\Omega)$ for all $|\alpha| \leq m$, which means that $u \in V_1$, i.e. the assertion (4) holds true, too. Hence the proof of Theorem 1 is completed by verification of the claim (3.6). This we shall do by means of the following three lemmas.

Lemma 2. *For each α with $|\alpha| = m$ there exists a function h_α in $L^1(\Omega)$ such that $g_\alpha(D^\alpha u_n) \rightharpoonup h_\alpha$ in $L^1(\Omega)$ for some subsequence of $\{u_n\}$.*

Proof. It is sufficient to show that $\{g_\alpha(D^\alpha u_n)\}$ is uniformly equi-integrable for each $|\alpha| = m$. By the conditions $(H_1)^*$ and (H_2) each g_α can be written in the form $g_\alpha = \varphi_\alpha + c_\alpha$, where $\varphi_\alpha(t) = g_\alpha(t) - g_\alpha(0)$ and $c_\alpha = g_\alpha(0)$. Thus $\varphi_\alpha(t)t \geq 0$ for all $t \in R$. For any $\delta > 0$ we have

$$\varphi_\alpha(D^\alpha u_n(x)) \leq \max_{|t| \leq \delta^{-1}} |\varphi_\alpha(t)| + \delta \varphi_\alpha(D^\alpha u_n(x))D^\alpha u_n(x),$$

where $\{\varphi_\alpha(D^\alpha u_n)D^\alpha u_n\}$ is uniformly bounded in $L^1(\Omega)$ by a constant c_4 for all $|\alpha| = m$. For any measurable subset E of Ω we then have

$$\int_E |\varphi_\alpha(D^\alpha u_n)|\, dx \leq \int_E K_\delta\, dx + \delta c_4 = K_\delta \mu(E) + \delta c_4,$$

where K_δ is a constant depending on δ. Given $\varepsilon > 0$ let δ be such that $\delta c_4 < \varepsilon/2$ and let $\mu(E) < \varepsilon/2K_\delta$. Hence we can see that $\{\varphi_\alpha(D^\alpha u_n)\}$ and therefore also $\{g_\alpha(D^\alpha u_n)\}$ are equi-integrable. $\square$

In the following lemma we denote, for convenience, $h_\alpha = g_\alpha(D^\alpha u)$ for each α with $|\alpha| \leq m-1$. Thus we can write $g_\alpha(D^\alpha u_n) \rightharpoonup h_\alpha$ in $L^1(\Omega)$ for all $|\alpha| \leq m$, where in fact $g_\alpha(D^\alpha u_n) \to h_\alpha$ in $L^1(\Omega)$ for all $|\alpha| \leq m-1$.

Lemma 3. $\limsup \int_\Omega \sum_{|\alpha| \leq m} \{g_\alpha(D^\alpha u_n) - h_\alpha\}D^\alpha u_n\, dx \leq 0$.

Proof. Since $\{u_n\} \subset K_1 = K \cap W$, we may substitute $v = u_j$ $(1 \leq j \leq n)$ in (3.2) to get

$$(3.7) \qquad \frac{1}{n}\big(S(u_n), u_j - u_n\big) + \int_\Omega \sum_{|\alpha| \leq m} g_\alpha(D^\alpha u_n)(D^\alpha u_j - D^\alpha u_n)\, dx$$

$$\geq (f, u_j - u_n)$$

and hence

$$\int_\Omega \sum_{|\alpha| \leq m} g_\alpha(D^\alpha u_n)D^\alpha u_n\, dx \leq \frac{1}{n}\big(S(u_n), u_j - u_n\big) + (f, u_n - u_j)$$

$$+ \int_\Omega \sum_{|\alpha| \leq m} g_\alpha(D^\alpha u_n)D^\alpha u_j\, dx.$$

As $(S(u_n), u_n) \geqq 0$ we have further

$$\int_\Omega \sum_{|\alpha| \leqq m} \{g_\alpha(D^\alpha u_n) - h_\alpha\} D^\alpha u_n \, dx \leqq \frac{1}{n} (S(u_n), u_j) + (f, u_n - u_j)$$

$$+ \int_\Omega \sum_{|\alpha| \leqq m} g_\alpha(D^\alpha u_n) D^\alpha u_j \, dx - \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_n \, dx.$$

Letting $n \to \infty$ and keeping j fixed we can conclude

$$(3.8) \qquad \limsup_n \int_\Omega \sum_{|\alpha| \leqq m} \{g_\alpha(D^\alpha u_n) - h_\alpha\} D^\alpha u_n \, dx \leqq (f, u - u_j)$$

$$+ \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_j \, dx - \liminf_n \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_n \, dx.$$

On the other hand, we can deduce from (3.7) also that

$$\int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_j \, dx \geqq (f, u_j - u) - \limsup_n \int_\Omega \sum_{|\alpha| \leqq m} |g_\alpha(D^\alpha u_n) D^\alpha u_n| \, dx$$

for all $j \in N$, where we already know that $\sum_{|\alpha| \leqq m} \|g_\alpha(D^\alpha u_n) D^\alpha u_n\|_{L^1} \leqq c_4$ for all $n \in N$. Hence

$$-\liminf_n \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_n \, dx \leqq c_4.$$

Finally we can select from the sequence $\{u_j\}$ such a subsequence $\{u_{j_k}\}$ that

$$\lim_{k \to \infty} \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_{j_k} \, dx = \liminf_n \int_\Omega \sum_{|\alpha| \leqq m} h_\alpha D^\alpha u_n \, dx,$$

which enables us to conclude the assertion of Lemma 3 from (3.8). □

Lemma 4. $D^\alpha u_n(x) \to D^\alpha u(x)$ *a.e. in* Ω *for all* $|\alpha| = m$ *for some subsequence of* $\{u_n\}$.

Proof. We denote

$$q_n(x) = \sum_{|\alpha| = m} \{g_\alpha(D^\alpha u_n(x)) - g_\alpha(D^\alpha u(x))\} (D^\alpha u_n(x) - D^\alpha u(x)).$$

In view of (H_2) the convergence of $D^\alpha u_n(x)$ to $D^\alpha u(x)$ almost everywhere for all $|\alpha| = m$ is implied by $q_n(x) \to 0$ a.e. in Ω (cf. [14], Lemma 6). In fact, as $q_n(x) \geqq 0$ for all $x \in \Omega$, it is sufficient to show that

$$(3.9) \qquad \limsup_n \int_{\Omega_k} q_n(x) \, dx \leqq \varepsilon_k,$$

where $\varepsilon_k \to 0$ as $k \to \infty$ and $\{\Omega_k\}$ is a growing sequence of the subsets of Ω such that $\mu(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$.

To this end we define for any $k = 1, 2, \ldots$

$$\Omega_k = \{x \in \Omega : |D^\alpha u(x)| \leqq k \text{ and } |h_\alpha(x)| \leqq k \text{ for all } |\alpha| \leqq m\}.$$

Then we observe that

(3.10)
$$\lim_n \int_{Q_k} \sum_{|\alpha|=m} g_\alpha(D^\alpha u)(D^\alpha u_n - D^\alpha u)\,dx = 0,$$

(3.11)
$$\lim_n \int_{\Omega_k} \sum_{|\alpha|=m} h_\alpha(D^\alpha u_n - D^\alpha u)\,dx = 0$$

and

(3.12)
$$\lim_n \int_{\Omega_k} \sum_{|\alpha|=m} (h_\alpha - g_\alpha(D^\alpha u_n))D^\alpha u\,dx = 0$$

for any fixed k. Moreover, by the inequality (1.3) of (H_3) we have for all $|\alpha| \leq m-1$,

$$\{\varphi_\alpha(D^\alpha u) - \varphi_\alpha(D^\alpha u_n)\}D^\alpha u_n \leq K_1 \varphi_\alpha(D^\alpha u)D^\alpha u + K_2,$$

where $\varphi_\alpha(D^\alpha u)D^\alpha u \in L^1(\Omega)$. Since $\{\varphi_\alpha(D^\alpha u_n(x)) - \varphi_\alpha(D^\alpha u(x))\}D^\alpha u_n(x) \to 0$ a.e. in Ω for all $|\alpha| \leq m-1$ and since its positive part is dominated by a function in $L^1(\Omega)$, we can deduce by the dominated convergence theorem that

$$\limsup \int_\Omega \{\varphi_\alpha(D^\alpha u_n) - \varphi_\alpha(D^\alpha u)\}D^\alpha u_n\,dx \leq 0.$$

Since also

(3.13) $\{r_\alpha(D^\alpha u) - r_\alpha(D^\alpha u_n)\}D^\alpha u_n \to 0$ in $L^1(\Omega)$ for all $|\alpha| \leq m-1$,

we get (with the convention $h_\alpha = g_\alpha(D^\alpha u)$ for all $|\alpha| \leq m-1$)

(3.14)
$$\limsup \int_\Omega \sum_{|\alpha| \leq m-1} \{h_\alpha - g_\alpha(D^\alpha u_n)\}D^\alpha u_n\,dx \leq 0.$$

As a consequence of (3.10), (3.11), (3.12) and (3.14) we can derive

$$\limsup \int_{\Omega_k} q_n(x)\,dx = \limsup \int_{\Omega_k} \sum_{|\alpha|=m} g_\alpha(D^\alpha u_n)(D^\alpha u_n - D^\alpha u)\,dx$$

$$= \limsup \int_{\Omega_k} \sum_{|\alpha|=m} \{g_\alpha(D^\alpha u_n) - h_\alpha\}D^\alpha u_n\,dx$$

$$\leq \limsup \int_\Omega \sum_{|\alpha| \leq m} \{g_\alpha(D^\alpha u_n) - h_\alpha\}D^\alpha u_n\,dx$$

$$+ \limsup \int_{\Omega \setminus \Omega_k} \sum_{|\alpha|=m} \{h_\alpha - g_\alpha(D^\alpha u_n)\}D^\alpha u_n\,dx.$$

Taking into account that the functions g_α for each $|\alpha| = m$ satisfy the inequality (1.3) of (H_3), we can apply the argument of [13] p. 20 to show that

(3.15)
$$\sum_{|\alpha|=m} \{h_\alpha(x) - g_\alpha(D^\alpha u_n(x))\}D^\alpha u_n(x) \leq \Gamma(x)$$

for all $n \in N$, Γ being some function in $L^1(\Omega)$. From (3.15) and Lemma 3 it then follows that

$$\limsup \int_{\Omega_k} q_n(x)\,dx \leq \int_{\Omega \setminus \Omega_k} \Gamma(x)\,dx,$$

where $\varepsilon_k = \int_{\Omega \setminus \Omega_k} \Gamma(x)dx \to 0$ as $k \to \infty$, which completes the proof of Lemma 4 and also the proof of Theorem 1. $\square$

For various boundary value problems for the operator (1.1) subject to the boundary constraints associated to the subspace V with $W_0^{m,p}(\Omega) \subset V \subset W^{m,p}(\Omega)$, where p is determined by the condition $(H_1)^*$, we have the following theorem:

Theorem 2. *Let Ω be a bounded domain in $\mathbf{R}^N$ having the cone property and let the functions g_α satisfy the conditions $(H_1)^*$, (H_2) and (H_3). Then for each $f \in V^*$ there exists u in V_1 such that*

$$(3.16) \qquad \int_\Omega \sum_{|\alpha| \le m} g_\alpha(D^\alpha u) D^\alpha w \, dx = (f, w) \quad \text{for all} \quad w \in W,$$

while

$$(3.17) \qquad \int_\Omega \sum_{|\alpha| \le m} g_\alpha(D^\alpha u) D^\alpha u \, dx \le (f, u).$$

Proof. We may choose $K = V$ in the proof of Theorem 1. Setting $v = w + u_n$ in (3.2) with w arbitrary in W we get

$$\frac{1}{n}(S(u_n), w) + \int_\Omega \sum_{|\alpha| \le m} g_\alpha(D^\alpha u_n) D^\alpha w \, dx = (f, w).$$

Letting here $n \to \infty$ we obtain (3.16). Furthermore, if we substitute $w = 0$ in (3.1), we obtain (3.17). $\square$

Remark 1. When the two norms $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ are equivalent in V (i.e. Poincaré's inequality holds), the coercivity condition $(1.2)^*$ of $(H_1)^*$ can be relaxed for the functions g_α with $|\alpha| \le m-1$. Such is the case when $V = W_0^{m,p}(\Omega)$, for example.

Next we shall tackle the general case where the conditions (H_1), (H_2) and (H_3) are assumed. Let V be a closed subspace of $W^{m,1}(\Omega)$ containing $W_0^{m,1}(\Omega)$. Selecting $s > N$ we define $W = W^{m+1,s}(\Omega) \cap V$. Let further $E_M(\Omega)$, $W^m E_M(\Omega)$ and $E_{\overline{M}}(\Omega)$ be the spaces determined through the function M appearing in the condition (H_1). We note that each $f \in W^{-m} E_{\overline{M}}(\Omega) = \{f = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(\Omega) \text{ for all } |\alpha| \le m\}$ defines an element in W^* by

$$(f, w) = \int_\Omega \sum_{|\alpha| \le m} f_\alpha D^\alpha w \, dx,$$

while

$$|(f, w)| \le \text{const.} \|w\|_W \sum_{|\alpha| \le m} \|f_\alpha\|_{\overline{M}}.$$

Now we are in a position to prove

Theorem 3. *Let Ω be a bounded domain in $\mathbf{R}^N$ having the cone property, let the functions g_α satisfy the conditions (H_1), (H_2) and (H_3) and let K be a closed con-*

vex subset of V containing the origin. Then for each $f \in W^{-m}E_{\overline{M}}(\Omega)$ the variational inequality

(3.1) $a(u, w-u) \geqq (f, w-u)$ *for all* $w \in K \cap W$

admits a solution u in $K \cap V_1 \cap W^m L_M(\Omega)$.

Proof. We shall only indicate the changes which are necessary in the run of the proof of Theorem 1.

The inequality (3.2) obviously holds for all $v \in K_1 = K \cap W$. Setting $w=0$ in (3.2) we get for all $n \in N$,

$$\frac{1}{n}\big(S(u_n), u_n\big) + \int_{\Omega} \sum_{|\alpha| \leq m} g_\alpha(D^\alpha u_n)D^\alpha u_n \, dx \leq \int_{\Omega} \sum_{|\alpha| \leq m} f_\alpha D^\alpha u_n \, dx.$$

By (H$_1$) and Young's inequality ([12] p. 135) we have further

$$\frac{1}{n}\|u_n\|_W^s + \int_{\Omega} \sum_{|\alpha| \leq m} M(D^\alpha u_n) \, dx - c_1 \leq \int_{\Omega} \sum_{|\alpha| \leq m} \overline{M}(2f_\alpha) + M\left(\frac{1}{2}D^\alpha u_n\right) dx$$

$$\leq c_5 + \frac{1}{2}\int_{\Omega} \sum_{|\alpha| \leq m} M(D^\alpha u_n) \, dx,$$

where we have used the fact that $f_\alpha \in E_{\overline{M}}(\Omega)$ for all $|\alpha| \leq m$ and where $c_5 > 0$ is a constant. As a consequence we can conclude that the real-valued sequences $\{(1/n)\|u_n\|_W^s\}$, $\{\sum_{|\alpha| \leq m} \int_{\Omega} M(D^\alpha u_n)dx\}$ and $\{\sum_{|\alpha| \leq m} \int_{\Omega} g_\alpha(D^\alpha u_n)D^\alpha u_n dx\}$ are uniformly bounded. By the de la Vallée—Poussin theorem ([12] p. 174), $\{D^\alpha u_n\}$ is uniformly equi-integrable in $L^1(\Omega)$ for each $|\alpha| \leq m$. On the other hand, as $\{D^\alpha u_n\}$ is bounded also in $L_M(\Omega)$ for each $|\alpha| \leq m$, we can establish the existence of a subsequence of $\{u_n\}$ such that $D^\alpha u_n \to D^\alpha u$ in $L_M(\Omega)$ for the topology $\sigma\big(L_M(\Omega), E_{\overline{M}}(\Omega)\big)$ and also $D^\alpha u_n \to D^\alpha u$ in $L^1(\Omega)$ for each $|\alpha| \leq m$. We can then show the desired result that the limit element $u \in W^m L_M(\Omega) \cap K$ is also in V_1 and solves (3.1) by verifying again the assertions (1) to (4). The proofs of these assertions remain true without any essential change. We mention the following two facts only. Firstly, (3.5) and (3.13) hold by Vitali's convergence theorem because of the equi-integrability of $\{D^\alpha u_n\}$. Secondly, $(f, u_n) \to (f, u)$ as $n \to \infty$, since $D^\alpha u_n \to D^\alpha u$ in $L_M(\Omega)$ for $\sigma\big(L_M(\Omega), E_{\overline{M}}(\Omega)\big)$ for all $|\alpha| \leq m$ and $f = (f_\alpha) \in W^{-m}E_{\overline{M}}(\Omega)$.   □

For the boundary value problems subject to the boundary constraints which are associated to the subspace V with $W_0^{m,1}(\Omega) \subset V \subset W^{m,1}(\Omega)$ the following generalisation to Theorem 2 can be derived from Theorem 3.

Theorem 4. *Let Ω be a bounded domain in R^N having the cone property and let the functions g_α satisfy the conditions* (H$_1$), (H$_2$) *and* (H$_3$). *Then for each $f \in W^{-m}E_{\overline{M}}(\Omega)$ there exists u in $V_1 \cap W^m L_M(\Omega)$ such that* (3.16) *and* (3.17) *hold.*

Remark 2. When $\|\cdot\|_{m,1}$ and $|\cdot|_{m,1}$ are equivalent norms in V (for example $V = W_0^{m,1}(\Omega)$), the coercivity condition (1.2) of (H$_1$) can be relaxed for the functions g_α with $|\alpha| \leq m-1$.

4. Examples and applications

In this section we shall indicate briefly some examples of the problems which can be solved by our existence theorems of the previous section. We start with mentioning some specific functions g_α which satisfy the conditions (H$_1$) or (H$_1$)*, (H$_2$) and (H$_3$).

1) $g_\alpha(t) = |t|^{p_\alpha} \operatorname{sgn} t,$

where $0 < p_\alpha < \infty$ for all $|\alpha| \leq m$. The Sobolev space $W^{m,p}(\Omega)$ is determined by $p = \min \{p_\alpha + 1 : |\alpha| \leq m\}$. In this case the partial differential operator A defined by (1.1) has polynomial growth which may depend upon the direction. This kind of operator is called anisotropic in the literature (cf. [7] p. 191). Obviously the functions g_α satisfy the conditions (H$_1$)*, (H$_2$) and (H$_3$).

2) $g_\alpha(t) = |t|^{p_\alpha} e^{|t|^{q_\alpha}} \operatorname{sgn} t,$

where $0 < p_\alpha, q_\alpha < \infty$ for all $|\alpha| \leq m$. Then $W^{m,p}(\Omega)$ is defined as above and the operator A is anisotropic with an exponential growth. The conditions (H$_1$)*, (H$_2$) and (H$_3$) are satisfied.

3) $g_\alpha(t) = \begin{cases} t^{p_\alpha}, & \text{if } t \geq 0, \\ -|t|^{q_\alpha} \log(1 + |t|), & \text{if } t < 0, \end{cases}$

where $0 < p_\alpha, q_\alpha < \infty$ for all $|\alpha| \leq m$. The conditions (H$_1$)*, (H$_2$) and (H$_3$) are again satisfied. Note further that the operator A is not odd (cf. [8], [10]).

4) As an example of functions g_α satisfying (H$_1$) but not (H$_1$)* we mention

$$g_\alpha(t) = \log(1 + |t|) \operatorname{sgn} t \quad \text{for all} \quad |\alpha| \leq m.$$

The Sobolev—Orlicz space $W^m L_M(\Omega)$ is then defined by the function

$$M(s) = \int\limits_0^s g_\alpha(t)\, dt.$$

The corresponding operator A has slow growth, the coercivity condition (1.2) of (H$_1$) being still satisfied.

5) The functions g_α may, of course, be any combinations of the types mentioned above. A constant c_α can be added to each g_α with $|\alpha| = m$ and each g_α with $|\alpha| \leq m - 1$ may be added by any bounded function k_α.

We finally conclude with some examples of specific boundary value problems and convex sets K appearing in the applications.

a) If $V = W_0^{m,p}(\Omega)$ with $1 \leq p < \infty$, Theorem 2 or Theorem 4, respectively, establishes the existence of a weak solution u for the Dirichlet problem

(4.1)
$$\begin{cases} \sum\limits_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha\big(g_\alpha(D^\alpha u)\big) = f \quad \text{in} \quad \Omega, \\ D^\alpha u = 0 \quad \text{on } \partial\Omega \text{ for all } |\alpha| \leq m - 1 \end{cases}$$

for each prescribed $f \in W^{-m,p'}(\Omega)$ (resp. $f \in W^{-m}E_{\overline{M}}(\Omega)$), provided the functions g_α satisfy the conditions $(H_1)^*$ (resp. (H_1)), (H_2) and (H_3).

Similarly, if $V = W^{m,p}(\Omega)$, the existence of a weak solution for the boundary value problem with Neumann type boundary constraints follows.

b) Let $\psi \in W^{1,\infty}(\Omega)$ be a given function such that $\psi(x) \leqq 0$ on the boundary $\partial\Omega$. Let

$$K = \{v \in W_0^{1,p}(\Omega): v(x) \geqq \psi(x) \text{ a.e. in } \Omega\}$$

and let the functions g_α satisfy the conditions (H_1) to (H_3). The "obstacle problem" asks for a solution u in K of the variational inequality

$$(4.2) \qquad \int_\Omega \sum_{|\alpha| \leqq 1} g_\alpha (D^\alpha u)(D^\alpha w - D^\alpha u)\, dx \geqq (f, w - u) \quad \text{for all} \quad w \in K \cap W,$$

where $W = W^{2,s}(\Omega) \cap W_0^{1,p}(\Omega)$, $s > \max\{N, p\}$, with $1 < p < \infty$ if $(H_1)^*$ is satisfied, $p = 1$ if only (H_1) is met. When $\psi(x) \leqq 0$ in the whole set Ω, $0 \in K$ and the existence of a solution u in $K \cap V_1$ follows immediately from Theorem 1 for each $f \in W^{-1,p}(\Omega)$ and from Theorem 3 for each $f \in W^{-1}E_{\overline{M}}(\Omega)$, respectively. If $0 \notin K$, we may perform a change of variable $u \to \hat{u} = u - \psi^+$, where $\psi^+ = \max\{\psi, 0\}$.

Then we set

$$\hat{K} = \{\hat{u} \in W_0^{1,p}(\Omega): \hat{u}(x) \geqq \psi(x) - \psi^+(x) \text{ in } \Omega\},$$

whence K contains the origin. The problem (4.2) is now equivalent to finding $\hat{u} \in \hat{K}$ such that

$$(4.3) \qquad \int_\Omega \sum_{|\alpha| \leqq 1} g_\alpha (D^\alpha \hat{u} + D^\alpha \psi^+)(D^\alpha \hat{w} - D^\alpha \hat{u})\, dx \geqq (f, \hat{w} - \hat{u}) \quad \text{for all} \quad \hat{w} \in \hat{K} \cap W.$$

Setting $\hat{g}_\alpha(x, t) = g_\alpha(t + D^\alpha \psi^+(x))$ and formulating the analogous conditions for the functions of the type $g_\alpha(x, t)$ the existence theorems remain valid and the solvability of (4.3) follows.

c) Let $K = \{v \in W^{m,1}(\Omega): v(x) \geqq 0 \text{ on } \partial\Omega\}$. Then the variational inequality (3.1) is equivalent to a boundary value problem with certain boundary constraints (cf. [17] p. 99). When $K = \{v \in W_0^{1,1}(\Omega): |\nabla v(x)| \leqq 1 \text{ a.e. in } \Omega\}$, (3.1) constitutes a problem of the type appearing in the theory of elasto-plasticity (cf. [17], p. 100).

5. Uniqueness theorems

Our purpose in this section is to deal with the uniqueness of solutions of the variational problems for the operator (1.1) discussed in Section 3. To this end we shall assume in what follows that all coefficients g_α are strictly increasing. For convenience we shall restrict ourselves to the second order operators and to the case where the strict form of the sign condition is fulfilled. Thus we can reformulate the conditions for the functions $g_0, g_1, \ldots, g_N$ as follows.

(E_1) *For each* $i=0, 1, ..., N$ *the function* $g_i \colon \boldsymbol{R} \to \boldsymbol{R}$ *is continuous and strictly increasing, and satisfies* $g_i(t)t \geqq 0$ *for all* $t \in \boldsymbol{R}$. *There exists an Orlitz function* M *and a constant* $c_1 > 0$ *such that for each* $i = 1, 2, ..., N$,

$$g_i(t)t \geqq M(t) - c_1 \quad \text{for all} \quad t \in \boldsymbol{R}.$$

(E_2) $$g_0(t)t \geqq M(t) - c_1 \quad \text{for all} \quad t \in \boldsymbol{R}.$$

For each $i = 0, 1, ..., N$ we define the function G_i by

(5.1) $$G_i(s) = \int_0^s g_i(t)\, dt,$$

which is convex obeying the inequality

(5.2) $$G_i(s) - G_i(t) \geqq g_i(t)(s - t) \quad \text{for all} \quad s, t \in \boldsymbol{R}.$$

When Ω is a bounded subset of $\boldsymbol{R}^N$ having the cone property, V is a closed subspace of $W^{1,1}(\Omega)$ containing $W_0^{1,1}(\Omega)$, $W = V \cap W^{2,s}(\Omega)$ with $s > N$, and K is any closed convex subset of V containing the origin, it follows from the conditions (E_1) and (E_2) and from Theorem 3 that the variational inequality

(5.3) $$\int_\Omega \sum_{i=0}^N g_i(D_i u)(D_i w - D_i u)\, dx \geqq (f, w - u) \quad \text{for all} \quad w \in K \cap W$$

admits a solution u in $K \cap V_1 \cap W^1 L_M(\Omega)$ for each given f in $W^{-1}E_{\overline{M}}(\Omega)$, where

$$V_1 = \{v \in V \colon g_i(D_i v)D_i v \in L^1(\Omega) \text{ and } g_i(D_i v) \in L^1(\Omega) \text{ for } i = 0, 1, ..., N\}.$$

On the other hand, by (5.2) each solution u of (5.3) is also a solution of the inequality

(5.4) $$\int_\Omega \sum_{i=0}^N \{G_i(D_i w) - G_i(D_i u)\}\, dx \geqq (f, w - u) \quad \text{for all} \quad w \in K \cap W.$$

Similarly, the solutions of the boundary value problem

(5.5) $$\int_\Omega \sum_{i=0}^N g_i(D_i u) D_i w\, dx = (f, w) \quad \text{for all} \quad w \in W$$

with the property

(5.6) $$\int_\Omega \sum_{i=0}^N g_i(D_i u) D_i u\, dx \leqq (f, u)$$

are also solutions of (5.4) with $K = V$. Our study on the uniqueness of solutions for (5.3) is based on the inequality (5.4) which, however, should hold true for all w in $K \cap V_2$, where

$$V_2 = \{v \in V \colon G_i(D_i v) \in L^1(\Omega) \text{ for all } i = 0, 1, ..., N\}.$$

Note that V_1 is contained in V_2, since $G_i(t) \leqq g_i(t)t$ for all $t \in \boldsymbol{R}$ and $i = 0, 1, ..., N$. As $G_i(t) \geqq 0$ for all $t \in \boldsymbol{R}$, (5.4) is obviously satisfied for all $w \in V$ with $w \notin V_2$. The following simple lemma (cf. [3]) shows that the main problem in establishing

the uniqueness for (5.3) or for the pair {(5.5), (5.6)}, respectively, is to extend (5.4) for all $w \in K \cap V_2$.

Lemma 5. *Let the functions g_i $(i=0, 1, ..., N)$ satisfy the conditions (E_1) and (E_2). Then the variational inequality*

$$(5.7) \qquad \int_\Omega \sum_{i=0}^N \{G_i(D_i v) - G_i(D_i u)\} dx \geqq (f, v-u) \quad \text{for all} \quad v \in K \cap V_2$$

admits at most one solution u in $K \cap V_1$.

Proof. Let u_1 and u_2 be two solutions of (5.7) in $K \cap V_1$. Since $V_1 \subset V_2$ and since each G_i is convex, $(1/2)(u_1+u_2) \in V_2$. Thus we can set $v=(1/2)(u_1+u_2)$ in the inequalities

$$\int_\Omega \sum_{i=0}^N \{G_i(D_i v) - G_i(D_i u_1)\} dx \geqq (f, v-u_1),$$

$$\int_\Omega \sum_{i=0}^N \{G_i(D_i v) - G_i(D_i u_2)\} dx \geqq (f, v-u_2),$$

and we obtain by addition,

$$\int_\Omega \sum_{i=1}^N \left\{ \frac{G_i(D_i u_1) - G_i(D_i u_2)}{2} - G_i\left(\frac{D_i u_1 + D_i u_2}{2}\right) \right\} dx \leqq 0.$$

The conclusion $u_1 = u_2$ follows now from the strict convexity of the functions G_i for each $i=0, 1, ..., N$. □

A domain Ω in $\mathbf{R}^N$ is said to be *starlike,* if for any $r > 1$, $\Omega_r = \{x \in \mathbf{R}^N : rx \in \Omega\} \subset\subset \Omega$, i.e. $\bar{\Omega}_r \subset \Omega$ and $\bar{\Omega}_r$ is compact. We shall deal below with the starlike domains only. We are able to prove the uniqueness of solutions for the boundary value problems with the Dirichlet boundary conditions $(V = W_0^{1,1}(\Omega))$ and with the Neumann type boundary conditions $(V = W^{1,1}(\Omega))$, and for the obstacle problem (Example b, in Section 4). For simplicity, to avoid the discussion of various cases separately, we shall assume that f is a given function in $L^\infty(\Omega)$. We start with

Theorem 5. *Let Ω be a bounded starlike domain in $\mathbf{R}^N$ having the cone property and let the functions g_i satisfy the condition (E_1). If $V = W_0^{1,1}(\Omega)$, then for each $f \in L^\infty(\Omega)$ there exists a unique u in V_1 such that*

$$(5.8) \qquad -\sum_{i=0}^N D_i(g_i(D_i u)) + g_0(u) = f \quad \text{in} \quad \Omega$$

(in the sense of distributions), while (5.6) also holds.

Proof. Let u be a solution of (5.8) satisfying (5.6). Such a solution exists by Theorem 4 and Remark 2. Thus

$$\int_\Omega \sum_{i=0}^N g_i(D_i u) D_i w \, dx = (f, w) \quad \text{for all} \quad w \in C_0^\infty(\Omega),$$

which, in view of (5.2) together with (5.6), implies that

(5.9) $\int_\Omega \sum_{i=0}^{N} \{G_i(D_i w) - G_i(D_i u)\}\,dx \geqq (f, w-u)$ for all $w \in C_0^\infty(\Omega).$

To be able to apply Lemma 5 we must only show that (5.9) remains valid for all w in V_2.

Let $v \in W_0^{1,1}(\Omega)$ with $G_i(D_i v) \in L^1(\Omega)$ for all $i=0, 1, \ldots, N$. For any δ with $0<\delta<1$ we define $\Omega_\delta = \{x \in R^N : \delta^{-1}x \in \Omega\}$, whence $\Omega_\delta \subset\subset \Omega$. Let further

$$z_\delta(x) = \begin{cases} v(\delta^{-1}x), & \text{if } x \in \Omega_\delta, \\ 0, & \text{if } x \in \Omega \setminus \Omega_\delta \end{cases}$$

and $v_\delta(x)=\delta z_\delta(x)$ for all $x \in \Omega$. Then z_δ and v_δ belong to $W_0^{1,1}(\Omega)$ and $z_\delta \to v$ in $W_0^{1,1}(\Omega)$ as $\delta \to 1$ ([12] p. 70). Hence also $v_\delta \to v$ in $W_0^{1,1}(\Omega)$ as $\delta \to 1$. Furthermore, as

$$\int_\Omega G_0(v(x))\,dx = \delta^{-N} \int_{\Omega_\delta} G_0(v(\delta^{-1}x))\,dx = \delta^{-N} \int_\Omega G_0(z_\delta(x))\,dx,$$

$G_0(z_\delta) \to G_0(v)$ and $\delta^{-N}G_0(z_\delta) \to G_0(v)$ in $L^1(\Omega)$ as $\delta \to 1$. Since $0 \leqq G_0(\delta s) \leqq G_0(s)$ for all $s \in R$ and $0<\delta<1$, $G_0(v_\delta(x)) \leqq G_0(z_\delta(x))$ for all $x \in \Omega$. On the other hand, $G_0(v_\delta(x)) \to G_0(v(x))$ a.e. in Ω. Hence it follows by the dominated convergence theorem that $G_0(v_\delta) \to G_0(v)$ in $L^1(\Omega)$ as $\delta \to 1$. Similarly one can show that $G_i(D_i v_\delta) \to G_i(D_i v)$ in $L^1(\Omega)$ for each $i=1, 2, \ldots, N$.

The next step is to approximate each v_δ by test functions. We denote

$$v_{\delta\varepsilon}(x) = \int_\Omega j_\varepsilon(x-y) v_\delta(y)\,dy,$$

the mollification of v_δ for any $\varepsilon>0$ ([1] p. 29). For each given δ, $v_{\delta\varepsilon} \in C_0^\infty(\Omega)$ and $v_{\delta\varepsilon} \to v_\delta$ in $W_0^{1,1}(\Omega)$ as $\varepsilon \to 0$ ([1] p. 52). We must show that $G_i(D_i v_{\delta\varepsilon}) \to G_i(D_i v_\delta)$ in $L^1(\Omega)$ for all $i=0, 1, \ldots, N$. Indeed, by Jensen's inequality (see [12] p. 123),

$$0 \leqq G_0(v_{\delta\varepsilon}(x)) = G_0\left(\int_\Omega j_\varepsilon(x-y)v_\delta(y)\,dy\right)$$

$$\leqq \int_\Omega j_\varepsilon(x-y) G_0(v_\delta(y))\,dy = (G_0(v_\delta))_\varepsilon(x),$$

where $(G_0(v_\delta))_\varepsilon \to G_0(v_\delta)$ in $L^1(\Omega)$ ([1] p. 30). Therefore $G_0(v_{\delta\varepsilon}) \to G_0(v_\delta)$ in $L^1(\Omega)$ again by the dominated convergence theorem. For each $i=1,2, \ldots, N$ we similarly have

$$G_i(D_i v_{\delta\varepsilon}(x)) = G_i((D_i v_\delta)_\varepsilon(x)) \leqq (G_i(D_i v_\delta)_\varepsilon)(x),$$

and the convergence $G_i(D_i v_{\delta\varepsilon}) \to G_i(D_i v_\delta)$ in $L^1(\Omega)$ can be derived accordingly.

The final step of the proof is to set $w=v_{\delta\varepsilon}$ in (5.9), to send firstly $\varepsilon \to 0$ and then $\delta \to 1$ to obtain the assertion. $\square$

For the Neumann type boundary value problems some modifications are necessary.

Theorem 6. *Let Ω be a bounded starlike domain in $\mathbf{R}^N$ having the cone property and let the functions g_i satisfy the conditions (E_1) and (E_2). Let $V = W^{1,1}(\Omega)$ and $W = V \cap W^{2,s}(\Omega)$ with $s > N$. Then for each $f \in L^\infty(\Omega)$ there exists a unique u in $V_1 \cap W^1 L_M(\Omega)$ such that (5.5) and (5.6) hold.*

Proof. Let u be a solution in $V_1 \cap W^1 L_M(\Omega)$ of the pair $\{(5.5), (5.6)\}$. A solution exists by Theorem 4. As above in the proof of Theorem 5 we can conclude that

$$(5.10) \qquad \int_\Omega \sum_{i=0}^N \{G_i(D_i w) - G_i(D_i u)\} \, dx \geqq (f, w - u) \quad \text{for all} \quad w \in W^{2,s}(\Omega).$$

In view of Lemma 5 it will be sufficient to find, for each $v \in W^{1,1}(\Omega)$ with $G_i(D_i v) \in L^1(\Omega)$ for all $i = 0, 1, ..., N$, a suitable approximation in $W^{2,s}(\Omega)$. For any $\delta > 1$ we define $\Omega_\delta = \{x \in \mathbf{R}^N : \delta^{-1} x \in \Omega\}$ so that $\Omega \subset \subset \Omega_\delta$. Let $z_\delta(x) = v(\delta^{-1} x)$ for all $x \in \Omega_\delta$. Then $z_\delta \to v$ in $W^{1,1}(\Omega)$ as $\delta \to 1$ and, furthermore,

$$\int_\Omega G_0(v(x)) \, dx = \delta^{-N} \int_{\Omega_\delta} G_0(v(\delta^{-1} x)) \, dx \geqq \delta^{-N} \int_\Omega G_0(z_\delta(x)) \, dx,$$

implying that $G_0(z_\delta) \in L^1(\Omega)$ and $G_0(z_\delta) \to G_0(v)$ in $L^1(\Omega)$ as $\delta \to 1$. For each $i = 1, 2, ..., N$, we have further $D_i z_\delta(x) = \delta^{-1} D_i v(\delta^{-1} x)$ and hence

$$\int_\Omega G_i(D_i v(x)) \, dx = \delta^{-N} \int_{\Omega_\delta} G_i(D_i v(\delta^{-1} x)) \, dx \geqq \delta^{-N} \int_\Omega G_i(D_i v(\delta^{-1} x)) \, dx$$

$$\geqq \delta^{-N} \int_\Omega G_i(\delta^{-1} D_i v(\delta^{-1} x)) \, dx = \delta^{-N} \int_\Omega G_i(D_i z_\delta(x)) \, dx.$$

Consequently, $G_i(D_i z_\delta) \to G_i(D_i v)$ in $L^1(\Omega)$ for each $i = 1, 2, ..., N$ follows by a previous argument.

Each z_δ can now be approximated by the mollification $z_{\delta\varepsilon}$ such that $z_{\delta\varepsilon} \to z_\delta$ in $W^{1,1}(\Omega)$ as $\varepsilon \to 0$, because $\Omega \subset \subset \Omega_\delta$ (see [1] p. 52). It is also obvious that $z_{\delta\varepsilon} \in W^{2,s}(\Omega)$ for any $\varepsilon > 0$ and $\delta > 1$. We can complete the proof as the proof of Theorem 5 by showing that $G_i(D_i z_{\delta\varepsilon}) \to G_i(D_i z_\delta)$ in $L^1(\Omega)$ for all $i = 0, 1, ..., N$ and letting then $\varepsilon \to 0$ and finally $\delta \to 1$ in (5.10) with $w = z_{\delta\varepsilon}$. $\quad \square$

We shall close our discussion with the following uniqueness theorem for the obstacle problem.

Theorem 7. *Let Ω be a bounded starlike domain in $\mathbf{R}^N$ having the cone property, let the functions g_i satisfy the condition (E_1), let $K = \{v \in W_0^{1,1}(\Omega) : v(x) \geqq \psi(x) \text{ a.e. in } \Omega\}$, where ψ is a given function in $W^{2,\infty}(\Omega)$ with $\psi \leqq 0$ in Ω, and let $V = W_0^{1,1}(\Omega)$, $W = V \cap W^{2,s}(\Omega)$ with $s > N$. Then for each $f \in L^\infty(\Omega)$ there exists a unique u in $K \cap V_1$ such that*

$$(5.11) \qquad \int_\Omega \sum_{i=0}^N g_i(D_i u)(D_i w - D_i u) \, dx \geqq (f, w - u) \quad \text{for all} \quad w \in K \cap W.$$

Proof. By Theorem 3 and Remark 2 a solution $u \in K \cap V_1$ of (5.11) exists. By the previous argument we get from (5.11) the inequality

$$(5.12) \qquad \int_{\Omega} \sum_{i=0}^{N} \{G_i(D_i w) - G_i(D_i u)\} \, dx \geqq (f, w - u) \quad \text{for all} \quad w \in K \cap W.$$

For a given $v \in K$ with $G_i(D_i v) \in L^1(\Omega)$ for all $i = 0, 1, \ldots, N$, we define v_δ and $v_{\delta \varepsilon}$ with $0 < \delta < 1$ and $\varepsilon > 0$ as in the proof of Theorem 5. Since these approximations are not necessarily in K, we also consider a map $w \to w_\psi$ in $W^{1,1}(\Omega)$ defined by

$$w_\psi(x) = \max \{w(x), \psi(x)\} = [w - \psi]^+(x) + \psi(x).$$

As $\psi(x) \leqq 0$ a.e. in Ω, $0_\psi = 0$. Moreover, it is known that

$$D_i w_\psi(x) = \begin{cases} D_i w(x), & \text{if} \quad w(x) > \psi(x), \\ D_i \psi(x), & \text{if} \quad w(x) \leqq \psi(x) \end{cases}$$

and that the map $w \to w_\psi$ is continuous in $W^{1,1}(\Omega)$. Then it is also clear that $v_\psi = v$, $(v_\delta)_\psi \in K$ and $(v_{\delta \varepsilon})_\psi \in K$. Since $v_{\delta \varepsilon} \to v_\delta$ and $v_\delta \to v$ in $W^{1,1}(\Omega)$ as $\varepsilon \to 0$ and $\delta \to 1$, respectively, $(v_{\delta \varepsilon})_\psi \to (v_\delta)_\psi$ and $(v_\delta)_\psi \to v_\psi = v$ in $W^{1,1}(\Omega)$. Next we show that $G_i(D_i((v_{\delta \varepsilon})_\psi)) \to G_i(D_i v)$ in $L^1(\Omega)$ for each $i = 0, 1, \ldots, N$. Indeed, it is readily seen that $D_i((v_{\delta \varepsilon})_\psi)(x) \to D_i v(x)$ a.e. in Ω (for a subsequence). By continuity also $G_i(D_i((v_{\delta \varepsilon})_\psi)(x)) \to G_i(D_i v(x))$ a.e. in Ω. From the proof of Theorem 5 we already know that $G_i(D_i v_{\delta \varepsilon}) \to G_i(D_i v)$ in $L^1(\Omega)$. Hence there exist functions $h_0, h_1, \ldots, h_N$ in $L^1(\Omega)$ such that $G_i(D_i v_{\delta \varepsilon}(x)) \leqq h_i(x)$ a.e. in Ω. Consequently,

$$G_i(D_i((v_{\delta \varepsilon})_\psi)(x)) \leqq G_i(D_i v_{\delta \varepsilon}(x)) + G_i(D_i \psi(x))$$

$$\leqq h_i(x) + G_i(D_i \psi(x)),$$

where $h_i + G_i(D_i \psi) \in L^1(\Omega)$ for all $i = 0, 1, \ldots, N$. Hence $G_i(D_i((v_{\delta \varepsilon})_\psi)) \to G_i(D_i v)$ in $L^1(\Omega)$, by the dominated convergence theorem.

Because $(v_{\delta \varepsilon})_\psi$ is not necessarily in $W^{2,s}(\Omega)$ and therefore not in W, the final step of the proof is to find for each fixed $(v_{\delta \varepsilon})_\psi$ an approximation in W. Indeed, let $w = v_{\delta \varepsilon} \in C_0^\infty(\Omega)$ be given. For any $n \in N$ we define a function $\varphi_n : R \to R$ by

$$\varphi_n(t) = \begin{cases} 0, & \text{if} \quad t \leqq \dfrac{1}{2n}, \\[2ex] nt^2 - \dfrac{1}{4n} - t, & \text{if} \quad \dfrac{1}{2n} < t \leqq \dfrac{1}{n}, \\[2ex] t - \dfrac{3}{4n}, & \text{if} \quad \dfrac{1}{n} < t. \end{cases}$$

The mollification of φ_n is denoted by $\varphi_{n\sigma}$, where $0 < \sigma < 1/4n$. Then $\varphi_{n\sigma} \in C^\infty(R)$, $\varphi_{n\sigma} \geqq 0$ and $\varphi_{n\sigma} = 0$ for all $t \leqq 1/4n$. Moreover, there exists a function $\varrho \in C_0^\infty(\Omega)$ such that $0 \leqq \varrho(x) \leqq 1$ in Ω and $\varrho(x) = 1$ in the support of w. Now we can set

$w_{n\sigma} = \{\varphi_{n\sigma}(w - \psi) + \psi\}\varrho$. Hence it is obvious that $w_{n\sigma} \in W_0^{1,1}(\Omega) \cap W^{2,\infty}(\Omega)$, $w_{n\sigma} \geqq \psi\varrho \geqq \psi$, i.e. $w_{n\sigma}$ is also in K. As φ_n and φ_n' are continuous and $w - \psi$ is bounded in $W^{1,\infty}(\Omega)$, we can easily conclude that $w_{n\sigma} \to w_\psi \varrho = w_\psi$ in $W^{1,1}(\Omega)$ as $\sigma \to 0$ and $n \to \infty$, while $\varrho(x) = 1$ in supp (w_ψ). Since $|\varphi_{n\sigma}(t)| \leqq |t| + 1$ and $|\varphi_{n\sigma}'(t)| \leqq 1$ for all t, we can apply the dominated convergence theorem to conclude that $G_0(w_{n\sigma}) \to G_0(w_\psi)$ and $G_i(D_i w_{n\sigma}) \to G_i(D_i w_\psi)$ for all $i = 1, 2, \ldots, N$ in $L^1(\Omega)$. Our proof is complete if we substitute $w_{n\sigma}$ with $w = v_{\delta\varepsilon}$ into (5.12) and let $\sigma \to 0$, $n \to \infty$, $\varepsilon \to 0$ and finally $\delta \to 1$. $\quad\square$

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