

LIPSCHITZ AND QUASICONFORMAL APPROXIMATION AND EXTENSION

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1. Introduction

D. Sullivan has recently given an important variation of the theory of R. D. Edwards and R. C. Kirby [EK] on the deformation of embeddings. The essential difference between these theories is that Sullivan replaces the n -torus T^n by a more complicated closed manifold Q^n and the natural covering map $e: R^n \rightarrow T^n$ by a covering map $e': B^n \rightarrow Q^n$ where B^n is the open unit ball in R^n . Sullivan's theory is particularly useful in the categories of quasiconformal and lipschitz maps. For example, he proved in both categories the annulus theorem in all dimensions. Moreover, he established approximation theorems, which implied the existence of a lipschitz structure on every topological n -manifold without boundary for $n \neq 4$, and also the hauptvermutung for these manifolds in both categories. For $n \leq 3$ these results can also be obtained by PL methods, see [Tu₁] and [Vä₂].

In this paper, we first develop Sullivan's theory for manifolds which may have boundary. For example, we prove in both categories the hauptvermutung for n -manifolds with boundary, $n \neq 4, 5$. In the quasiconformal case our proof makes use of the extension of a quasiconformal map from R^n to R_+^{n+1} , proved by the authors in [TV₂] with the aid of Sullivan's theory.

Since the presentation of [Su] is very sketchy, a large part of this article (most of Sections 2—4) is devoted to a fairly detailed exposition of Sullivan's theory. We take on faith the most difficult part, namely the existence of the Sullivan groups (defined in 2.9), and we assume that the reader is familiar with the basic ideas of [EK], but otherwise our presentation is reasonably self-contained.

We also keep track of the dilatations and the bilipschitz constants of the maps, and obtain quantitative versions of Sullivan's results. These are applied in Section 5 to give estimates for these constants in certain extension problems. For example, we obtain a dilatation estimate in the quasiconformal Schoenflies theorem, previously known only in a very special case. In Section 7 we consider the approximation of quasiconformal and certain more general homeomorphisms by homeomorphisms which satisfy a bilipschitz condition, either locally in the euclidean metric

or globally in the quasihyperbolic metric. This metric and the associated notion of a solid homeomorphism are considered in detail in Section 6.

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2. Basic constructions

2.1. Notation and terminology. The one-point compactification of the euclidean n -space R^n is written as $\bar{R}^n = R^n \cup \infty$. If $m < n$, we identify R^m with the subspace $R^m \times 0$ of $R^n = R^m \times R^{n-m}$. Let (X, d) be a metric space. If $a \in X$ and $r > 0$, we let $B_d(a, r)$ or $B(a, r)$ denote the open ball $\{x \in X: d(x, a) < r\}$. More generally, if $\emptyset \neq A \subset X$, we write $B(A, r) = \{x \in X: d(x, A) < r\}$ where $d(x, A)$ is the distance between x and A . If A and B are nonempty subsets of X , we let $d(A, B)$ denote the distance between A and B and $d(A)$ the diameter of A . If $X = R^n$, we let d denote the standard euclidean metric, and write

$$B^n(x, r) = B(x, r), \quad B^n(r) = B(0, r), \quad B^n = B(0, 1),$$

$$S^{n-1}(x, r) = \partial B(x, r), \quad S^{n-1}(r) = S^{n-1}(0, r), \quad S^{n-1} = S^{n-1}(0, 1).$$

The closed unit cube $[-1, 1]^n$ of R^n is denoted by I^n , and the *open* cube $(-r, r)^n$ by $I^n(r)$. However, we set $I = [0, 1]$, although $I^1 = [-1, 1]$. The upper half space of R^n is $R_+^n = \{x \in R^n: x_n \geq 0\}$. If A is any subset of R^n , we set $A_+ = A \cap R_+^n$.

If f and g are maps into a metric space (Y, d) , defined in a set A , we write

$$d(f, g; A) = \sup \{d(f(x), g(x)): x \in A\}.$$

Let (X, d) and (Y, d') be metric spaces. A map $f: X \rightarrow Y$ is L -lipschitz, $L \geq 0$, if

$$d'(f(x), f(y)) \leq Ld(x, y)$$

for all $x, y \in X$. If $L \geq 1$ and if, in addition,

$$d'(f(x), f(y)) \geq d(x, y)/L,$$

f is L -bilipschitz. If each point x of X has a neighborhood U such that $f|U$ is lipschitz, L -lipschitz, bilipschitz or L -bilipschitz, f is said to be locally lipschitz, locally L -lipschitz, locally bilipschitz or locally L -bilipschitz, respectively. We let LIP denote the category of metric spaces and locally lipschitz (= LIP) maps. A locally bilipschitz embedding or immersion is said to be a LIP embedding or a LIP immersion, respectively.

Let $A \subset \bar{R}^n$ be a set such that $A \subset \text{cl int } A$. If $n \geq 2$, an embedding $f: A \rightarrow \bar{R}^n$ is said to be K -quasiconformal or K -QC if its restriction to each component of $\text{int } A$ is K -QC in the sense of [Vä₁]. If $n = 1$, f is said to be K -quasiconformal if

its restriction to each component of $\text{int } A$ is K -quasisymmetric [LeV₂, II.7.1]. We allow the possibility that f is sense-reversing. The concept “locally quasiconformal” (LQC) is defined in the obvious way.

We can define LQC manifolds and LIP manifolds in the well-known manner using atlases (see also [LuV, § 3] and [LT, 4.6]). Observe that every LIP manifold is an LQC manifold.

If $A \subset B$, we let id denote the inclusion map $A \rightarrow B$.

If X and Y are metric spaces, we shall consider the space $C(X, Y)$ of all continuous maps $f: X \rightarrow Y$ endowed with the compact-open topology. We let $H(X) \subset C(X, X)$ denote the group of all homeomorphisms $f: X \rightarrow X$.

If A is a subset of a topological space X , we let “near A ” mean “in a neighborhood of A ”.

If $A \subset R^n$, if $f: A \rightarrow R^p$ is a map and if $x_0 \in A$ is an accumulation point of A , we set

$$L(x_0, f) = \limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}.$$

If α is a rectifiable path in R^n , $l(\alpha)$ will denote its euclidean length.

The 1-dimensional case needs often a special treatment, since the properties of 1-dimensional quasisymmetric maps are different from the properties of higher-dimensional quasiconformal maps. On the other hand, it is usually easy to give a direct proof for the 1-dimensional case. To avoid technicalities, we make the following *convention*: The symbol n is always an integer at least two, unless otherwise stated.

The following elementary lemma is well-known (see, for example, [Fe, p. 64]):

2.2. Lemma. *Suppose that A is a convex set in R^p and that $f: A \rightarrow R^m$ is a map such that $L(x, f) \leq L_0$ for every $x \in A$. Then f is L_0 -lipschitz. \square*

2.3. Möbius transformations. We let $GM(m)$ denote the group of all Möbius transformations of \bar{R}^m . Thus $GM(m)$ is generated by similarity maps and inversions in spheres (in fact, by inversions alone). The image of a p -sphere under a Möbius transformation is always a p -sphere, if we regard a p -plane as a p -sphere through ∞ .

We let Möb_m denote the subgroup of $GM(m)$ consisting of maps g such that $gB^m = B^m$. Then Möb_m is generated by inversions in all orthogonal spheres of S^{m-1} . The group of all sense-preserving maps in Möb_m is written as Möb_m^+ . There is a natural topology in $GM(m)$, which can be defined in several ways. It is the topology of pointwise convergence and also the compact-open topology. Moreover, the induced topology of Möb_m can be defined by the metric $d(f, g; B^m)$.

2.4. Hyperbolic metric. Let $0 \leq p \leq m-1$, and let $T \subset \bar{R}^m$ be a p -plane (affine subspace). The hyperbolic metric σ_T of $R^m \setminus T$ is defined by the element of length $d\sigma_T = |dx|/d(x, T)$. This means that if $a, b \in R^m \setminus T$, $\sigma_T(a, b)$ is the infimum of the

line integrals

$$(2.5) \quad \int_{\alpha} \frac{|dx|}{d(x, T)}$$

taken over all rectifiable paths α joining a and b in $R^m \setminus T$. If H is a half space whose boundary contains T , and if $a, b \in H$, it is sufficient to consider paths α in H . If $p = m - 1$, $\sigma_T(a, b)$ is defined only if a and b belong to the same component of $R^m \setminus T$. The metric σ_T is invariant under the Möbius transformations g which respect T , that is, $gT = T$. Indeed, this is easy to verify if g is a similarity or if $g(x) = x/|x|^2$, and the general case follows from these.

Let $S \subset R^m$ be a p -sphere of radius r . The hyperbolic metric σ_S in $\bar{R}^m \setminus S$ is defined as follows: Choose $h \in GM(m)$ such that hS is a p -plane T . Then

$$\sigma_S(a, b) = \sigma_T(h(a), h(b)).$$

If $p = m - 1$, $\sigma_S(a, b)$ is again defined only if a and b belong to the same component of $\bar{R}^m \setminus S$. The metric σ_S is independent of the choice of h , and it is invariant under the Möbius transformations which respect S . It can also be defined explicitly by the element of length

$$(2.6) \quad d\sigma_S(x) = \frac{2r|dx|}{d(x, S)D(x, S)},$$

where

$$D(x, S) = \max \{|x - y| : y \in S\}.$$

In particular, if $S = S^{m-1}$, the hyperbolic metric of B^m is defined by

$$d\sigma_S(x) = \frac{2|dx|}{1 - |x|^2}.$$

The formula (2.6) follows by direct computation. However, we shall only need the fact that $d\sigma_S(x)d(x, S)/|dx|$ is bounded away from 0 and ∞ near S . We shall often omit the subscript S from σ_S if there is no danger of misunderstanding.

If $p = m - 1$, a hyperbolic ball $B_\sigma(x, r)$ is always a euclidean ball or a half space or the exterior of a euclidean ball.

2.7. Möbius coordinates. Let m and k be positive integers, and set $n = m + k$. For every $z \in \bar{R}^n \setminus S^{m-1}$ we define the Möbius coordinates $\xi \in B^m$ and $\eta \in \bar{R}^k$ as follows: There is a unique orthogonal k -sphere S_1^k of S^{n-1} through z which intersects B^m orthogonally. (Again, k -planes are considered as k -spheres through ∞ .) Then ξ is the unique point of $S_1^k \cap B^m$. Furthermore, there is a unique m -sphere S_2^m through S^{m-1} and z . Let C be the z -component of $S_2^m \setminus S^{m-1}$. Then $(0, \eta)$ is the unique point of $C \cap (0 \times R^k)$. Each point z is uniquely determined by its Möbius coordinates, and we write $z = [\xi, \eta]$. All this is easy to see if $m = k = 1$, and the general case follows by considering two-dimensional sections through $0, z$, and the orthogonal projection of z in R^m . If $z \in S^{m-1}$, we set $\xi = z$, but η is not defined.

2.8. **Lemma.** *Let $g \in \text{Möb}_m$. Then the map g^* defined by $g^*[\xi, \eta] = [g(\xi), \eta]$ is in Möb_n .*

Proof. If g is a reflection in an orthogonal $(m-1)$ -sphere of S^{m-1} , then g^* is the reflection in the $(n-1)$ -sphere with the same center and the same radius. Hence $g^* \in \text{Möb}_n$. The general case follows from this. \square

2.9. *Sullivan groups.* Let G be a subgroup of Möb_m^+ . We consider G as a transformation group of B^m . Assume that G has the following properties:

- (1) G is discrete.
- (2) G acts freely on B^m . This means that $g(x) \neq x$ whenever $x \in B^m$, $g \in G$ and $g \neq \text{id}$.

Each orbit Gx , $x \in B^m$, is discrete and closed in B^m . Let Q be the orbit space B^m/G with the quotient topology. The natural map $\pi_0: B^m \rightarrow Q$ is a covering map, and Q is an m -manifold without boundary. Moreover, π_0 defines a DIFF (and hence LIP and LQC) structure on Q . The hyperbolic metric σ_Q of Q is defined as follows: If $p, q \in Q$, $\sigma_Q(p, q)$ is the hyperbolic distance of $\pi_0^{-1}(p)$ and $\pi_0^{-1}(q)$ as subsets of B^m . Then $\pi_0: B^m \rightarrow Q$ is locally a hyperbolic isometry. Indeed, set $r(x) = \sigma(x, (Gx \setminus x))/4$. Then π_0 embeds $B_{\sigma_Q}(x, r(x))$ isometrically onto

$$B_{\sigma_Q}(\pi_0(x), r(x)).$$

The LIP structure of Q is also defined by the metric σ_Q . Each pair $p, q \in Q$ can be joined by a hyperbolic geodesic.

We say that G is a *Sullivan group* if, in addition, it has the following two properties:

- (3) $B^m/G = Q$ is compact.
- (4) If $q \in Q$, there is a LIP immersion $\alpha: Q \setminus q \rightarrow R^m$.

The orbit space Q will be called a *Sullivan manifold*. By a deep result of Sullivan [Su], a Sullivan group exists for every m . We fix for each positive integer m a Sullivan group G^m and the corresponding Sullivan manifold $Q^m = B^m/G^m$. The point $\pi_0(0) \in Q^m$ will be denoted by q_0 .

For every m we fix $r_m \in (0, 1/2)$ such that $\pi_0: B^m \rightarrow Q^m$ is a hyperbolic isometry in a neighborhood of each closed hyperbolic ball of radius r_m . This is possible by the compactness of Q^m . Then, if $0 < r \leq r_m$ and $q \in Q^m$, the pre-image of $B_{\sigma_Q}(q, r)$ is a disjoint union of hyperbolic balls in B^m , and π_0 maps each of these isometrically onto $B_{\sigma_Q}(q, r)$. We let D^m denote the hyperbolic ball $B_{\sigma}(0, r_m) \subset B^m$ and $s_m = \tanh(r_m/2)$ the euclidean radius of D^m .

2.10. *Some constructions.* Let m and k be positive integers. Set $n = m + k$ and $A_k^n = \bar{B}^n \setminus S^{m-1}$. Then each $z \in A_k^n$ has Möbius coordinates $\xi \in B^m$ and $\eta \in \bar{B}^k$; see 2.7. We thus obtain a natural homeomorphism $u: A_k^n \rightarrow B^m \times \bar{B}^k$ defined by $u[\xi, \eta] = (\xi, \eta)$. Setting $\pi = \pi_0 \times \text{id}: B^m \times \bar{B}^k \rightarrow Q_m \times \bar{B}^k$ and $e = \pi u$ we obtain the commuta-

tive diagram

$$\begin{array}{ccc}
 A_k^n & \xrightarrow{u} & B^m \times \bar{B}^k \\
 e \downarrow & \searrow \pi & \\
 Q^m \times \bar{B}^k & &
 \end{array}$$

Here π and e are covering maps. Observe that $e(0)=(q_0, 0)$. The map e plays an important role in the straightening of k -handles on an n -manifold. It corresponds to the natural covering map $R^m \times I^k \rightarrow T^m \times I^k$ of the Edwards—Kirby theory. For 0-handles we define $A_0^n = B^n$ and $e = \pi_0: B^n \rightarrow Q^n$. For n -handles we set $A_n^n = \bar{B}^n$ and $e = \text{id}: \bar{B}^n \rightarrow \bar{B}^n$.

In A_k^n we have the euclidean metric d of R^n and the hyperbolic metric σ of $\bar{R}^n \setminus S^{m-1}$. In $B^m \times \bar{B}^k$ we use the metric ϱ defined by

$$\varrho(x, x') = \max(\sigma(x_1, x'_1), |x_2 - x'_2|).$$

Replacing σ by σ_Q we obtain a metric on $Q^m \times \bar{B}^k$, also denoted by ϱ . Then π is an isometry in each set of the form $B_\sigma(x, r_m) \times \bar{B}^k$.

2.11. Lemma. *The homeomorphism $u: (A_k^n, \sigma) \rightarrow (B^m \times \bar{B}^k, \varrho)$ is bilipschitz.*

Proof. Set $F = \bar{B}^m(1/2)$ and $E = u^{-1}[F \times \bar{B}^k]$. Since u is a diffeomorphism, $u|E$ is L -bilipschitz for some L . Define the action of Möb_m on $B^m \times \bar{B}^k$ by $g(x_1, x_2) = (gx_1, x_2)$. Then every $g: B^m \times \bar{B}^k \rightarrow B^m \times \bar{B}^k$ is a ϱ -isometry. Since $gu = ug$ for all $g \in \text{Möb}_m$, $u|gE$ is L -bilipschitz for every $g \in \text{Möb}_m$. Since the sets gE cover A_k^n , it easily follows that u is L -lipschitz and that u^{-1} is $2L$ -lipschitz. \square

2.12. Lemma. *Suppose that $0 \leq k \leq n-1$ and that $f: A_k^n \rightarrow A_k^n$ is a homeomorphism such that f is locally L -bilipschitz in the hyperbolic metric σ of $\bar{R}^n \setminus S^{m-1}$ and such that $\sigma(f, \text{id}) \leq M < \infty$. Then f is L_1 -bilipschitz in the euclidean metric of R^n with $L_1 = 2Le^M$.*

Proof. It suffices to show that f is L_1 -lipschitz. Setting $\delta(x) = d(x, S^{m-1})$ we obtain by (2.6)

$$L(x, f) \leq 2L\delta(f(x))/\delta(x)$$

for all $x \in A$. By Lemma 2.2, it suffices to show that $\delta(f(x))/\delta(x) \leq e^M$. Let $x \in A_k^n$ and let α be a rectifiable path joining x and $f(x)$ in A_k^n . Since $\delta: A_k^n \rightarrow R^1$ is 1-lipschitz, [Vä₁, 5.3] implies

$$\int_\alpha \frac{|dy|}{\delta(y)} \leq \int_{\delta \circ \alpha} \frac{|dt|}{t} \leq \log \frac{\delta(f(x))}{\delta(x)}.$$

Hence

$$\frac{\delta(f(x))}{\delta(x)} \leq e^{\sigma(f, \text{id})} \leq e^M. \quad \square$$

2.13. *Lifting of maps.* Since $B^m \times \bar{B}^k$ is simply connected and $\pi: B^m \times \bar{B}^k \rightarrow Q^m \times \bar{B}^k$ is a covering map, every (continuous) map $f: Q^m \times \bar{B}^k \rightarrow Q^m \times \bar{B}^k$ can be lifted to a map $\hat{f}: B^m \times \bar{B}^k \rightarrow B^m \times \bar{B}^k$ such that $\pi\hat{f} = f\pi$. The lifting \hat{f} is unique as soon as we know $\hat{f}(0)$, which can be chosen arbitrarily in $\pi^{-1}f(q_0, 0)$.

The set

$$\mathcal{P}_0 = \{f \in H(Q^m \times \bar{B}^k) : \varrho(f, \text{id}) < r_m/2\}$$

is a neighborhood of id in $H(Q^m \times \bar{B}^k)$. In what follows, we shall only consider liftings of maps $f \in \mathcal{P}_0$. We let \hat{f} denote the unique lifting of $f \in \mathcal{P}_0$ satisfying $\hat{f}(0) \in B_\sigma(0, r_m/2) \times \bar{B}^k$. The map \hat{f} is a self-homeomorphism of $B^m \times \bar{B}^k$. Indeed, if $f \in \mathcal{P}_0$, then also $f^{-1} \in \mathcal{P}_0$, and $(f^{-1})^\wedge = (\hat{f})^{-1}$. The lifting \hat{f} can also be directly defined as follows: For $x = (x_1, x_2) \in B^m \times \bar{B}^k$ let

$$\pi_x: B_\sigma(x_1, r_m) \times \bar{B}^k \rightarrow B_{\sigma_Q}(\pi_0(x_1), r_m) \times \bar{B}^k$$

be the isometry defined by π . Then $f(\pi(x)) \in \text{im } \pi_x$, and we can define $\hat{f}(x) = \pi_x^{-1}f\pi(x)$. From this description it is easy to see that $\varrho(\hat{f}_1, \hat{f}_2) = \varrho(f_1, f_2)$ for all $f_1, f_2 \in \mathcal{P}_0$. In particular, $\varrho(\hat{f}, \text{id}) < r_m/2$ for every $f \in \mathcal{P}_0$. Furthermore, if the action of Möb_m on $B^m \times \bar{B}^k$ is defined by $g(x_1, x_2) = (gx_1, x_2)$, then $g\hat{f} = \hat{f}g$ for every $f \in \mathcal{P}_0$ and $g \in G^m$.

We define the action of Möb_m on A_k^n by $g[\zeta, \eta] = [g(\zeta), \eta]$. By Lemma 2.8 this action is conformal. For $f \in \mathcal{P}_0$ we set $f^* = u^{-1}\hat{f}u: A_k^n \rightarrow A_k^n$. Then f^* is an e -lifting of f , that is, $ef^* = fe$. From the above considerations and from Lemma 2.11 we easily obtain the following properties of f^* :

2.14. **Lemma.** *The correspondence $f \mapsto f^*$ is a continuous map $\mathcal{P}_0 \rightarrow H(A_k^n)$. If $f \in \mathcal{P}_0$ and $g \in G^m$, $gf^* = f^*g$. There is a constant M_n depending only on n such that $\sigma(f^*, \text{id}) \leq M_n$ for all $f \in \mathcal{P}_0$. Every f^* can be extended by $\text{id}|S^{m-1}$ to a homeomorphism $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$. \square*

2.15. *Local and global bilipschitz constants.* It is usually much easier to show that a given embedding is locally L -bilipschitz than L -bilipschitz. For example, let G be open in R^n , and let $f: G \rightarrow R^n$ be an embedding. Suppose that G can be expressed as a locally finite union of sets A_j such that $f|A_j$ is L -bilipschitz for every j . Then $f|\bar{A}_j \cap G$ is also L -bilipschitz, and the local convexity of R^n implies that f is locally L -bilipschitz. However, f need not be bilipschitz even if $G = G_1 \cup G_2$ with $f|G_j$ bilipschitz. However, we show that if we restrict the embeddings to a compact subset of G , the difference between L -bilipschitz and locally L -bilipschitz maps becomes fairly small. For this, we need the following result of F. John [Jo, Theorems II and III]:

2.16. **Lemma.** *Suppose that $f: B^n \rightarrow R^n$ is a locally L -bilipschitz immersion with $f(0) = 0$. Then*

- (1) $B^n(1/L) \subset fB^n \subset B^n(L)$,
- (2) $f|B^n(1/L^2)$ is L -bilipschitz. \square

2.17. Lemma. *Let G be open in R^n , and let A be compact in G . Let $f: G \rightarrow R^n$ be a locally L -bilipschitz embedding. Then $(f|_A)^{-1}$ is L_1 -lipschitz with L_1 depending only on G, A and L . If G is connected, $f|_A$ is L_1 -bilipschitz.*

Proof. We may assume that $G \neq R^n$. Set $r = d(A, \partial G)$, and let $x_1, x_2 \in A$. If $|x_1 - x_2| \leq r/L^2$, Lemma 2.16 implies $|f(x_1) - f(x_2)| \geq |x_1 - x_2|/L$. Suppose that $|x_1 - x_2| \geq r/L^2$. Then the balls $B^n(x_j, r/2L^2)$ are disjoint. By Lemma 2.16, $B^n(f(x_j), r/2L^3) \subset fB^n(x_j, r/2L^2)$, which implies

$$|f(x_1) - f(x_2)| \geq r/L^3 \geq r|x_1 - x_2|/L^3 d(A).$$

Hence $(f|_A)^{-1}$ is L_1 -lipschitz with $L_1 = \max(L, L^3 d(A)/r)$.

Next assume that G is connected. For $a, b \in G$ let $d_1(a, b)$ be the infimum of the lengths of all rectifiable paths α joining a and b in G . Then d_1 is a metric of G defining its usual topology. Since A is compact, $d_1(A) < \infty$. Suppose that $x, y \in A$. If $|x - y| < r$, the segment xy lies in G , which implies $|f(x) - f(y)| \leq L|x - y|$. Assume that $|x - y| \geq r$. Choose a path α joining x and y in G such that $l(\alpha) < 1 + d_1(A)$. Since f is locally L -bilipschitz, $l(f\alpha) \leq Ll(\alpha) \leq L(1 + d_1(A))$. Hence

$$|f(x) - f(y)| \leq L(1 + d_1(A)) \leq L(1 + d_1(A))|x - y|/r.$$

Consequently, f is L_2 -lipschitz with $L_2 = \max(L, L(1 + d_1(A))/r)$. \square

3. Deformation of embeddings near the identity

3.1. In this section we present Sullivan’s version of the Edwards—Kirby theory. Let M be a manifold, possibly with boundary. If $X \subset M$, an embedding $f: X \rightarrow M$ is said to be *clean* if $f^{-1}\partial M = X \cap \partial M$. The family of all clean embeddings $f: X \rightarrow M$ is denoted by $E(X; M)$. If $A \subset X$, we set

$$E(X, A; M) = \{f \in E(X; M) : f|_A = \text{id}\}.$$

We first give a quantitative canonical Schoenflies theorem for embeddings near id.

3.2. Lemma. *Let $A = \bar{B}^n \setminus B^n(1/2)$, and let*

$$\mathcal{P} = \{f \in E(A; R^n) : d(f, \text{id}) < 1/30\}.$$

Then there exist a continuous map $\varphi: \mathcal{P} \rightarrow E(\bar{B}^n; R^n)$, a universal constant a_0 , and a constant b_n depending only on n such that:

- (1) $\varphi(f)|_{S^{n-1}} = f|_{S^{n-1}}$ for all $f \in \mathcal{P}$.
- (2) $\varphi(\text{id}) = \text{id}$.
- (3) If f is K -QC, $\varphi(f)$ is $b_n K^3$ -QC.
- (4) If f is locally L -bilipschitz, $\varphi(f)$ is locally $a_0 L^3$ -bilipschitz.
- (5) If f is L -bilipschitz, $\varphi(f)$ is $a_0 L^3$ -bilipschitz.

Proof. The map $\varphi(f)$ is constructed by the furling idea of M. Brown. We can follow the proof of [GVM₁, Lemma 9, p. 241]. However, since we do not assume that $S^{n-1} \subset \text{im } e$, we choose $q=0.9$, and define the radial map λ by $\lambda[0, b]=[0, a]$, $\lambda[b, q]=[a, q]$, $\lambda[q, \infty]=[q, \infty]$. Moreover, we must have $2c=b/a$ in order that the construction be valid in the LIP category. If we choose $a=0.6$, $b=0.8$, $c=2/3$ and $d=0.65$, the construction is possible for all $e \in \mathcal{P}$. Then $\varphi=(e \mapsto \hat{e})$ satisfies (1), (3), (4) and (5). Replacing $\varphi(e)$ by $\varphi(e)\varphi(\text{id})^{-1}$, we find φ satisfying all conditions of the lemma. \square

3.3. Lemma. (*Straightening of handles near id.*) *Let $m \geq 0$ and $k \geq 0$ be integers such that $m+k=n \geq 2$. Then there exist a neighborhood \mathcal{P} of id in $E(I^n, I^m \times \partial I^k; R^m \times I^k)$ and a continuous map*

$$\psi: \mathcal{P} \times I \rightarrow E(I^n, I^m \times \partial I^k; R^m \times I^k)$$

with the following properties:

- (1) $\psi(\text{id}, t) = \text{id}$ for all $t \in I$.
- (2) $\psi(h, 0) = h$ for all $h \in \mathcal{P}$.
- (3) $\psi(h, t)|\partial I^n = h|\partial I^n$ for all $h \in \mathcal{P}$ and $t \in I$.
- (4) $\psi(h, 1)|(I^m \times \partial I^k) \cup (\bar{I}^m(1/2) \times I^k) = \text{id}$ for all $h \in \mathcal{P}$.
- (5) If $h \in \mathcal{P}$ is K -QC, then $\psi(h, t)$ is K_1 -QC with K_1 depending only on K and n .
- (6) If $h \in \mathcal{P}$ is locally L -bilipschitz, then $\psi(h, t)$ is locally L_1 -bilipschitz with L_1 depending only on L and n .
- (7) If $h \in \mathcal{P}$ is L -bilipschitz, then $\psi(h, t)$ is L_1 -bilipschitz with L_1 depending only on L and n .

Proof. Recall the notation $D^m = B_\sigma(0, r_m) = B^m(s_m)$ from 2.9. With the aid of an auxiliary bilipschitz homeomorphism of R^n , we may replace the cubes I^m and I^k by closed unit balls \bar{B}^m and \bar{B}^k , and the cubes $\bar{I}^m(1/2)$ of (4) by the ball $\bar{B}_\sigma(0, r_m/2)$. Performing a preliminary deformation given by [EK, Proposition 3.2], it suffices to consider embeddings $h \in E(\bar{B}^m \times \bar{B}^k, \bar{B}^m \times \partial B^k; R^m \times \bar{B}^k)$ such that $h = \text{id}$ in $C = B^m(9/10) \times (\bar{B}^k \setminus B^k(1/2))$. This deformation multiplies the dilatation and the bilipschitz constant of h at most by a factor depending only on n . Let $u: A_k^n \rightarrow B^m \times \bar{B}^k$ be the homeomorphism defined in 2.10. We first show that for every $h \in (\bar{B}^m \times \bar{B}^k, \bar{B}^m \times \partial B^k \cup C; R^m \times \bar{B}^k)$ which is sufficiently close to the identity, there is a homeomorphism $g: R^n \rightarrow R^n$, continuously dependent on h , such that (i) $g = \text{id}$ if $h = \text{id}$, (ii) $g = u^{-1}hu$ in $u^{-1}[\bar{D}^m \times \bar{B}^k]$, (iii) $g = \text{id}$ in $R^n \setminus B^n$, and (iv) g respects the QC and bilipschitz properties of h .

If $m=0$, $h: \bar{B}^n \rightarrow \bar{B}^n$ is a homeomorphism with $h|S^{n-1} = \text{id}$. Hence we can choose $g = h \cup (\text{id}|R^n \setminus B^n)$. Suppose that $m \geq 1$. Let us first recall the Sullivan covering map $\pi_0: B^m \rightarrow Q^m$ from 2.9. It is injective in a neighborhood of the ball \bar{D}^m . Choose a point $p \in Q^m \setminus \pi_0 \bar{D}^m$ and a LIP immersion $\alpha_0: Q^m \setminus p \rightarrow B^m(9/10)$. By the LIP Schoenflies theorem [LuV, 7.7], we may assume that $\alpha_0 \pi_0 = \text{id}$ in a neighborhood of \bar{D}^m . Set $\alpha = \alpha_0 \times \text{id}: (Q^m \setminus p) \times \bar{B}^k \rightarrow \bar{B}^m \times \bar{B}^k$. Then $\alpha e = u$ in a neigh-

neighborhood of $u^{-1}[\bar{D}^m \times \bar{B}^k]$. Choose $t > 0$ such that $4t < \min(r_m, \sigma_Q(p, \pi_0 \bar{D}^m))$, and set $D_j = B_{\sigma_Q}(p, jt)$ for $j \geq 4$. We can now construct the following variation of the main diagram of Edwards—Kirby.

$$\begin{array}{ccc}
 R^n & \xrightarrow{g} & R^n \\
 \cup & & \cup \\
 A_k^n & \xrightarrow{h_4} & A_k^n \\
 \downarrow e & & \downarrow e \\
 Q^m \times \bar{B}^k & \xrightarrow{h_3} & Q^m \times \bar{B}^k \\
 \cup & & \parallel \\
 Q^m \times \bar{B}^k \setminus D_3 \times B^k(3/4) & \xrightarrow{h_2|} & Q^m \times \bar{B}^k \\
 \cap & & \parallel \\
 Q^m \times \bar{B}^k \setminus D_2 \times B^k(5/8) & \xrightarrow{h_2} & Q^m \times \bar{B}^k \\
 \cup & & \cup \\
 (Q^m \setminus D_2) \times \bar{B}^k & \xrightarrow{h_1} & (Q^m \setminus D_1) \times \bar{B}^k \\
 \downarrow \alpha| & & \downarrow \alpha| \\
 \bar{B}^m \times \bar{B}^k & \xrightarrow{h} & R^m \times \bar{B}^k
 \end{array}$$

We start from the bottom with h in a neighborhood \mathcal{P} of id. On each step, we may replace \mathcal{P} by a smaller neighborhood. We first find h_1 as in [EK, p. 68]. The embedding h_2 is the extension of h_1 via the identity, and the homeomorphism h_3 is found by the canonical Schoenflies theorem 3.2. The homeomorphism h_4 is the lifting h_3^* given by 2.13, and g is the extension of h_4 via the identity. If \mathcal{P} is a sufficiently small neighborhood of id, the properties (i), (ii) and (iii) are easily verified.

If $K \geq 1$ or $L \geq 1$, we let K_1, K_2, \dots or L_1, L_2, \dots denote constants depending only on K and n or L and n , respectively. Suppose that h is K -QC. We divide $Q^m \times \bar{B}^k$ into the following three sets:

$$F_1 = (Q_m \setminus \bar{D}_3) \times \bar{B}^k, \quad F_2 = \bar{D}_3 \times \bar{B}^k(3/4), \quad F_3 = \bar{D}_3 \times (\bar{B}^k \setminus B^k(3/4)).$$

Set $E_j = e^{-1}F_j, j=1, 2, 3$. Since Q^m is compact, we can choose a set U , open in A_k^n , such that \bar{U} is compact in A_k^n and such that $eU = Q^m \times \bar{B}^k$. Then $\alpha e|U \setminus e^{-1}[\bar{D}_1 \times \bar{B}^k]$ is locally L_1 -bilipschitz in the euclidean metric. Since $\alpha e h_4 = h \alpha e$ in E_1 , $h_4|E_1 \cap U$ is K_1 -QC with $K_1 = KL_1^{4n-4}$. Since h_4 is compatible with G^m by 2.14 and since G^m acts conformally on A_k^n , $h_4|E_1$ is K_1 -QC.

The set E_2 has a countable number of components $E_{2j}, j \in N$, and e maps each E_{2j} homeomorphically onto F_2 . In each E_{2j} , h_4 is constructed by Lemma 3.2 using an auxiliary bilipschitz embedding which is defined in a neighborhood of E_{2j} and which maps E_{2j} onto \bar{B}^n . Hence the dilatation of $h_4|E_{2j}$ depends only on K, n and j . However, if $i \neq j$, there is $g \in G^m$ such that $h_4|E_{2i} = g h_4 g^{-1}|E_{2j}$. Hence $h_4|E_2$ is K_2 -QC.

Finally, $h_4|E_3 = \text{id}$, which implies that h_4 is K_3 -QC with $K_3 = \max(K_1, K_2)$, cf. [Vä₁, 35.1]. Hence g is K_3 -QC.

Suppose that h is locally L -bilipschitz. Arguing as in the QC case we conclude that h_4 is locally L_1 -bilipschitz in the hyperbolic metric of $\bar{R}^n \setminus S^{m-1}$. On the other hand, $\sigma(h_4, \text{id}) \leq M_n$ by 2.14. It follows from 2.12 that h_4 is (globally) L_2 -bilipschitz in the euclidean metric. Hence g is L_2 -bilipschitz.

Define the Alexander isotopy $g_t: R^n \rightarrow R^n$ by $g_t(x) = tg(x/t)$ for $0 < t \leq 1$, $g_0 = \text{id}$. Choose a bilipschitz embedding $b: \bar{B}^n \rightarrow \bar{B}^m(3/4) \times \bar{B}^k$ such that $v = u$ in $u^{-1}[\bar{B}^m(1/2) \times \bar{B}^k]$. Let $f_t: R^n \rightarrow R^n$ be the isotopy defined by $f_t = vg_t^{-1}v^{-1}$ in $v\bar{B}^n$ and by $f_t = \text{id}$ outside $v\bar{B}^n$. The desired deformation ψ is obtained by setting

$$\psi(h, t) = f_t h: \bar{B}^m \times \bar{B}^k \rightarrow R^m \times \bar{B}^k.$$

If h is in a sufficiently small neighborhood \mathcal{P} of id , $\psi(h, t)$ is in

$$E(\bar{B}^n \times \bar{B}^k, \bar{B}^m \times \partial B^k; R^m \times \bar{B}^k),$$

and the properties (1)–(7) of Lemma 3.3 are easily checked (remember that $\bar{D}^m \subset B^m(1/2)$). \square

3.4. Theorem. *Let G be open in \bar{R}^n , and let A, A', B, B', C, C' be subsets of G satisfying the following conditions: A and C are closed in G , B is compact, and $A \subset \text{int } A', B \subset \text{int } B', C \subset \text{int } C'$. Then there exist a neighborhood \mathcal{P} of id in $E(G, C'; \bar{R}^n)$, an open neighborhood $A'' \subset A'$ of A in G and a continuous map $\psi: \mathcal{P} \times I \rightarrow E(G, C; \bar{R}^n)$ such that the following conditions are satisfied for all $t \in I$ and $h \in \mathcal{P}$:*

- (1) $\psi(\text{id}, t) = \text{id}$.
- (2) $\psi(h, 0) = h$.
- (3) $\psi(h, t) = h$ in $G \setminus B'$.
- (4) $\psi(h, 1)|B \cup C = \text{id}$.
- (5) *If h is LQC in $\text{int } A'$, $\psi(h, t)$ is LQC in A'' . If $h|_{\text{int } A'}$ is K -QC, $\psi(h, t)|_{A''}$ is K_1 -QC with K_1 depending only on K and n .*

(6) *If $G \subset R^n$ and if $h|_{A'}$ is a LIP embedding, then $\psi(h, t)|_{A''}$ is a LIP embedding. If $h|_{A'}$ is locally L -bilipschitz, $\psi(h, t)|_{A''}$ is locally L_1 -bilipschitz with L_1 depending only on L and n .*

(7) *If $G \subset R^n$ and if $h|_{A'}$ is L -bilipschitz, $\psi(h, t)|_{A''}$ is L_1 -bilipschitz with L_1 depending only on L and n .*

Proof. We may assume that $\infty \notin B$. Choose a compact PL manifold $P \subset \text{int } B'$ such that $B \subset \text{int } P$ and a fine handle decomposition of P . As in [EK, pp. 71–73], we can find a neighborhood \mathcal{P} and a map ψ satisfying (1)–(4) by successive applications of Lemma 3.3, starting with zero handles. In order that (5)–(7) also be true, we must choose P and the handle decomposition in a regular manner. For example, we can start with the cell decomposition T_0 of R^n consisting of all cubes with vertices in $2^k \mathbf{Z}^n$ and with side length 2^k , where k is a small negative integer. We let P be the underlying polyhedron of a finite subcomplex T_1 of T_0 such that $B \subset \text{int } P$ and such that $Q \subset B'$ whenever $Q \in T_0$ and Q meets P . Then

$P \subset \text{int } B'$. Choose a simplicial subdivision T_2 of T_1 without new vertices [RS, 2.9]. Then the second barycentric subdivision T_2'' defines a handle decomposition \mathcal{H} of P as in [RS, p. 82]. Let $i(H)$ be the index of a handle $H \in \mathcal{H}$. Define $H \cong H'$ if $H \cap H' \neq \emptyset$ and $i(H) \cong i(H')$. Then $H \cong H' \cong H''$ implies $H \cong H''$. Let \mathcal{H}_0 be the set of the handles $H \in \mathcal{H}$ such that there is $H' \in \mathcal{H}$ with $H \cong H'$ and $H' \cap C \neq \emptyset$. We may assume that $P_0 = \cup \mathcal{H}_0 \subset C'$. Let H_1, \dots, H_s be the handles of $\mathcal{H} \setminus \mathcal{H}_0$ arranged in order of increasing index. For each j we choose a PL embedding $\mu_j: R^n \rightarrow B \setminus C$ such that $\text{im } \mu_j$ is a small neighborhood of H_j and such that for $i(H_j) = k$, $m = n - k$ and $H_j^* = \mu_j I^m$ we have:

- (i) $\mu_j [I^m(1/2) \times I^k] = H_j$.
- (ii) $\mu_j [R^m \times I^k] \cap (P_0 \cup H_1 \cup \dots \cup H_{j-1}) = \mu_j [R^m \times \partial I^k]$.
- (iii) There are similarity maps $\alpha_j: R^n \rightarrow R^n$ such that the embeddings $\alpha_j \mu_j$ belong to a finite set depending only on n .

The desired isotopy $\psi(h, t)$ is a composition $\varphi_s * \dots * \varphi_1$, where

$$\begin{aligned} \varphi_j(h, t) &= \mu_j \psi_k(\mu_j^{-1} h \mu_j | I^m, t) \mu_j^{-1} \text{ in } H_j^*, \\ \varphi_j(h, t) &= h \text{ outside } H_j^*, \end{aligned}$$

and ψ_k is given by Lemma 3.3 with $k = i(H_j)$.

The conditions (5)–(7) follow from the corresponding conditions of Lemma 3.3. In (5) we also need the fact that a set of finite $(n - 1)$ -dimensional measure is removable for K -QC homeomorphisms [Vä₁, 35.1]. In (7) we use the following elementary lemma: Let $G \subset R^n$ be open, let $A \subset G$ be compact, and let $f: G \rightarrow R^n$ be an embedding such that $f|_A$ and $f|_{G \setminus A}$ are L -bilipschitz. Then f is L -bilipschitz. \square

3.5. Theorem. *In Theorem 3.4, the conditions (2), (3), (4) can be replaced by the following ones:*

- (2') $\psi(h, 0) = \text{id}$.
- (3') $\psi(h, t)|_{G \setminus B'} = \text{id}$.
- (4') $\psi(h, 1) = h$ in $B \cup C$.

Proof. We may assume that B' is compact. Choose a closed neighborhood A_1 of A in $\text{int } A'$ and a compact neighborhood B_1 of B in $\text{int } B'$. Apply 3.4 with A and B replaced by A_1 and B_1 . We obtain a map $\psi_0: \mathcal{P} \times I \rightarrow E(G, C; \bar{R}^n)$ and an open neighborhood $A_0'' \subset A'$ of A . Choose an open neighborhood A'' of A with $\text{cl}_G A'' \subset A_0''$. Replacing \mathcal{P} by a smaller neighborhood we may assume that $hB \subset B_1$ and that $h[A'' \cap B'] \subset \psi_0(h, t)A_0''$ for all $h \in \mathcal{P}$ and $t \in I$. Then the map $\psi(h, t) = \psi_0(h, t)^{-1}h$ has the required properties. \square

3.6. Theorem. *Let $G \subset R^n$ be open, let $B \subset G$ be compact, let $B' \subset G$ be a neighborhood of B , and let $\varepsilon > 0$. Then there is $\delta > 0$ such that for every LIP embedding $h: G \rightarrow R^n$ with $d(h, \text{id}) \leq \delta$ there is a LIP homeomorphism $h': R^n \rightarrow R^n$ such that*

- (1) $d(h', \text{id}) \leq \varepsilon$,
- (2) $h'|_B = h|_B$,
- (3) $h'|_{R^n \setminus B'} = \text{id}$.

Furthermore, if h is K -QC, h' is K_1 -QC with K_1 depending only on K and n . If h is locally L -bilipschitz, h' is L_1 -bilipschitz with L_1 depending only on L and n .

Proof. We may assume that B' is compact. We apply Theorem 3.5 with $A = A' = G$ and $C = C' = \emptyset$ and obtain a map $\psi: \mathcal{P} \times I \rightarrow E(G; R^n)$. Since $\psi(\text{id}, t) = \text{id}$ for every $t \in I$, we may assume, replacing \mathcal{P} by a smaller neighborhood, that $|\psi(h, t)(x) - x| < \varepsilon$ for all $h \in \mathcal{P}$, $t \in I$ and $x \in G$. There is $\delta > 0$ such that $h \in \mathcal{P}$ whenever $h \in E(G; R^n)$ and $d(h, \text{id}) \leq \delta$. We can then define h' by $h'|_G = \psi(h, 1)$ and by $h'|_{R^n \setminus G} = \text{id}$. \square

3.7. *Remark.* The qualitative part of Theorem 3.6 is the same as [TV₂, 2.19]. This is all that was needed from Sullivan's theory in the proof of the QC extension theorem from dimension n to $n + 1$. However, it is possible to simplify this proof by making also use of the quantitative part of Theorem 3.6; see 7.1. We shall use this simplification in the proof of Theorem 7.4, which can be considered as a generalization of the result in [TV₂]. For this purpose, we give a quantitative version of [TV₂, 2.20]. For this and for later purposes, it is convenient to introduce the following notion:

3.8. *Solid families.* Let X and Y be metric spaces. A family \mathcal{F} of embeddings $f: X \rightarrow Y$ is said to be *solid* if $\text{cl } \mathcal{F}$ is a compact family of embeddings. If $Y = R^n$ and if $X \subset R^n$ is either open or compact, \mathcal{F} is solid if and only if it is pointwise bounded, equicontinuous and inversely equicontinuous. More precisely:

- (1) For every $x_0 \in X$, the set $\{f(x_0) : f \in \mathcal{F}\}$ is bounded.
- (2) For every $x_0 \in X$ and $\varepsilon > 0$, there is a neighborhood U of x_0 such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in U$ and $f \in \mathcal{F}$.
- (3) For every $x_0 \in X$ and for every neighborhood U of x_0 , there is $\varepsilon' > 0$ such that $|f(x) - f(x_0)| \geq \varepsilon'$ whenever $x \in X \setminus U$ and $f \in \mathcal{F}$.

Indeed, by Ascoli's theorem, (1) and (2) are true if and only if $\text{cl } \mathcal{F}$ is compact in $C(X, R^n)$, and (3) means that the members of $\text{cl } \mathcal{F}$ are embeddings.

The corresponding statement is true if we replace R^n by \bar{R}^n and the euclidean metric by the spherical metric. However, in this case the condition (1) is superfluous.

Let $U \subset R^n$ be open and let $E_{\text{LIP}}(U; R^n)$ be the family of all LIP embeddings $f: U \rightarrow R^n$. By a result of Sullivan [Su], $E_{\text{LIP}}(U; R^n)$ is dense in $E(U; R^n)$ if $n \neq 4$. We shall prove this in Section 4 (Corollary 4.5). We say that an embedding $f: U \rightarrow R^n$ is LIP approximable if $f \in \text{cl } E_{\text{LIP}}(U; R^n)$, which is always true if $n \neq 4$.

We next give a quantitative version of [TV₂, 2.20]. This will be needed in Section 7.

3.9. **Lemma.** *Let U, U', V, W be open sets in R^n such that*

$$W \subset V \subset U, \quad \bar{U}' \subset U, \quad \bar{W} \cap U \subset V,$$

and \bar{U}' is compact. Let \mathcal{F} be a solid family of embeddings $g: U \rightarrow R^n$. For $n=4$ we also assume that the members of \mathcal{F} are LIP approximable. Let $\varepsilon > 0$, $K \geq 1$, $L \geq 1$. Then there are $\delta > 0$, $K' \geq 1$, $L' \geq 1$ with the following properties:

Let $h: V \rightarrow R^n$ be a LIP embedding, and let $g \in \mathcal{F}$ be such that $d(g, h; V) \leq \delta$. Then there is a LIP embedding $h': U \rightarrow R^n$ such that

- (1) $d(h', g; U) \leq \varepsilon$,
- (2) $h' = h$ in $W \cap U'$,
- (3) $h'|U'$ is K' -QC if h is K -QC,
- (4) $h'|U'$ is L' -bilipschitz if h is locally L -bilipschitz.

Here δ depends only on $\tau = (U, U', V, W, \mathcal{F}, \varepsilon)$, K' depends on τ and K , and L' depends on τ and L .

Proof. We shall closely follow the proof of [TV₂, 2.20]. Choose open sets A and A_1 such that $\bar{U}' \subset A$, $\bar{A} \subset A_1$, $\bar{A}_1 \subset U$ and \bar{A}_1 is compact. Since \mathcal{F} is solid, there is $\varepsilon_1 > 0$ such that $|g(x) - g(y)| \leq \varepsilon/2$ whenever $g \in \mathcal{F}$, $|x - y| \leq \varepsilon_1$ and $x, y \in A$. Next, by Theorem 3.6, there is $\delta_1 > 0$ such that if $f: A \cap V \rightarrow R^n$ is a LIP embedding with $d(f, \text{id}; A \cap V) \leq \delta_1$, then there is a LIP homeomorphism $f': R^n \rightarrow R^n$ such that

$$(3.10) \quad d(f', \text{id}) \leq \varepsilon_1, \quad f' = f \text{ in } W \cap U', \quad f'|R^n \setminus (A \cap V) = \text{id}.$$

Finally, we choose $\delta > 0$ such that $y \in gA_1$ and $|g^{-1}(x) - g^{-1}(y)| \leq \delta_1$ whenever $g \in \mathcal{F}$, $x \in gA$, $y \in R^n$ and $|x - y| \leq 3\delta$. We show that the lemma is true with this δ .

Since $\text{cl } \mathcal{F}$ is compact and since its members are LIP approximable, there is a finite family \mathcal{F}' of LIP embeddings $U \rightarrow R^n$ such that for every $g \in \text{cl } \mathcal{F}$ we can choose $g' \in \mathcal{F}'$ with $d(g', g; A_1) \leq \min(\varepsilon/2, \delta/2)$. The family \mathcal{F}' depends only on τ . Hence there are K_0 and L_0 , depending only on τ , such that for every $g' \in \mathcal{F}'$, $g'|A_1$ is K_0 -QC and L_0 -bilipschitz.

Let $h: V \rightarrow R^n$ and $g \in \mathcal{F}$ be as in the lemma. Since $d(g', g; A_1) \leq \delta/2$ and $B(gA, 3\delta) \subset gA_1$, we have $B(gA, 5\delta/2) \subset g'A_1$, which implies $B(g'A, 2\delta) \subset g'A_1$. Since $d(g', h; A \cap V) < 2\delta$, we can define an embedding $f: A \cap V \rightarrow R^n$ by $f = g'^{-1}h|A \cap V$. Then $f[A \cap V] \subset A_1$. Moreover, $d(f, \text{id}; A \cap V) \leq \delta_1$. To show this, let $x \in A \cap V$. Then $|g(f(x)) - g(x)| \leq \delta + |g'(f(x)) - g'(x)| = \delta + |h(x) - g'(x)| \leq 3\delta$. By the choice of δ , this yields $|f(x) - x| \leq \delta_1$.

If (a) h is K -QC, f is KK_0 -QC. If (b) h is locally L -bilipschitz, f is locally LL_0 -bilipschitz. By the choice of δ_1 , there is a LIP homeomorphism $f': R^n \rightarrow R^n$ satisfying (3.10). Moreover, by Theorem 3.6, f' is K_1 -QC in case (a) and L_1 -bilipschitz in case (b), with K_1 depending on τ and K , L_1 on τ and L . Then $h' = g'(f'|U): U \rightarrow R^n$ is the desired embedding. In case (a) it is K_0K_1 -QC and in case (b) L_0L_1 -bilipschitz in U' . In $W \cap U'$ we have $h' = g'f' = g'f = h$, and thus (2) is true. Furthermore,

$$d(h', g; U) \leq d(g'f', gf'; A) + d(gf', g; A).$$

Since $f'A = A$, $d(g'f', gf'; A) = d(g, g'; A) \leq \varepsilon/2$. Since $d(f', \text{id}; A) \leq \varepsilon_1$, the choice of ε_1 implies that $d(gf', g; A) \leq \varepsilon/2$, and we obtain (1). \square

3.11. *Stable homeomorphisms and the annulus conjecture.* Let C be one of the categories TOP, LQC, LIP. A C -homeomorphism $f: R^n \rightarrow R^n$ is said to be C -stable

if f can be written as a composite map $f_1 \dots f_k$ where each $f_j: R^n \rightarrow R^n$ is a C -homeomorphism such that $f_j|_{U_j} = \text{id}$ for some open $U_j \neq \emptyset$ [BG]. The *stable homeomorphism conjecture* for C states that every sense-preserving C -homeomorphism $f: R^n \rightarrow R^n$ is C -stable. Sullivan [Su] proved this for $C = \text{LQC}$ and LIP in all dimensions; we give this in 3.12 below. For the case $n \leq 3$, see also [Vä₃] and [Kk, 4.5 and 4.6]. For $C = \text{TOP}$, the conjecture is known to be true for $n \neq 4$ by Kirby and Siebenmann; the case $n = 4$ is unsolved.

To formulate the annulus conjecture we need some terminology. A set $S \subset R^n$ is an $(n-1)$ -dimensional C -sphere if there is a C -homeomorphism $f: S \rightarrow S^{n-1}$. For $C = \text{LQC}$ this means that f is quasisymmetric in the sense of [TV₁]. We say that S is *locally C -flat* if every point $x \in S$ has a neighborhood $U \subset R^n$ such that there is a C -homeomorphism $h: (U, U \cap S) \rightarrow (B^n, B^{n-1})$. By a basic flattening theorem, every locally C -flat $(n-1)$ -dimensional C -sphere $S \subset R^n$ is C -flat, that is, there is a C -homeomorphism $f: (R^n, S) \rightarrow (R^n, S^{n-1})$. For $C = \text{TOP}$ this is a well-known result of M. Brown. For $C = \text{LIP}$ it is given in [LuV, 7.8]. The case $C = \text{LQC}$ follows from Theorem 4.10 and from the Schoenflies theorem.

The *annulus conjecture* for C states that if S_1 and S_2 are locally C -flat $(n-1)$ -dimensional C -spheres in R^n and if S_1 lies in the bounded component of $R^n \setminus S_2$, then the closure of the domain bounded by $S_1 \cup S_2$ is C -homeomorphic to the standard annulus $\bar{B}^n \setminus B^n(1/2)$. The stable homeomorphism conjecture implies the annulus conjecture; see [BG] or [Kr].

3.12. Theorem. *The stable homeomorphism conjecture and the annulus conjecture are true in the categories LQC and LIP in all dimensions.*

Proof. Let C be either LQC or LIP , and let \mathcal{F} be the set of all C -stable C -homeomorphisms $f: R^n \rightarrow R^n$. It suffices to show that \mathcal{F} is the group $H_C^+(R^n)$ of all sense-preserving C -homeomorphisms $f: R^n \rightarrow R^n$. The case $k=0$ of Lemma 3.3 implies that id is an interior point of \mathcal{F} in $H_C^+(R^n)$. Since \mathcal{F} is a subgroup, it is both open and closed in $H_C^+(R^n)$. Hence it suffices to show that $H_C^+(R^n)$ is connected. We show that every $f \in H_C^+(R^n)$ can be joined by a path to id . There is $x_0 \in R^n$ at which f is differentiable with a positive jacobian. We may assume that $x_0 = 0 = f(x_0)$. Set $g_t(x) = f(tx)/t$ for $t > 0$ and $g_0 = f'(0)$. Then the path $t \mapsto g_t$ joins the linear map $f'(0)$ to f . Since $\det f'(0) > 0$, it follows from elementary linear algebra that $f'(0)$ and id can be joined by a path. \square

3.13. Conventions. In the rest of this section, C will denote either LQC or LIP . We say that a homeomorphism $f: S^{n-1} \rightarrow S^{n-1}$ is LQC if it is quasisymmetric in the sense of [TV₁] and [Vä₄].

3.14. Theorem. *Let $f: S^{n-1} \rightarrow S^{n-1}$ be a sense-preserving C -homeomorphism and let $0 < a < 1$. Then f can be extended to a C -homeomorphism $g: \bar{B}^n \rightarrow \bar{B}^n$ such that $g|_{B^n(a)} = \text{id}$.*

Proof. It follows easily from 3.12 that f can be written as the composition $f_1 \dots f_k$ of C -homeomorphisms $f_j: S^{n-1} \rightarrow S^{n-1}$ such that each f_j is the identity in some nonempty open subset of S^{n-1} . Hence we may assume that $f|V = \text{id}$ for some open nonempty $V \subset S^{n-1}$. It suffices to find an extension g such that $g|U = \text{id}$ for some open $U \subset B^n$, $U \neq \emptyset$. Indeed, we can then choose a C -homeomorphism $h: \bar{B}^n \rightarrow \bar{B}^n$ such that $B^n(a) \subset hU$ and $h|S^{n-1} = \text{id}$, and the desired map is hgh^{-1} .

If $C = \text{LIP}$, we let g be the cone of f . Suppose that $C = \text{LQC}$. By [TV₂, 3.15.4], we can choose a QC extension $g_1: \bar{B}^n \rightarrow \bar{B}^n$ of f . Choose a QC homeomorphism $\varphi: R^n \rightarrow R^n$ such that $\varphi|S^{n-1} \setminus V = \text{id}$, $B^n \subset \varphi B^n$ and $|\varphi(0)| > 1$. Setting

$$\begin{aligned} g(x) &= \varphi^{-1} g_1 \varphi(x) & \text{if } |\varphi(x)| \leq 1, \\ g(x) &= x & \text{if } |\varphi(x)| > 1, \end{aligned}$$

we obtain the required map $g: \bar{B}^n \rightarrow \bar{B}^n$. \square

3.15. *Notation.* For $0 < a < b$ we let $A(a, b)$ denote the closed annulus $\bar{B}^n(b) \setminus B^n(a)$.

3.16. **Theorem.** *Let $0 < a < b$, let $0 < r < b$, and let $f: A(a, b) \rightarrow R^n$ be a sense-preserving C -embedding such that the bounded component of $R^n \setminus fS^{n-1}(b)$ contains $\bar{B}^n(r) \cup fS^{n-1}(a)$. Then there is a C -embedding $g: \bar{B}^n(b) \rightarrow R^n$ such that $g = f$ near $\partial B^n(b)$ and $g| \bar{B}^n(r) = \text{id}$.*

Proof. Choose r_1 and a_1 such that $r < r_1 < a_1 < b$, $a < a_1$ and $\bar{B}^n(r_1) \cap fA(a_1, b) = \emptyset$. Let R be the domain bounded by $S^{n-1}(r_1) \cup fS^{n-1}(a_1)$. By the annulus theorem 3.12, there is a sense-preserving C -homeomorphism $h: A(r_1, a_1) \rightarrow \bar{R}$ with $hS^{n-1}(r_1) = S^{n-1}(r_1)$. By Theorem 3.14, $h^{-1}f|S^{n-1}(a_1)$ can be extended to a C -homeomorphism $\varphi: A(r_1, a_1) \rightarrow A(r_1, a_1)$ such that $\varphi|S^{n-1}(r_1) = \text{id}$. Similarly, $h|S^{n-1}(r_1)$ can be extended to a C -homeomorphism $h_1: \bar{B}^n(r_1) \rightarrow \bar{B}^n(r_1)$ with $h_1| \bar{B}^n(r) = \text{id}$. The desired $g: \bar{B}^n(b) \rightarrow R^n$ is obtained by setting

$$\begin{aligned} g(x) &= f(x) & \text{if } a_1 \leq |x| \leq b, \\ g(x) &= h(\varphi(x)) & \text{if } r_1 \leq |x| \leq a_1, \\ g(x) &= h_1(x) & \text{if } |x| \leq r_1. \quad \square \end{aligned}$$

3.17. **Theorem.** *Let $0 < a < b < c < d$ and let $f: A(a, b) \cup A(c, d) \rightarrow R^n$ be a sense-preserving C -embedding. Suppose also that $fA(a, b) \cup fS^{n-1}(c)$ lies in the bounded component of $R^n \setminus fS^{n-1}(d)$ and that $fS^{n-1}(a)$ lies in the bounded component of $R^n \setminus fS^{n-1}(b)$. Then there is a C -embedding $g: A(a, d) \rightarrow R^n$ such that $g = f$ near $\partial A(a, d)$.*

Proof. Replacing a by a slightly larger number and d by a slightly smaller number, we may assume that $fS^{n-1}(a)$ and $fS^{n-1}(d)$ are locally C -flat. By the annulus theorem 3.12, we may assume that $|f(x)| = |x|$ for all $x \in \partial A(a, d)$.

By Theorem 3.16, there is a C -homeomorphism $g_1: \bar{B}^n(d) \rightarrow \bar{B}^n(d)$ such that $g_1=f$ near $\partial B^n(d)$ and $g_1=\text{id}$ near $\partial B^n(a)$. Together with an auxiliary inversion, Theorem 3.16 also implies that there is a C -homeomorphism $g_2: R^n \setminus B^n(a) \rightarrow R^n \setminus B^n(a)$ such that $g_2=f$ near $\partial B^n(a)$ and $g_2=\text{id}$ near $R^n \setminus B^n(d)$. Then $g=g_2g_1|A(a, d)$ is the desired embedding. \square

4. LQC and LIP approximation of homeomorphisms

4.1. In this section we show that if $C=\text{LIP}$ or LQC , and if M and N are C -manifolds without boundary of dimension $n \neq 4$, then every homeomorphism $f: M \rightarrow N$ can be approximated by C -homeomorphisms. We next prove the same result for manifolds with boundary provided that $n \neq 4, 5$. In particular, we obtain the LIP and LQC hauptvermutung for these manifolds. For manifolds without boundary, these results are due to Sullivan [Su]; the LIP case for manifolds with boundary has been proved by J. Luukkainen (unpublished). The key of the proof is the following result whose PL analogue is true only for $k \neq 3$ (by Kirby and Siebenmann).

4.2. **Lemma.** (*C-straightening of TOP handles.*) *Let $m \geq 0$ and $k \geq 0$ be integers with $m+k=n \geq 2$, $n \neq 4$, let C be either LQC or LIP, let U be an open neighborhood of I^n , and let $f \in E(U; R^n)$ be a C -embedding near $I^m \times \partial I^k$. Let $\varepsilon > 0$. Then there is an isotopy $\varphi: I \rightarrow E(U; R^n)$ such that:*

- (1) $\varphi_0=f$.
- (2) $\varphi_t=f$ near $U \setminus \text{int } I^n$ for all $t \in I$.
- (3) $|\varphi_t(x)-f(x)| \leq \varepsilon$ for all $t \in I$ and $x \in U$.
- (4) φ_1 is a C -embedding near $I^m \times \partial I^k \cup \bar{I}^m(1/2) \times I^k$.

Proof. By the Schoenflies theorem, we may assume that $U=R^n$ and that f is a homeomorphism onto R^n .

Choose $\delta \in (0, 1)$ such that $|f(x)-f(y)| \leq \varepsilon/2$ whenever $x, y \in I^n(2)$ and $|x-y| \leq \delta$. Then $\mathcal{P}_1 = \{g \in H(R^n): d(g, \text{id}; I^n) < \delta\}$ is a neighborhood of id in $H(R^n)$. Choose $r > 0$ such that f is a C -embedding in $A' = B^n(I^m \times \partial I^k, 2r)$. Apply Theorem 3.4 with $G=R^n$, $A = \bar{B}^n(I^m \times \partial I^k, r)$, B a compact neighborhood of $\bar{I}^m(1/2) \times \bar{I}^k(1-r)$ in $\text{int } I^n$, and B' a compact neighborhood of B in $\text{int } I^n$. We obtain a neighborhood \mathcal{P} of id in $H(R^n)$ and a map $\psi: \mathcal{P} \times I \rightarrow H(R^n)$. Since $\psi(\text{id}, t) = \text{id}$, we may assume that $\text{im } \psi \subset \mathcal{P}_1$, replacing \mathcal{P} by a smaller neighborhood.

Since $n \neq 4$, there is a PL homeomorphism $g \in H(R^n)$ such that $d(g, f; R^n) < \varepsilon/2$ and $g^{-1}f = h \in \mathcal{P}$. If $k=0$, we need here the deep stable homeomorphism theorem of Kirby—Siebenmann, unless f is known to be stable. Set $h_t = \psi(h, t)$. We show that $\varphi_t = gh_t$ is the desired isotopy. First $\varphi_0 = gh_0 = gh = f$. Outside B' we have $\varphi_t = gh = f$. If $x \in B'$, then $|h_t(x) - x| < \delta$, and we obtain

$$|\varphi_t(x) - f(x)| \leq |g(h_t(x)) - f(h_t(x))| + |f(h_t(x)) - f(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

In B , $\varphi_1 = gh_1 = g$ is PL, hence a C -embedding. In A , h_t is a C -embedding for every t . In particular, $\varphi_1|_A = gh_1|_A$ is a C -embedding. Consequently, φ_1 is a C -embedding in $\text{int } A \cup \text{int } B$, which is a neighborhood of $I^m \times \partial I^k \cup \bar{I}^m(1/2) \times I^k$. \square

4.3. *Notation.* If M and N are n -manifolds without boundary, we let $E(M; N)$ denote the set of all embeddings $f: M \rightarrow N$.

4.4. **Theorem.** *Let C be either LQC or LIP, let M and N be n -dimensional C -manifolds without boundary, $n \neq 4$, let $A, B \subset M$ be closed, and let $B' \subset M$ be a neighborhood of B . Let $f: M \rightarrow N$ be an embedding such that f is a C -embedding near A , and let $\varepsilon: M \rightarrow (0, \infty)$ be continuous. Then there is a continuous map $\varphi: I \rightarrow E(M; N)$ such that:*

- (1) $\varphi_0 = f$.
- (2) $\varphi_t = f$ in $M \setminus B'$ and near A .
- (3) φ_1 is a C -embedding near $A \cup B$.
- (4) $d(\varphi_t(x), f(x)) < \varepsilon(x)$ for all $t \in I$ and $x \in M$, where d is a given metric of N .

Proof. 1. *Special case.* M is open in R^n , $N = R^n$, and B is compact. Choose an open neighborhood A' of A such that $f|_{A'}$ is a C -embedding. Choose a compact PL manifold $P \subset \text{int } B'$ such that $B \subset \text{int } P$. Choose a fine handle decomposition \mathcal{H} of P as in [RS, p. 82]. Let $i(H)$ denote the index of a handle $H \in \mathcal{H}$. Define $H \leq H'$ if $H \cap H' \neq \emptyset$ and $i(H) \leq i(H')$. Then $H \leq H' \leq H''$ implies $H \leq H''$. Let \mathcal{H}_0 be the set of the handles $H \in \mathcal{H}$ such that there is $H' \in \mathcal{H}$ with $H \leq H'$ and $H' \cap A \neq \emptyset$. We may assume that $P_0 = \cup \mathcal{H}_0 \subset A'$. Let H_1, \dots, H_s be the handles of $\mathcal{H} \setminus \mathcal{H}_0$ arranged in order of increasing index. For each j we choose a PL embedding $\mu_j: R^n \rightarrow B' \setminus A$ such that for $i(H_j) = k$, $m = n - k$ and $H_j^* = \mu_j I^m$ we have

$$\mu_j[\bar{I}^m(1/2) \times I^k] = H_j,$$

$$H_j^* \cap (P_0 \cup H_1 \cup \dots \cup H_{j-1}) = \mu_j[I^m \times \partial I^k].$$

We may assume that $H_j^* \cap H_k^* = \emptyset$ whenever $i(H_j) = i(H_k)$ and $j \neq k$.

The isotopy φ is defined in s steps as follows: Set $f_0 = f$, and suppose that the $(j-1)$ th step has produced an embedding $f_{j-1}: M \rightarrow N$. We apply Lemma 4.2 with the substitution $f \rightarrow f_{j-1}\mu_j$ and with a small ε . This yields an isotopy $\varphi^j: I \rightarrow E(M; N)$ by $\varphi_t^j = \varphi_t' \mu_j^{-1}$ in H_j^* and by φ_t^j elsewhere. Combining the isotopies $\varphi^1, \dots, \varphi^s$ gives the desired φ .

2. *General case.* Let A' be as in the special case. Choose a closed $B_0 \subset B \setminus A$ such that $\text{cl}(B \setminus B_0) \subset A'$. Choose a locally finite covering \mathcal{B} of B_0 by open subsets of M such that for every $W \in \mathcal{B}$, W and fW can be C -embedded in R^n and \bar{W} is compact in $B' \setminus A$. By [Mu, 2.7] we may assume that \mathcal{B} can be expressed as $\mathcal{B}_0 \cup \dots \cup \mathcal{B}_n$, where each collection \mathcal{B}_j is disjoint. For every $W \in \mathcal{B}$ we choose a smaller open set W' such that $\bar{W}' \subset W$ and such that the sets W' cover B_0 . The isotopy

φ is constructed in $n+1$ steps so that the j^{th} step is active only in $\cup \mathcal{B}_j$. We give the first two steps. Set $E = \text{cl}(B \setminus B_0)$.

Let $W \in \mathcal{B}_0$. We apply the special case with the substitution $f \mapsto f|W$, $A \mapsto E \cap W$, $B \mapsto \bar{W}'$, $B' \mapsto$ a compact neighborhood of \bar{W}' in W , and with a small ε . This yields an isotopy of $f|W$ for every $W \in \mathcal{B}_0$. These isotopies can be extended by $f|M \setminus (\cup \mathcal{B}_0)$ to an isotopy of f , which produces an embedding $f_1: M \rightarrow N$.

In the second step, we take $W \in \mathcal{B}_1$ and apply the special case with $f \mapsto f_1|W$, $A \mapsto (E \cup (\cup \{\bar{W}' : W \in \mathcal{B}_0\})) \cap W$, $B \mapsto \bar{W}'$, and B' and ε as above. \square

4.5. Corollary. *Let M and N be n -dimensional C -manifolds without boundary, $n \neq 4$, and let $\varepsilon: M \rightarrow (0, \infty)$ be continuous. Then for every homeomorphism $f: M \rightarrow N$ there is a C -homeomorphism $g: M \rightarrow N$ such that $d(f(x), g(x)) < \varepsilon(x)$ for all $x \in M$. In particular, the *hauptvermutung* is true for these manifolds in the categories LQC and LIP. \square*

4.6. Theorem. *Every topological n -manifold without boundary, $n \neq 4$, has a LIP manifold structure (and hence an LQC manifold structure).*

Proof. The theorem follows in a well-known manner from Theorem 4.4. See, for example, [Kr, Theorem 10]. \square

4.7. Lemma. *Let C be either LQC or LIP. Let A be closed in B^{n-1} , let $B \subset B^{n-1}$ be compact, and let $f: B_+^n \rightarrow R_+^n$ be a clean embedding such that $f|B^{n-1}$ is a C -embedding and such that f is a C -embedding near A in B_+^n . Then, for every $\varepsilon > 0$ there is an isotopy $\varphi: I \rightarrow E(B_+^n; R_+^n)$ such that:*

- (1) $\varphi_0 = f$.
- (2) $\varphi_t|B^{n-1} = f|B^{n-1}$ for every $t \in I$.
- (3) For some $b \in (0, 1)$, $\varphi_t = f$ in $B_+^n \setminus B_+^n(b)$ for every $t \in I$.
- (4) φ_1 is a C -embedding near $A \cup B$.
- (5) $|\varphi_t(x) - f(x)| < \varepsilon$ for all $x \in B_+^n$ and $t \in I$.

Proof. Choose numbers $0 < b_1 < b_2 < b < 1$ such that $B \subset B^{n-1}(b_1)$. By the Schoenflies theorem, we can extend $f|B^{n-1}(b)$ to a C -homeomorphism $g: R^{n-1} \rightarrow R^{n-1}$ such that g is QC if $C = \text{LQC}$. For $n = 2$ this follows from [LeV₂, II, Lemma 7.2], Let $F: R_+^n \rightarrow R_+^n$ be a C -homeomorphism with $F|R^{n-1} = g$. For $C = \text{LIP}$, we may choose $F = g \times \text{id}$; for $C = \text{LQC}$ we apply [TV₂, 3.12]. Then $W = F^{-1}fB_+^n(b)$ is a neighborhood of $\bar{B}^{n-1}(b_2)$ in R_+^n . Choose $r > 0$ and set $a_j = (r^2 + b_j^2)^{1/2}$ and $V_j = B^n(-re_n, a_j) \cap R_+^n$ for $j = 1, 2$. Then $V_2 \subset W$ if r is large enough. Let $\alpha: I \rightarrow E(R_+^n \setminus \bar{B}^{n-1}(b_1); R_+^n \setminus \bar{B}^{n-1}(b_1))$ be the obvious vertical isotopy through LIP embeddings such that $\alpha_0 = \text{id}$, $\alpha_t|R_+^n \setminus V_2 = \text{id}$, and $\text{im } \alpha_1 = R_+^n \setminus \bar{V}_1$. Then it is easy to verify that the lemma is true for a sufficiently large r and with φ as follows:

$$\varphi_t = F\alpha_t F^{-1}f\alpha_t^{-1} \text{ in } B_+^n \cap (\text{im } \alpha_t)$$

$$\varphi_t = F \text{ in } B_+^n \setminus \text{im } \alpha_t. \quad \square$$

4.8. **Theorem.** *Theorem 4.4, Corollary 4.5 and Theorem 4.6 are true for n -manifolds with boundary, $n \neq 4, 5$. In Theorem 4.4, $E(M; N)$ denotes the set of all embeddings $g: M \rightarrow N$ such that $g^{-1}\partial N = \partial M$, and we assume that $f \in E(M; N)$. If $f|_{\partial M}$ is a C -embedding, Theorem 4.4 is also true for $n = 5$, and we can choose φ so that $\varphi_t|_{\partial M} = f|_{\partial M}$ for all $t \in I$.*

Proof. We prove the special case of 4.4; the rest of the proof is essentially unchanged. We assume that in 4.4 M is open in R^n_+ , that $N = R^n_+$ and that B is compact. We may assume that B' is compact. Choose an open neighborhood A' of A such that $f|_{A'}$ is a C -embedding. Applying 4.4 to $f_0 = f|_{M \cap R^{n-1}}$ (for $n = 2$ see Remark 4.9) we find a small isotopy $\varphi': I \rightarrow E(M \cap R^{n-1}; R^{n-1})$ of f_0 which is active only in a compact neighborhood D of $(B \cap R^{n-1}) \setminus A'$ in $(\text{int } B' \cap R^{n-1}) \setminus A$ and which produces an embedding $\varphi'_1: M \cap R^{n-1} \rightarrow R^{n-1}$ which is a C -embedding near $(A \cup B) \cap R^{n-1}$. Using a spindle neighborhood of D in M , φ' can be extended in a well-known manner to a small isotopy $\varphi: I \rightarrow E(M; N)$ of f such that $\varphi_t = f$ in $M \setminus B'$ and near A for all $t \in I$. If $f|_{\partial M}$ is a C -embedding, we let φ be the constant isotopy.

Choose a covering of $(B \cap R^{n-1}) \setminus A'$ by half balls U_1, \dots, U_s such that $\bar{U}_j \subset B \setminus A$ and such that $\varphi_1|_{U_j \cap R^{n-1}}$ is a C -embedding. Applying Lemma 4.7 s times gives a small isotopy $\psi: I \rightarrow E(M; N)$ of φ_1 which is active only in $U_1 \cup \dots \cup U_s$, satisfies $\psi_t|_{\partial M} = \varphi_1|_{\partial M}$ for all $t \in I$, and produces a map ψ_1 which is a C -embedding near $A \cup (B \cap R^{n-1})$ in M .

Choose compact B_1, B_2 such that $B = B_1 \cup B_2$, $B_1 \subset \text{int } R^n_+$, and ψ_1 is a C -embedding near B_2 . Apply 4.4 with the substitution $M \mapsto \text{int } M$, $N \mapsto \text{int } R^n_+$, $A \mapsto (A \cup B_2) \cap \text{int } M$, $B \mapsto B_1$, $B' \mapsto$ a compact neighborhood of B_1 in $B' \cap \text{int } M$, $f \mapsto \psi_1|_{\text{int } M}$. We obtain an isotopy χ of ψ_1 , and the desired isotopy is the combination $\chi * \psi * \varphi$. \square

4.9. *Remark.* Suppose that $n = 2$. Then in the proof of 4.8 we need the special case of Theorem 4.4 for $n = 1$. For $C = \text{LIP}$ this is very easy. We prove the case $C = \text{LQC}$. We may assume that $M \subset R^1$ is an open interval, that B is a compact subinterval, that $N = R^1$ and that f is increasing. Replacing A by a closed neighborhood, we may assume that A is the union of a locally finite family of closed intervals. Replacing B by a finite family of disjoint subintervals and treating each of these separately, we may assume that $B = [0, 1]$, $B' = [-1, 2]$, and that $B' \cap A$ is either empty or $[-1, r]$ or $[-1, r] \cup [1 - r, 2]$ for some $r \in (0, 1/2)$. It suffices to find an embedding $f_1: M \rightarrow R^1$ such that (1) $f_1 = f$ in $M \setminus B'$ and near A , (2) f_1 is LQC near $A \cup B$ and (3) $d(f_1, f; B') < \varepsilon$. Indeed, we can then define $\varphi_t = (1 - t)f + tf_1$.

Suppose, for example, that $B' \cap A = [-1, r]$. Choose $s \in (r, 1/2)$ such that f is QC near $[0, s]$. There is a QC extension $g: R^1 \rightarrow R^1$ of $f|_{[0, s]}$ such that g is C^1 outside $[0, s]$; see [LeV₁, p. 12]. Choose $s_1 \in (s, 1/2)$ such that $g(s_1) < f(3/2)$. Define $f_1: M \rightarrow R^1$ as follows: $f_1(x) = f(x)$ if $x \leq s$ or $x \geq 3/2$, $f_1(x) = g(x)$ if

$s \leq x \leq s_1$, and f_1 is a suitable PL function on $[s_1, 3/2]$. Then f_1 satisfies (1) and (2), and (3) can easily be arranged.

4.10. Theorem. *Let M and N be LQC n -manifolds and let $f: \partial M \rightarrow \partial N$ be an LQC embedding. Then f can be extended to an LQC embedding of a neighborhood of ∂M .*

Proof. Since ∂M and ∂N have TOP collars, f can be extended to an embedding of a neighborhood of ∂M . If $n \neq 4$, the theorem follows directly from the last statement of Theorem 4.8. However, since the last isotopy χ of the proof of Theorem 4.8 is not needed in the present case, the theorem is also true for $n=4$. \square

4.11. Remark. The LIP version of 4.10 is clear, since the boundary of a LIP manifold has a LIP collar [LuV, 7.5]. The concept "LQC collar" seems to be meaningless, since there is no natural way to form products in the category LQC, cf. [Tu₂, Example 4]. Theorem 4.10 acts as a substitute for collaring in the LQC manifold theory.

5. Extension

5.1. In this section we consider the following problem: Let G be open in \bar{R}^n , and let $U \subset G$ be an open set such that $U \cup \partial G$ is a neighborhood of ∂G in \bar{G} . Suppose that $f: G \rightarrow \bar{R}^n$ is an embedding such that $f|U$ is K -QC. Does there exist an embedding $g: G \rightarrow \bar{R}^n$ such that $g=f$ near ∂G and such that g is K' -QC with some K' depending only on G, U and K ? The following example shows that without any restrictions the answer is negative: Let $G = B^n(4) \setminus \bar{B}^n$, $U = (B^n(2) \setminus \bar{B}^n) \cup (B^n(4) \setminus \bar{B}^n(3))$. For $0 < a < 1$ we let $f: G \rightarrow R^n$ be a radial embedding such that $f(x) = ax$ for $1 < |x| < 2$ and $f(x) = x$ for $3 < |x| < 4$. Then $f|U$ is 1-QC, but the dilatation of g must converge to ∞ as $a \rightarrow 0$.

We shall show that the answer is positive if U is connected. Moreover, for $n=4$ we must also assume that f is QC. In Theorem 5.7 we shall show that the connectedness condition can be replaced by a condition concerning the size of the boundary components of fG . As special cases, we obtain quantitative versions of the Schoenflies theorem and the annulus theorem.

Corresponding results are also true in the category LIP.

5.2. Lemma. *Let $G \subset \bar{R}^n$ be open, let $f: G \rightarrow \bar{R}^n$ be an embedding and let $g: G \rightarrow \bar{R}^n$ be an immersion such that $f=g$ near ∂G . Then $gG=fG$, and g is an embedding.*

Proof. We may assume that G is connected and that f and g are defined on \bar{G} . Then the topological degree $\mu(y, f, G)$ is ± 1 for $y \in fG$ and $\mu(y, f, G) = 0$ for $y \in \bar{R}^n \setminus f\bar{G}$. Moreover, $\mu(y, f, G) = \mu(y, g, G)$ for all $y \in \bar{R}^n \setminus f\partial G$ and $\mu(y, g, G) = \pm \text{card } g^{-1}(y)$. See, for example, [RR, § II.2]. The lemma follows. \square

In the following theorem, all closures and boundaries are taken in \bar{R}^n :

5.3. Theorem. *Let G be a domain in \bar{R}^n , let U and V be open in G such that $U \cup \partial G$ and $V \cup \partial G$ are neighborhoods of ∂G in \bar{G} , $\bar{V} \cap G \subset U$, and U is connected. Let $f: G \rightarrow \bar{R}^n$ be an embedding such that $f|U$ is K -QC and such that either $n \neq 4$ or f is QC. Then there is a K' -QC embedding $g: G \rightarrow \bar{R}^n$ such that $g|V = f|V$ and K' depends only on G, U, V and K .*

Proof. Choose a connected open set $U_1 \subset G$ such that $\bar{V} \cap G \subset U_1$ and $\bar{U}_1 \cap G \subset U$. Then $C = \partial U_1 \cap G$ is a compact subset of $U \setminus \bar{V}$. Choose open sets G_1, G_2, G_3, G_4 such that $\bar{G}_{i+1} \subset G_i, C \subset G_4$, and $\bar{G}_1 \subset U \setminus \bar{V}$. We apply Theorem 3.4 with the substitution $G \mapsto G_2, A \mapsto G_2, A' \mapsto G_2, B \mapsto \bar{G}_4, B' \mapsto G_3, K \mapsto K^2$. We obtain a neighborhood \mathcal{P} of id in $E(G_2; \bar{R}^n)$, a continuous map $\psi: \mathcal{P} \times I \rightarrow E(G_2; \bar{R}^n)$ and a number $K_1 \cong K$, depending only on K and n . Letting q denote the spherical metric of \bar{R}^n we choose $\varepsilon_1 > 0$ such that the set

$$N_q(\text{id}, G_2, \varepsilon_1) = \{f \in E(G_2; \bar{R}^n): q(f(x), x) < \varepsilon_1 \text{ for all } x \in G_2\}$$

is contained in \mathcal{P} . Set $\varepsilon = \min(\varepsilon_1, q(G_2, \partial G_1))$, and fix three distinct points a_1, a_2, a_3 in U . Let \mathcal{F} be the family of all K -QC embeddings $u: U \rightarrow \bar{R}^n$ such that (1) $u(a_j) = a_j$ for $j = 1, 2, 3$, and (2) u has an extension to a QC embedding $v: G \rightarrow \bar{R}^n$. Then \mathcal{F} depends only on G, U and K . Since U is connected, \mathcal{F} is solid (3.8 and [Vä₁, 19.5 and 21.1]). Hence there is $\varepsilon' > 0$ such that $q(u(x), u(y)) \cong \varepsilon'$ whenever $x, y \in \bar{G}_1, q(x, y) \cong \varepsilon$ and $u \in \mathcal{F}$. Furthermore, there is a finite family $\{u_1, \dots, u_s\} \subset \mathcal{F}$ such that \mathcal{F} is covered by the sets $N_q(u_j, \bar{G}_1, \varepsilon'), 1 \leq j \leq s$. For each $j \in [1, s]$ we choose a QC extension $v_j: G \rightarrow \bar{R}^n$ of u_j . Choose $K_0 = K_0(G, U, K)$ such that each v_j is K_0 -QC.

Let $f: G \rightarrow \bar{R}^n$ be an embedding such that $f|U$ is K -QC.

Case 1. f is QC. We show that g can be chosen to be $K_0 K_1$ -QC. By an auxiliary Möbius transformation we may assume that $f(a_j) = a_j$ for $j = 1, 2, 3$. Then $f|U \in \mathcal{F}$. Hence $f \in N_q(u_j, \bar{G}_1, \varepsilon')$ for some $j \in [1, s]$. Since $q(u_j \bar{G}_2, u_j \partial G_1) \cong \varepsilon', fG_2 \subset u_j G_1$. Setting $h = u_j^{-1} f|G_2$ we thus obtain a K^2 -QC embedding $h: G_2 \rightarrow G_1$, which belongs to $N_q(\text{id}, G_2, \varepsilon)$ and hence to \mathcal{P} . Then the K_1 -QC embedding $h_1 = \psi(h, 1): G_2 \rightarrow \bar{R}^n$ satisfies the conditions $h_1|G_4 = \text{id}$ and $h_1|G_2 \setminus G_3 = h|G_2 \setminus G_3$. Define $g: G \rightarrow \bar{R}^n$ as follows:

$$g(x) = f(x) \quad \text{for } x \in U_1 \setminus \bar{G}_3,$$

$$g(x) = u_j(h_1(x)) \quad \text{for } x \in (G_2 \cap U_1) \cup G_4,$$

$$g(x) = v_j(x) \quad \text{for } x \in G \setminus \bar{U}_1.$$

Then g is a well-defined immersion. Since $g = f$ in V, g is an embedding by Lemma 5.2. Since g is $K_0 K_1$ -QC, Case 1 is proved.

Case 2. $n \neq 4$. Applying Theorem 4.4 with the substitution $M \mapsto G, A \mapsto \bar{U}_1 \cap G, B \mapsto \text{cl}(G \setminus U_1), B' \mapsto G \setminus V, f \mapsto f$, we obtain a QC embedding $\phi_1: G \rightarrow \bar{R}^n$ such

that $\varphi_1|_{U_1}=f|_{U_1}$ is K -QC. Hence this case reduces to Case 1 with U replaced by U_1 . \square

5.4. Quantitative QC Schoenflies theorem. *Let $0 < r < 1$, and let $f: \bar{B}^n \setminus B^n(r) \rightarrow \bar{R}^n$ be a K -QC embedding. Then there is a K' -QC embedding $g: \bar{B}^n \rightarrow \bar{R}^n$ such that $g|_{\partial B^n} = f|_{\partial B^n}$ and K' depends only on K , n and r .*

Proof. Choose $s \in (r, 1)$. By the QC Schoenflies theorem, there is a QC embedding $f_1: \bar{B}^n \rightarrow \bar{R}^n$ such that $f_1(x) = f(x)$ for $s \leq |x| \leq 1$. The theorem follows from 5.3. \square

5.5. Remark. Theorem 5.4 was proved by M. Näätänen [Nä] in the special case where $f\partial B^n$ is a sphere. A shorter proof for this case was given by D. B. Gauld and M. K. Vamanamurthy [GVm₂]. If $n \geq 3$, it follows from the extension theorem of [TV₂] that K' can in this case be chosen to be independent of r [TV₂, 3.14]. \square

5.6. Nonconnected U . Suppose that U in Theorem 5.3 is not connected. Let \mathcal{F}_0 be the family of all K -QC embeddings $u: U \rightarrow \bar{R}^n$ which have a QC extension $v: G \rightarrow \bar{R}^n$. In the proof of 5.3 we used the normalization $u(a_j) = a_j$, which gave a solid subfamily $\mathcal{F} \subset \mathcal{F}_0$. This is not possible if U is not connected. However, there are other restrictions which give rise to solid families. As an example we prove the following result:

5.7. Theorem. *Let G, U, V be as in Theorem 5.3 except that U is not necessarily connected. Let $\alpha > 0$ and let $f: G \rightarrow \bar{R}^n$ be an embedding such that $f|_U$ is K -QC and such that either $n \neq 4$ or f is QC. Suppose that for each component U_1 of U , the spherical diameter of $(\text{cl } fU_1) \setminus fG$ is at least α . Then there is a K' -QC embedding $g: G \rightarrow \bar{R}^n$ such that $g|_V = f|_V$ and K' depends only on G, U, V, K and α .*

Proof. By Theorem 5.3 we may assume that U is not connected. Let \mathcal{F} be the family of all K -QC embeddings $u: U \rightarrow \bar{R}^n$ such that u has a QC extension $v: G \rightarrow \bar{R}^n$ and such that for each component U_1 of U , $q((\text{cl } uU_1) \setminus uG) \geq \alpha$ where q is the spherical metric of \bar{R}^n . It suffices to show that \mathcal{F} is solid, since the proof of 5.3 will then be valid also in the present case.

From [Vä₁, 19.2] it follows that \mathcal{F} is equicontinuous, and hence $\text{cl } \mathcal{F}$ is compact in $C(U, \bar{R}^n)$. Suppose that (f_j) is a sequence in \mathcal{F} converging to a map $f: U \rightarrow \bar{R}^n$ uniformly on compact sets. Let U_1 be a component of U . Then $f|_{U_1}$ is either constant or an embedding [Vä₁, 21.3]. Assume that $f|_{U_1}$ is constant. Choose a compact set $A \subset U_1$ such that no component of $\bar{R}^n \setminus A$ meets both $\partial U_1 \cap G$ and $\partial U_1 \setminus G$. Then for every j , no component of $\bar{R}^n \setminus f_j A$ meets both $\partial f_j U_1 \cap f_j G$ and $\partial f_j U_1 \setminus f_j G$. Hence there are disjoint sets W_1 and W_2 which are unions of components of $\bar{R}^n \setminus f_j A$ such that $q(W_1) \geq \alpha$ and $q(W_2) \geq \alpha$. This contradicts the fact that the maps $f_j|_A$ converge uniformly to a constant. Hence $f|_{U_1}$ is an embedding for every component U_1 of U . From this it easily follows that f is an embedding. \square

5.8. Quantitative QC annulus theorem. *Suppose that in Theorem 3.17, $f: A(a, b) \cup A(c, d) \rightarrow R^n$ is K -QC and that $d(fS^{n-1}(d))/d(fS^{n-1}(a)) \cong M$. Then $g: A(a, d) \rightarrow R^n$ can be chosen to be K' -QC with K' depending only on a, b, c, d, M, K and n .*

Proof. Using an auxiliary similarity map of R^n we may assume that $0 \in \text{im} f$ and that $d(fS^{n-1}(d)) = 1$. Thus $d(fS^{n-1}(a)) \cong 1/M$. Choose $a_1 \in (a, b)$ and $d_1 \in (c, d)$. By Theorem 3.17, there is a QC embedding $f_1: A(a, d) \rightarrow R^n$ such that $f_1 = f$ in $A(a, a_1) \cup A(d_1, d)$. The theorem now follows directly from Theorem 5.7. \square

5.9. Lipschitz versions. There are several ways to formulate the results of this section in the category LIP. First, one can consider either L -bilipschitz or locally L -bilipschitz embeddings. Second, one can use either the euclidean or the spherical metric. The proofs are fairly easy modifications of the proofs given in the QC case. As an example we give the bilipschitz version of Theorem 5.4:

5.10. Quantitative bilipschitz Schoenflies theorem. *Let $0 < r < 1$ and let $f: \bar{B}^n \setminus B^n(r) \rightarrow R^n$ be an L -bilipschitz embedding such that $fS^{n-1}(r)$ is contained in the bounded component of $R^n \setminus fS^{n-1}$. Then there is an L' -bilipschitz embedding $g: \bar{B}^n \rightarrow R^n$ such that $g|\partial B^n = f|\partial B^n$ and L' depends only on L, n and r . \square*

6. Quasihyperbolic metric and solid homeomorphisms

6.1. This section is a preparation for Section 7. We give some properties of the quasihyperbolic metric of a domain in R^n . This metric has been considered in [GP], [GO] and [Vu]. Our work partially overlaps with [Vu]. We also introduce the solid homeomorphisms which are more general than QC homeomorphisms but which share certain common properties with the latter class of maps.

6.2. Quasihyperbolic metric. In the rest of Section 6, D and D' will always denote proper subdomains of R^n , where $n \cong 2$ is a fixed integer. For $x \in D$ we set

$$\varrho_D(x) = d(x, \partial D)^{-1}.$$

The *quasihyperbolic metric* k_D of D is defined by the element of length $\varrho_D(x)|dx|$. This means that for $a, b \in D$ we have

$$k_D(a, b) = \inf_{\alpha} \int_{\alpha} \varrho_D ds$$

where the infimum is taken over all rectifiable paths α joining a and b in D . If ∂D is an affine subspace T of R^n , k_D is equal to the hyperbolic metric σ_T of D , defined in 2.4.

Gehring and Osgood [GO, Lemma 1] proved that each pair of points a, b in D can be joined by a quasihyperbolic geodesic α , that is,

$$k_D(a, b) = \int_{\alpha} \rho_D ds.$$

If $\emptyset \neq A \subset D$ with \bar{A} compact in D , we define the *relative size* of A in D as the number

$$r_D(A) = \frac{d(A)}{d(A, \partial D)}.$$

Observe that if $r_D(A) < 2$ and if $a, b \in A$, the segment ab lies in D . Moreover, if $r_D(A) < 1$, then A is contained in a ball in D .

We start with the following elementary and well-known property of R^n :

6.3. Lemma. *Let $E \subset R^n$ be a bounded set, and let $t > 0$ be such that $|a - b| \geq t$ whenever $a, b \in E$ and $a \neq b$. Then $\text{card } E \leq \kappa_n(d(E)/t)$ where $\kappa_n(x) = (1 + \alpha\sqrt{n})^n$.*

Proof. Choose a closed cube Q of side length $d(E)$ such that $E \subset Q$. Let k be the integer for which $k - 1 \leq d(E)\sqrt{n}/t < k$. Divide Q into k^n closed cubes Q_i of side length $d(E)/k$. Then $d(Q_i) < t$, and thus each Q_i contains at most one point of E . Hence $\text{card } E \leq k^n$. \square

6.4. Lemma. *Let $\mu > 0$ and let $A \subset D$ be a compact connected set with $r_D(A) \leq \mu$. Then there is a constant $c = c(\mu, n) \geq 1$ such that each pair of points $a, b \in A$ can be joined by a path α in D such that $l(\alpha) \leq c|a - b|$. In fact, one can choose $c = \kappa_n(2 + 2\mu)/4$ for all μ and $c = 1$ for $\mu < 2$.*

Proof. Set $t = d(A, \partial D)$, and let $a, b \in A$. If $|a - b| < 2t$, we can join a and b by a line segment in D . Suppose that $|a - b| \geq 2t$. Since A is connected, the neighborhood $B(A, t/2)$ is arcwise connected, Choose an arc C joining a and b in $B(A, t/2)$. Let \leq be the natural ordering of C with $a < b$. Define points $a = a_0 < a_1 < \dots < a_s = b$ of C by $a_{j+1} = \max(C \cap \bar{B}^n(a_j, t/2))$. Then $|a_i - a_j| \geq t/2$ whenever $0 \leq i < j \leq s - 1$. Furthermore, $d(C) \leq d(A) + t \leq (1 + \mu)t$. Hence Lemma 6.3 implies $s \leq \kappa_n(2 + 2\mu)$. Let α be the broken line with vertices a_0, a_1, \dots, a_s . Then α is in D , and

$$l(\alpha) \leq st/2 \leq \kappa_n(2 + 2\mu)|a - b|/4. \quad \square$$

6.5. Lemma. *Let $\mu > 0$ and let $A \subset D$ be compact and connected with $r_D(A) \leq \mu$. Then there is $c_1 = c_1(\mu, n) \geq 1$ such that*

$$\frac{1}{c_1} \leq \frac{k_D(a, b)d(A, \partial D)}{|a - b|} \leq c_1$$

whenever $a, b \in A$ and $a \neq b$. Moreover, $\lim_{\mu \rightarrow 0} c_1(\mu, n) = 1$, and $c_1(\mu, n)$ is increasing in μ .

Proof. Set $M=d(A)$ and $t=d(A, \partial D)$. Define $r=r(\mu)\in(0, 1)$ by

$$r = \min(\mu, 1/2).$$

Let D' be the neighborhood $B(A, rt)$. Then $M \leq \mu t$ and $r_D(A) \leq \mu/r$. By Lemma 6.4, there is a path α joining a and b in D' such that $l(\alpha) \leq c|a-b|$ with $c=c(\mu/r, n)$. Then

$$k_D(a, b) \leq \int_{\alpha} \varrho_D ds \leq \int_{\alpha} \frac{ds}{(1-r)t} \leq \frac{c|a-b|}{(1-r)t}.$$

This proves the second inequality of 6.5 with $c_1=c/(1-r)$. For $\mu \leq 1/2$ we have $c_1=1/(1-\mu)$, and hence $c_1 \rightarrow 1$ as $\mu \rightarrow 0$.

To prove the first inequality, let α be a rectifiable path joining a and b in D . If $\text{im } \alpha \subset D'$, then

$$(6.6) \quad \int_{\alpha} \varrho_D ds \leq \int_{\alpha} \frac{ds}{t+M+rt} \leq \frac{|a-b|}{(1+\mu+r)t}.$$

If $\text{im } \alpha \not\subset D'$, α has two subpaths α_1, α_2 joining A and $D \setminus D'$ in D' . Since $|a-b| \leq M \leq \mu t$, we obtain

$$\int_{\alpha} \varrho_D ds \leq \int_{\alpha_1 \cup \alpha_2} \frac{ds}{t+M+rt} \leq \frac{2r}{1+\mu+r} \leq \frac{2r|a-b|}{(1+\mu+r)\mu t}.$$

Together with (6.6) this gives the first inequality of 6.5 with

$$c_1 = (1+\mu+r) \max(1, \mu/2r).$$

For $\mu \leq 1/2$ we have $c_1=1+2\mu$, and hence $c_1 \rightarrow 1$ as $\mu \rightarrow 0$. \square

6.7. Corollary. *The quasihyperbolic metric of D is LIP equivalent to the euclidean metric.* \square

6.8. Remark. Since $c_1(\mu, n) \rightarrow 1$ as $\mu \rightarrow 0$, the identity map $i: (D, k_D) \rightarrow (D, d)$ is locally almost 1-quasisymmetric in the following sense: For every $x \in D$ and $\varepsilon > 0$ there is a neighborhood U of x such that $i|_U$ is η -quasisymmetric in the sense of [TV₁] with $\eta(t)=(1+\varepsilon)t$.

Lemma 6.5 also shows that $r_D(A)$ cannot be much smaller than the quasihyperbolic diameter $k_D(A)$ of A . More precisely, we have the following result:

6.9. Lemma. *Let $A \subset D$ be compact. Then*

$$r_D(A) \leq 2(e^{k_D(A)} - 1).$$

If, in addition, A is connected,

$$k_D(A) \leq c_1(r_D(A), n)r_D(A).$$

Proof. The first inequality follows from the inequality

$$k_D(a, b) \leq \log(1 + \varrho_D(a)|a-b|),$$

proved in [GP, (2.2)]. The second inequality follows directly from 6.5. See also [Vu, 2.18 and 2.19. (4)]. \square

6.10. *Solid homeomorphisms.* A homeomorphism $f: D \rightarrow D'$ is said to be *solid*, if f and f^{-1} are uniformly continuous in the quasihyperbolic metrics of D and D' . We say that a homeomorphism $f: D \rightarrow D'$ is φ -*solid* if $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism and if

$$\varphi^{-1}(k_D(a, b)) \cong k_{D'}(f(a), f(b)) \cong \varphi(k_D(a, b))$$

for all $a, b \in D$. From the convexity of the quasihyperbolic metric it easily follows that f is solid if and only if f is φ -solid for some φ . Moreover for every $r > 0$ one can choose φ so that $\varphi|_{[r, \infty)}$ is linear (see the proof of Lemma 6.21).

F. W. Gehring and B. G. Osgood [GO, Theorem 3] proved that every K -QC homeomorphism $f: D \rightarrow D'$ is φ_K -solid with

$$(6.11) \quad \varphi_K(t) = c \max(t, t^\alpha),$$

where $\alpha = K^{1/(1-n)}$ and $c = c(K, n)$. On the other hand, a solid homeomorphism need not be QC. For example, if $f, g: D \rightarrow D'$ are homeomorphisms such that g is solid and $f = g$ outside a compact set, then f is solid. The simple extension $F_f: R_+^{n+1} \rightarrow R_+^{n+1}$ of a QC map $f: R^n \rightarrow R^n$ considered in [TV₂] is solid in $H^{n+1} = \text{int } R_+^{n+1}$ but not usually QC (see 7.1). The following theorem, which is closely related to a result of Gehring [Ge, Theorem 4], shows, however, that quasiconformality can be characterized in terms of solidity:

6.12. **Theorem.** *A homeomorphism $f: D \rightarrow D'$ is quasiconformal if and only if there is a homeomorphism $\varphi: [0, \infty) \rightarrow [0, \infty)$ such that the restriction of f to every subdomain of D is φ -solid.*

Proof. If f is K -QC and if $G \subset D$ is a subdomain, then $f|_G$ is K -QC and hence φ_K -solid with φ_K as in (6.11). Conversely, suppose that $f|_G$ is φ -solid for every domain $G \subset D$. Let $x \in D$. For $r > 0$ we let $L(r)$ and $l(r)$ denote the maximum and minimum of $|f(y) - f(x)|$ over $y \in S^{n-1}(x, r) = S_r$. Set $G = D \setminus x$ and $G' = D' \setminus f(x) = fG$. Then for sufficiently small r we have $r_G(S_r) = 2$ and $L(r)/l(r) \cong r_{G'}(fS_r)$. Since $k_{G'}(fS_r) \cong \varphi(k_G(S_r))$, Lemma 6.9 implies that $L(r)/l(r)$ is bounded by a constant depending only on φ and n . Hence f is K -QC for some $K = K(\varphi, n)$ [Vä₁, 34.2]. \square

6.13. In the rest of this section we give alternative characterizations to the solidity of a homeomorphism in terms of solid families of embeddings (see 3.8). This explains the term. In Section 7 we show that for $n \neq 4$ solid homeomorphisms can be approximated by QC mappings which are bilipschitz in the quasihyperbolic metric.

By a *cube* in R^n we always mean a closed n -cube with sides parallel to the coordinate axes. For every cube Q , we let z_Q be its center and $2\lambda_Q$ its side length. Thus $Q = z_Q + \lambda_Q I^n$. Let $\alpha_Q: R^n \rightarrow R^n$ be the affine map

$$\alpha_Q(x) = z_Q + \lambda_Q x.$$

Thus $\alpha_Q I^n = Q$. For $t > 0$ we let $Q(t)$ be the *open* cube $\alpha_Q I^n(t) = z_Q + \lambda_Q I^n(t)$; the corresponding closed cube is then denoted by $\bar{Q}(t)$.

If $D \subset R^n$ is a proper subdomain and if $1 < r \leq s$, we let $\mathcal{K}_D(r, s)$ denote the family of all cubes $Q \subset D$ such that $Q(r) \subset D$ and $\bar{Q}(s) \not\subset D$. Furthermore, we set $\mathcal{K}_D(r) = \mathcal{K}_D(r, r)$. If $Q \in \mathcal{K}_D(r, s)$ and if $1 \leq t < r$, then obviously

$$(6.14) \quad \frac{2t}{s-t} \leq r_D(\bar{Q}(t)) \leq \frac{2t\sqrt{n}}{r-t}.$$

6.15. Lemma. *Suppose that $1 \leq t < r \leq s$ and that $Q \in \mathcal{K}_D(r, s)$. Then $\alpha_Q|_{\bar{I}^n(t)}: (\bar{I}^n(t), d) \rightarrow (D, k_D)$ is c_2 -bilipschitz with $c_2 = c_2(t, r, s, n)$.*

Proof. Let $c_1 = c_1(2t\sqrt{n}/(r-t), n)$ be the constant given by Lemma 6.5. If $x, y \in \bar{I}^n(t)$, we obtain from 6.5 and (6.14):

$$k_D(\alpha_Q(x), \alpha_Q(y)) \leq \frac{c_1 |\alpha_Q(x) - \alpha_Q(y)|}{d(\bar{Q}(t), \partial D)} \leq \frac{c_1 |x - y|}{r - t},$$

$$k_D(\alpha_Q(x), \alpha_Q(y)) \geq \frac{|\alpha_Q(x) - \alpha_Q(y)|}{c_1 d(\bar{Q}(t), \partial D)} \geq \frac{|x - y|}{c_1(s-t)\sqrt{n}}.$$

Hence we can choose $c_2 = \max(c_1/(r-t), c_1(s-t)\sqrt{n})$. \square

6.16. Notation. Let $f: D \rightarrow D'$ be a homeomorphism and let $Q \subset D$ be a cube. We set

$$d_Q^f = d(fQ, \partial D'), \quad \beta_Q^f(z) = \frac{z - f(z_Q)}{d_Q^f}, \quad f_Q = \beta_Q^f \circ f \circ \alpha_Q.$$

Thus β_Q^f is an affine map of R^n , and f_Q is an embedding defined in $\alpha_Q^{-1}D$. If $Q \in \mathcal{K}_D(r, s)$, this set always contains the open cube $I^n(r)$. Observe that $f_Q(0) = 0$.

6.17. Theorem. *Let $f: D \rightarrow D'$ be a homeomorphism. Then the following conditions are equivalent:*

- (1) f is solid.
- (2) There is $r > 1$ such that $\{f_Q|_{I^n}: Q \in \mathcal{K}_D(r)\}$ is solid.
- (3) If $1 \leq t < r < s$, then $\{f_Q|_{I^n(t)}: Q \in \mathcal{K}_D(r, s)\}$ is solid.

Proof. Trivially (3) implies (2). We show that (2) \Rightarrow (1) \Rightarrow (3).

Suppose that (2) is true. We first show that f is uniformly continuous in the quasihyperbolic metric. Let $\varepsilon > 0$. By (2), there is $M > 0$ such that $d(f_Q I^n) \leq M$

for all $Q \in \mathcal{K}_D(r)$. This implies $r_{D'}(fQ) \cong M$. By Lemma 6.5 we obtain

$$(6.18) \quad \frac{1}{c_1} \cong \frac{k_{D'}(f(a), f(b))d_Q^f}{|f(a) - f(b)|} \cong c_1$$

for all $a, b \in Q$, where $c_1 = c_1(M, n)$. By (2), there is $\delta_1 \in (0, 1)$ such that $|f_Q(x)| \cong \varepsilon/c_1$ whenever $|x| \cong \delta_1$ and $Q \in \mathcal{K}_D(r)$. Let $c_2 = c_2(1, r, r, n)$ be the constant of Lemma 6.15, and set $\delta = \delta_1/c_2$. We show that $k_{D'}(f(a), f(b)) \cong \varepsilon$ whenever $a, b \in D$ and $k_D(a, b) \cong \delta$.

Let $Q \in \mathcal{K}_D(r)$ be the cube with center $z_Q = a$. By Lemma 6.15, $k_D(a, \partial Q) \cong 1/c_2 \cong \delta$, and hence $b \in Q$. Thus $\alpha_Q^{-1}(a) = 0$ and $\alpha_Q^{-1}(b) = b' \in I^n$. By 6.15, $|b'| \cong c_2 k_D(a, b) \cong \delta_1$. Hence $|f_Q(b')| \cong \varepsilon/c_1$. By (6.18) this implies

$$k_{D'}(f(a), f(b)) \cong c_1 |f_Q(b')| \cong \varepsilon.$$

Hence f is uniformly continuous.

We next show that f^{-1} is uniformly continuous. Let $\delta' > 0$. By (2) there is $\varepsilon_1 > 0$ such that $|f_Q(x)| \cong \varepsilon_1$ whenever $Q \in \mathcal{K}_D(r)$, $x \in I^n$, and $|x| \cong \min(1, \delta'/c_2)$. We show that $k_{D'}(f(a), f(b)) \cong \varepsilon_1/c_1$ for all $a, b \in D$ such that $k_D(a, b) \cong \delta'$.

Choose $Q \in \mathcal{K}_D(r)$ such that $z_Q = a$. If $b \in Q$, then $\alpha_Q^{-1}(b) = b' \in I^n$, and Lemma 6.15 implies $|b'| \cong \delta'/c_2$. Hence $|f_Q(b')| \cong \varepsilon_1$. By (6.18) this yields

$$k_{D'}(f(a), f(b)) \cong |f_Q(b')|/c_1 \cong \varepsilon_1/c_1.$$

If $b \notin Q$, then (6.18) implies

$$k_{D'}(f(a), f(b)) \cong k_{D'}(f(a), f\partial Q) \cong d(0, f_Q\partial I^n)/c_1 \cong \varepsilon_1/c_1.$$

Hence f^{-1} is uniformly continuous.

Next suppose that (1) is true. Thus f is φ -solid for some homeomorphism $\varphi: [0, \infty) \rightarrow [0, \infty)$. We must show that the family $\mathcal{G} = \{f_Q|I^n(t) : Q \in \mathcal{K}_D(r, s)\}$ is solid. Since $f_Q(0) = 0$ and since $I^n(t)$ is connected, it suffices to show that \mathcal{G} is equicontinuous and inversely equicontinuous (see 3.8). Let $\varepsilon > 0$. For $Q \in \mathcal{K}_D(r, s)$ set $Q' = \bar{Q}(t)$. Then (6.14) and Lemma 6.9 imply $k_D(Q') \cong c_3$ for some $c_3 = c_3(r, t, n)$. Consequently, $k_{D'}(fQ') \cong \varphi(c_3)$. By Lemma 6.9 this yields $r_{D'}(fQ') \cong 2(e^{\varphi(c_3)} - 1) = c_4$. Hence, by Lemma 6.5,

$$(6.19) \quad \frac{1}{c'_1} \cong \frac{k_{D'}(f(a), f(b))d_{Q'}^f}{|f(a) - f(b)|} \cong c'_1$$

for all $a, b \in Q'$, where $c'_1 = c_1(c_4, n)$.

Let $x, y \in I^n(t)$. Since $d_{Q'}^f \cong d_Q^f$, (6.19) and 6.15 imply

$$\begin{aligned} |f_Q(x) - f_Q(y)| &\cong c'_1 k_{D'}(f(\alpha_Q(x)), f(\alpha_Q(y))) \\ &\cong c'_1 \varphi(k_D(\alpha_Q(x), \alpha_Q(y))) \\ &\cong c'_1 \varphi(c_2|x - y|), \end{aligned}$$

where $c_2 = c_2(t, r, s, n)$. Hence \mathcal{G} is equicontinuous.

We finally show that \mathcal{G} is inversely equicontinuous. Let $\delta > 0$, and let $x, y \in I^n(t)$ with $|x - y| \geq \delta$. We want to find a positive lower bound for $|f_Q(x) - f_Q(y)|$, valid for all $Q \in \mathcal{K}_D(r, s)$. By (6.19) and 6.15 we obtain

$$\begin{aligned} |f_Q(x) - f_Q(y)| &\cong k_{D'}(f(\alpha_Q(x)), f(\alpha_Q(y))) d_Q^f / c_1^f d_Q^f \\ &\cong \varphi^{-1}(\delta / c_2) d_Q^f / c_1^f d_Q^f. \end{aligned}$$

Hence it suffices to find a positive lower bound for d_Q^f / d_Q^f . First, we have $d_Q^f / d_Q^f \cong r_{D'}(fQ) / r_{D'}(fQ')$. Here $r_{D'}(fQ') \leq c_1$. Hence, if $r_{D'}(fQ) \geq 1$, $d_Q^f / d_Q^f \geq 1 / c_1$. Suppose that $r_{D'}(fQ) < 1$. Then Lemma 6.9 implies

$$r_{D'}(fQ) \cong k_{D'}(fQ) / c_1(1, n) \cong \varphi^{-1}(k_D(Q)) / c_1(1, n),$$

where, by Lemma 6.9 and by (6.14),

$$k_D(Q) \geq \log(1 + r_D(Q) / 2) \geq \log(1 + 1 / (s - 1)).$$

The theorem is proved. \square

6.20. *Remark.* The proof of Theorem 6.17 shows that a quantitative version is also true. Thus, if $f: D \rightarrow D'$ is a φ -solid homeomorphism and if $1 \leq t < r < s$, then $\{f_Q|I^n(t): Q \in \mathcal{K}_D(r, s)\}$ is contained in a solid family \mathcal{F} of embeddings $I^n(t) \rightarrow R^n$, depending only on φ, t, r, s and n . Conversely, if \mathcal{F} is a solid family of embeddings of I^n and if $\{f_Q|I^n: Q \in \mathcal{K}_D(r)\} \subset \mathcal{F}$, then f is φ -solid with φ depending only on \mathcal{F} and r .

6.21. **Lemma.** *Suppose that a homeomorphism $f: D \rightarrow D'$ is locally L -bilipschitz in the quasihyperbolic metric. Then f is L -bilipschitz in the quasihyperbolic metric.*

Proof. It suffices to show that f is L -lipschitz. Let $a, b \in D$, and choose a quasihyperbolic geodesic joining a and b . Divide α to subpaths $\alpha_1, \dots, \alpha_s$ such that f is L -bilipschitz on each in α_j . Let a_{j-1} and a_j be the end points of α_j . Then

$$\begin{aligned} k_{D'}(f(a), f(b)) &\leq \sum_{j=1}^s k_{D'}(f(a_{j-1}), f(a_j)) \\ &\leq L \sum_{j=1}^s k_D(a_{j-1}, a_j) \\ &= Lk_D(a, b). \quad \square \end{aligned}$$

7. Lipschitz approximation of solid and QC homeomorphisms

7.1. In this section we first show that if $n \neq 4$, every solid homeomorphism $D \rightarrow D'$ can be approximated arbitrarily closely in the quasihyperbolic metric by homeomorphisms which are bilipschitz in the quasihyperbolic metrics of D and D' (and hence QC). The method is a variation of the proof of the extension theorem

in [TV₂]. In fact, in [TV₂] we first extended a QC homeomorphism $f: R^n \rightarrow R^n$ to a homeomorphism $F_f: R_+^{n+1} \rightarrow R_+^{n+1}$ by

$$F_f(x, t) = (f(x), \max \{|f(y) - f(x)| : y \in \bar{B}^n(x, t)\}).$$

Then the restriction of F_f to $H = \text{int } R_+^{n+1}$ is solid but not necessarily QC. We constructed an approximation F of $F_f|_H$ which is bilipschitz in the hyperbolic metric and hence QC. Since $k_H(F, F_f) < \infty$, F and F_f have the same boundary values, and thus $F \cup f$ is the desired extension of f .

The proof in [TV₂] was based on the qualitative part of Lemma 3.9 of the present paper. However, it is possible to simplify the proof by making also use of the quantitative part of 3.9. Indeed, it is not necessary to introduce the points z'' in [TV₂, (2.8)], and Carleson's finiteness idea is only needed in the approximation of the maps $F_f|_Q$, not in the glueing process. We shall use this simplification in the proof of Theorems 7.4 and 7.18.

As an application of our result we extend to higher dimensions a result of Ahlfors concerning QC reflections in R^2 .

In general, the dilatation of the approximating homeomorphism depends on the closeness of the approximation. However, we show in 7.18 that every K -QC homeomorphism $D \rightarrow D'$, $n \neq 4$, can be approximated arbitrarily closely in the majorant topology by LIP homeomorphisms which are K_1 -QC for some $K_1 = K_1(K, n)$.

7.2. A cube decomposition. Let D be a proper subdomain of R^n . We define a decomposition $\mathcal{K}(D)$ of D into closed cubes as follows. For $k \in \mathbf{Z}$ let $\mathcal{L}(k)$ be the decomposition of R^n into closed n -cubes of side length 2^k and with vertices in $2^k \mathbf{Z}^n$. Set $\mathcal{L} = \cup \{\mathcal{L}(k) : k \in \mathbf{Z}\}$ and $\mathcal{K}^0(D) = \{Q \in \mathcal{L} : \bar{Q}(3) \subset D\}$ (see 6.13 for notation). Each cube of $\mathcal{K}^0(D)$ is contained in a maximal cube of $\mathcal{K}^0(D)$. Let $\mathcal{K}^1(D)$ be the family of all maximal cubes in $\mathcal{K}^0(D)$. We divide each cube of $\mathcal{K}^1(D)$ into $N = 2^n$ cubes bisecting the sides of Q . This gives the family $\mathcal{K}(D)$ of cubes in D .

We divide $\mathcal{K}(D)$ into disjoint subfamilies $\mathcal{K}_1(D), \dots, \mathcal{K}_N(D)$ as follows: Divide I^n into cubes Q_1, \dots, Q_N bisecting the sides. Then set $\mathcal{K}_j(D) = \{\alpha_Q Q_j : Q \in \mathcal{K}^1(D)\}$ where α_Q is as in 6.13.

Similar decompositions have been used, for example, by H. Whitney [Wh, p. 67], E. Stein [St, p. 167] and M. Kiikka [Kk, p. 7]. The following properties of $\mathcal{K}(D)$ are readily verified (notation as in 6.13):

7.3. Lemma. (1) $\cup \mathcal{K}(D) = D$.

(2) If $Q, R \in \mathcal{K}(D)$, $Q \neq R$ and $Q \cap R \neq \emptyset$, then $\text{int } Q \cap \text{int } R = \emptyset$ and $\lambda_Q / \lambda_R \in \{1/2, 1, 2\}$.

(3) $\mathcal{K}(D) \subset \mathcal{K}_D(5, 15)$.

(4) If $Q \in \mathcal{K}(D)$, then $1/7 \leq r_D(Q) \leq \sqrt{n}/2$.

(5) If $Q, R \in \mathcal{K}_i(D)$, then $Q \cap R = \emptyset$. \square

7.4. Theorem. *Let $n \neq 4$, let $\varepsilon > 0$, and let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. Then there is a number $L = L(\varphi, \varepsilon, n) \geq 1$ with the following property:*

Suppose that D and D' are proper subdomains of R^n and that $f: D \rightarrow D'$ is a φ -solid homeomorphism. Then there is a homeomorphism $F: D \rightarrow D'$ such that

- (1) $k_{D'}(F, f) \leq \varepsilon$,
- (2) F is L -bilipschitz in the quasihyperbolic metrics of D and D' ,
- (3) F is L^{2n-2} -QC.

Proof. The proof of Theorem 7.4 will be completed in 7.11. From now on, we assume that n, ε, φ are given as in the theorem and that $f: D \rightarrow D'$ is φ -solid.

7.5. Lemma. *There is $M = M(\varphi, n)$ such that*

$$1/M \leq d_Q^f/d_R^f \leq M, \quad d_Q^f \leq Md(f\bar{Q}(2), \partial D')$$

whenever $Q, R \in \mathcal{H}(D)$ with $Q \cap R \neq \emptyset$.

Proof. By 6.9 and 7.3.(4), $k_D(Q) \leq M_1$ for some $M_1 = M_1(n)$. Hence $k_{D'}(fQ) \leq \varphi(M_1)$. Since $d_R^f \leq d_Q^f + d(fQ) = (1 + r_{D'}(fQ))d_Q^f$, Lemma 6.9 implies $d_R^f \leq Md_Q^f$ with $M = 2e^{\varphi(M_1)}$. By symmetry, $d_Q^f \leq Md_R^f$. The last inequality follows from the fact that $\bar{Q}(2) \subset \cup \{R \in \mathcal{H}(D) : R \cap Q \neq \emptyset\}$. \square

7.6. Lemma. *There is a number $c_3 = c_3(\varphi, n) \geq 1$ such that*

$$\beta_Q^f|_{f\bar{Q}(2)}: (f\bar{Q}(2), k_{D'}) \rightarrow (R^n, d)$$

is c_3 -bilipschitz for every $Q \in \mathcal{H}(D)$.

Proof. Set $A = f\bar{Q}(2)$. Since $\bar{Q}(2) \in \mathcal{H}_D(5/2, 15/2)$, 6.9 and (6.14) imply $r_{D'}(A) \leq M_2 = M_2(\varphi, n)$. Hence we can apply Lemma 6.5 with $c_1 = c_1(M_2, n)$. For $a, b \in A$ we obtain

$$|\beta_Q^f(a) - \beta_Q^f(b)| = |a - b|/d_Q^f \leq c_1 k_{D'}(a, b) d(A, \partial D')/d_Q^f \leq c_1 k_{D'}(a, b).$$

By 7.5 we also get

$$|\beta_Q^f(a) - \beta_Q^f(b)| \geq k_{D'}(a, b) d(A, \partial D')/c_1 d_Q^f \geq k_{D'}(a, b)/Mc_1.$$

Hence the lemma is true with $c_3 = Mc_1$. \square

7.7. Constructions. We set

$$\begin{aligned} \mathcal{H}_i^*(D) &= \mathcal{H}_1(D) \cup \dots \cup \mathcal{H}_i(D), \\ V_i(D) &= \cup \{Q(1 + 2^{-i-1}) : Q \in \mathcal{H}_i^*(D)\}, \\ W_i(D) &= \cup \{Q(1 + 2^{-i-2}) : Q \in \mathcal{H}_i^*(D)\} \end{aligned}$$

with $V_0(D) = W_0(D) = \emptyset$. If $Q \in \mathcal{H}_i(D)$ and $1 < i \leq 3/2$, we set

$$\begin{aligned} V_Q(D, t) &= I^n(t) \cap \alpha_Q^{-1}V_{i-1}(D), \\ W_Q(D, t) &= I^n(t) \cap \alpha_Q^{-1}W_{i-1}(D). \end{aligned}$$

Then for every $t \in (1, 3/2]$ there is a finite family $S(n, t)$ such that $V_Q(D, t)$ and $W_Q(D, t)$ belong to $S(n, t)$ for every domain D and for every $Q \in \mathcal{K}(D)$.

By 6.17, 6.10 and 7.3.(3), there is a solid family $\mathcal{G} = \mathcal{G}(\varphi, n) \subset E(I^n(2); R^n)$ such that $f_Q|I^n(2) \in \mathcal{G}$ for every φ -solid homeomorphism $f: D \rightarrow D'$ and for every $Q \in \mathcal{K}(D)$.

We apply Lemma 3.9 with $U = I^n(3/2)$, $U' = I^n(4/3)$, $V = V_Q(D, 3/2)$, $W = W_Q(D, 3/2)$ for $Q \in \mathcal{K}(D)$, and with $\mathcal{F} = \{g|I^n(3/2): g \in \mathcal{G}\}$. Since the family $S(n, 3/2)$ is finite, we obtain:

7.8. Lemma. *Let $\varepsilon' > 0$ and $L \geq 1$. Then there are positive numbers $\delta = \delta(\varepsilon', \varphi, n) \leq \varepsilon'$ and $L' = L'(\varepsilon', \varphi, n, L) \geq L$ with the following properties:*

Let D be a proper subdomain of R^n , let $Q \in \mathcal{K}(D)$, let $h: V_Q(D, 3/2) \rightarrow R^n$ be a locally L -bilipschitz embedding, and let $g \in \mathcal{F}$ be such that $d(g, h; V_Q(D, 3/2)) \leq \delta$. Then there is a LIP embedding $h': I^n(3/2) \rightarrow R^n$ such that

- (1) $d(h', g; I^n(4/3)) \leq \varepsilon'$,
- (2) $h' = h$ in $W_Q(D, 4/3)$,
- (3) $h'|I^n(4/3)$ is L' -bilipschitz. \square

7.9. Constructions. Since \mathcal{G} is solid, there is a number $q = q(\varphi, n) > 0$ such that $|g(x) - g(y)| \geq q$ whenever $g \in \mathcal{G}$ and $x, y \in I^n(7/4)$ with $|x - y| \geq 1/8$. Let $c_2 = c_2(3/2, 5, 15, n)$, $c_3 = c_3(\varphi, n)$ and $M = M(\varphi, n)$ be as in 6.15, 7.6 and 7.5. Define numbers $\delta_N \geq \delta_{N-1} \geq \dots \geq \delta_0 > 0$ by $\delta_N = \min(q/(M+2), \varepsilon/c_3)$ and $\delta_{j-1} = \delta(\delta_j, \varphi, n)/M$ where $\delta(\cdot)$ is as in 7.8. We also define numbers $L_0 \leq \dots \leq L_N$ by $L_0 = 1$ and $L_j = c_2 c_3 L'(\delta_j, \varphi, n, c_2 c_3 L_{j-1})$, where $L'(\cdot)$ is as in 7.8. Observe that the sequences $(\delta_0, \dots, \delta_N)$ and (L_0, \dots, L_N) depend only on φ, n and ε . We show by induction that the following lemma is true for every integer $j \in [0, N]$:

7.10_j. Lemma. *There is an embedding $F_j: V_j(D) \rightarrow D'$ with the following properties:*

- (1) $d(F_j, f; Q(1+2^{-j-1})) \leq \delta_j d_Q^f$ for every $Q \in \mathcal{K}_j^*(D)$.
- (2) $F_j Q(1+2^{-j-1}) \subset f Q(3/2)$ for every $Q \in \mathcal{K}_j^*(D)$.
- (3) F_j is locally L_j -bilipschitz in the quasihyperbolic metrics of D and D' .

Proof. Since $V_0(D) = \emptyset$, 7.10₀ is true. Suppose that 7.10 _{$j-1$} is true. Thus we have an embedding $F_{j-1}: V_{j-1}(D) \rightarrow D'$. We define $F_j(x) = F_{j-1}(x)$ for $x \in W_{j-1}(D)$. Let $Q \in \mathcal{K}_j^*(D)$. Then $f_Q|I^n(3/2) \in \mathcal{F}$. Set $h_Q = \beta_Q^f F_{j-1} \alpha_Q|V_Q(D, 3/2)$. We first show that $d(h_Q, f_Q; V_Q(D, 3/2)) \leq \delta(\delta_j, \varphi, n)$. Let $x \in V_Q(D, 3/2)$. Then $x \in \alpha_Q^{-1} R(1+2^{-j})$ for some $R \in \mathcal{K}_{j-1}^*(D)$ with $R \cap Q \neq \emptyset$. By 7.5 and 7.10 _{$j-1$} we obtain

$$\begin{aligned} |h_Q(x) - f_Q(x)| &= |f(\alpha_Q(x)) - F_{j-1}(\alpha_Q(x))|/d_Q^f \\ &\leq M \delta_{j-1} = \delta(\delta_j, \varphi, n). \end{aligned}$$

Hence we can apply Lemma 7.8 with $g = f_Q$, $h = h_Q$ and $\varepsilon' = \delta_j$. We obtain a LIP embedding $h'_Q: I^n(3/2) \rightarrow R^n$ such that (a) $d(h'_Q, f_Q; I^n(4/3)) \leq \delta_j$ and (b) $h'_Q = h_Q$

in $W_Q(D, 4/3)$. By 6.15 and 7.6, h_Q is locally $c_2 c_3 L_{j-1}$ -bilipschitz in the euclidean metric. Hence $h'_Q|I^n(4/3)$ is $L_j/c_2 c_3$ -bilipschitz. Setting $F_j = (\beta'_Q)^{-1} h'_Q \alpha_Q^{-1}$ in $Q(1+2^{-j-1})$ we obtain a well-defined map $F_j: V_j(D) \rightarrow R^n$. We show that F_j satisfies the conditions (1), (2), (3) and that F_j is injective.

Let $Q \in \mathcal{K}_j^*(D)$. If $Q \in \mathcal{K}_{j-1}^*(D)$, (1) follows from 7.10 $_{j-1}$. If $Q \in \mathcal{K}_j(D)$, we obtain

$$d(F_j, f; Q(1+2^{-j-1})) = d_Q^f d(h'_Q, f_Q; I^n(1+2^{-j-1})) \cong \delta_j d_Q^f.$$

To prove (2), let again $Q \in \mathcal{K}_j^*(D)$. If $Q \in \mathcal{K}_{j-1}^*(D)$, (2) follows by induction. Suppose that $Q \in \mathcal{K}_j(D)$. Then $d(h'_Q, f_Q; I^n(4/3)) \cong \delta_j \cong \delta_N < q$. By the choice of q , this implies $h'_Q I^n(1+2^{-j-1}) \subset f_Q I^n(3/2)$. Hence (2) is true. Observe that (2) implies $\text{im } F_j \subset D'$.

If $Q \in \mathcal{K}_{j-1}^*(D)$, F_j is locally L_j -bilipschitz in $Q(1+2^{-j-1})$ by induction. If $Q \in \mathcal{K}_j(D)$, then 6.15, 7.6 and (2) imply that $F_j|Q(1+2^{-j-1})$ is L_j -bilipschitz. Hence F_j is a locally L_j -bilipschitz immersion. We finally show that F_j is injective. We know that $F_j|Q(1+2^{-j-1})$ is injective for every $Q \in \mathcal{K}_j^*(D)$. Moreover, if $Q, R \in \mathcal{K}_j^*(D)$ and $Q \cap R = \emptyset$, then (2) implies that

$$F_j Q(1+2^{-j-1}) \cap F_j R(1+2^{-j-1}) = \emptyset.$$

Hence it suffices to show that $F_j(x) \neq F_j(y)$ whenever $j \geq 2$, $x \neq y$, $x \in Q(1+2^{-j-1})$ and $y \in R(1+2^{-j-1})$ where $Q \in \mathcal{K}_j(D)$, $R \in \mathcal{K}_{j-1}^*(D)$ and $Q \cap R \neq \emptyset$. The equality $F_j = (\beta'_Q)^{-1} h'_Q \alpha_Q^{-1}$ is valid in $Q(4/3) \cap V_j(D)$. Hence we may assume that $y \notin Q(4/3)$. By the choice of q , we have $d(f_Q(\alpha_Q^{-1}(x)), f_Q \partial I^n(5/4)) \cong q$. Hence $|f(x) - f(y)| \cong q d_Q^f$. By (1) and 7.5 we obtain

$$\begin{aligned} |F_j(x) - F_j(y)| &\cong |f(x) - f(y)| - |F_j(x) - f(x)| - |F_j(y) - f(y)| \\ &\cong q d_Q^f - \delta_j d_Q^f - \delta_j d_R^f \\ &\cong d_Q^f (q - (M+1)\delta_N) \cong d_Q^f \delta_N > 0. \quad \square \end{aligned}$$

7.11. *Completion of the proof of Theorem 7.4.* We show that the theorem is true with $L = L_N$, defined in 7.9. Let $f: D \rightarrow D'$ be φ -solid. We show that the map F_N of 7.10 $_N$ is the required F . First, F_N is an embedding of D into D' , and F_N is locally L_N -bilipschitz in the quasihyperbolic metric. To prove the condition (1) of Theorem 7.4, let $x \in D$. Choose $Q \in \mathcal{K}(D)$ containing x . Then 7.10. (1) yields

$$|F_N(x) - f(x)| \cong \delta_N d_Q^f \cong \varepsilon d_Q^f / c_3.$$

By 7.6 and 7.10. (2), this implies $k_{D'}(F_N(x), f(x)) \cong \varepsilon$. Hence (1) is true. Moreover, (1) implies that F_N is a homeomorphism of D onto D' .

We already proved that F_N is locally L_N -bilipschitz in the quasihyperbolic metric. By 6.21, F_N is L_N -bilipschitz.

To prove that F_N is L_N^{2n-2} -QC, it suffices to show that the linear dilatation $H(x, F_N)$ is at most L_N^2 [Vä $_1$, 34.2]. This follows from (2) and from Remark 6.8. \square

7.12. Corollary. *Let $n \neq 4$, let $\varepsilon > 0$ and let $K \geq 1$. Then there is $L = L(K, \varepsilon, n) \geq 1$ such that if $D \subset R^n$ is a proper subdomain and if $f: D \rightarrow D'$ is a K -QC homeomorphism, then there is a homeomorphism $F: D \rightarrow D'$ such that*

- (1) $k_{D'}(F, f) \leq \varepsilon$,
- (2) F is L -bilipschitz in the quasihyperbolic metrics of D and D' ,
- (3) F is L^{2n-2} -QC. \square

7.13. Remark. It follows from Theorem 7.4 that solid homeomorphisms and QC homeomorphisms have similar boundary properties, at least for $n \neq 4$. For example, if b is an isolated boundary point of D and if $f: D \rightarrow D'$ is solid, then f has a limit b' at b , and b' is an isolated boundary point of D' . Furthermore, let $H^n = \text{int } R_+^n$, and let $f: H^n \rightarrow H^n$ be solid. The f can be extended to a homeomorphism $g: \bar{R}_+^n \rightarrow \bar{R}_+^n$, and the induced boundary map $g|_{\bar{R}^{n-1}}$ is QC. It is possible to give direct proofs for these and several other results on solid homeomorphisms. These proofs are also valid for $n=4$. Cf. [Th, 5.9.6].

7.14. Quasiconformal reflections. Let E be a closed connected set in R^n such that $R^n \setminus E$ has exactly two components D_1 and D_2 . A homeomorphism $f: R^n \rightarrow R^n$ is said to be a *reflection* in E if $f|_E = \text{id}$ and $fD_1 = D_2$. Then $fD_2 = D_1$, f is sense-reversing, and $\partial D_1 = E = \partial D_2$. If f is a reflection in E , we can always find a reflection g in E which is also an *involution*, that is, $gg = \text{id}$. For example, we can choose $g = (f|\bar{D}_1) \cup (f^{-1}|D_2)$.

L. V. Ahlfors [Ah, p. 80] proved that if $f: R^2 \rightarrow R^2$ is a QC reflection in E , there is a reflection F in E which is bilipschitz in the euclidean metric. We shall extend this result for all dimensions $n \neq 4$.

On the other hand, Ahlfors [Ah, p. 75] also proved that if E admits a QC reflection, $E \cup \infty$ is a QC circle. (By a classical result of L. E. J. Brouwer, it is always a topological circle.) This result cannot be extended to higher dimensions. Indeed, for every $n \geq 3$ there is a QC reflection $f: R^n \rightarrow R^n$ in a set E such that $E \cup \infty$ is a topological sphere but no nonempty open subset of E can be quasisymmetrically embedded into R^{n-1} . See [Tu₂, Example 2].

7.15. Theorem. *Suppose that $n \neq 4$ and that $f: R^n \rightarrow R^n$ is a K -QC reflection in $E \subset R^n$. Then there is a reflection $F: R^n \rightarrow R^n$ in E such that F is L -bilipschitz in the euclidean metric with L depending only on K and n .*

Proof. Let D_1, D_2 be the components of $R^n \setminus E$, and let $f_1: D_1 \rightarrow D_2$ and $f_2: D_2 \rightarrow D_1$ be the K -QC (sense-reversing) homeomorphisms defined by f . Applying 7.12 with $\varepsilon=1$ we find homeomorphisms $F_1: D_1 \rightarrow D_2$ and $F_2: D_2 \rightarrow D_1$ such that $k_{D_2}(F_1, f_1; D_1) \leq 1$, $k_{D_1}(F_2, f_2; D_2) \leq 1$, and such that F_1 and F_2 are L_1 -bilipschitz in the quasihyperbolic metrics of D_1 and D_2 with $L_1 = L_1(K, n)$. Then $F = F_1 \cup F_2 \cup (\text{id}|_E)$ is a reflection in E .

For $x \in R^n$ and $r > 0$ we let as usual $L(x, f, r)$ and $l(x, f, r)$ denote the maximum and minimum of $|f(y) - f(x)|$ over $y \in S^{n-1}(x, r)$. Then there is $H = H(K, n)$ such that $L(x, f, r) \leq Hl(x, f, r)$; see, for example, the proof of [Vä₁, 22.3]. We show that F is HeL_1 -bilipschitz, $e = 2.7182\dots$, in the euclidean metric. By Lemma 2.2, it suffices to show that

$$(7.16) \quad L(x, F) \leq HeL_1, \quad L(x, F^{-1}) \leq HeL_1$$

for all $x \in R^n$.

Let first $x \in R^n \setminus E$, say $x \in D_1$. Set $r = d(x, E)$, $r' = d(f(x), E)$, $r'' = d(F(x), E)$. Choose y and z in E such that $|y - x| = r$ and $|z - f(x)| = r'$. Setting $s = |x - z|$ we obtain

$$\begin{aligned} r' &\leq |f(x) - y| \leq L(y, f, r) \leq Hl(y, f, r) \leq Hr, \\ r' &= |f(x) - z| \geq l(z, f, s) \geq L(z, f, s)/H \geq s/H \geq r/H. \end{aligned}$$

Furthermore, since $k_{D_2}(F(x), f(x)) \leq 1$, [GP, (2.1)] implies

$$r'/e \leq r'' \leq er',$$

and (7.16) follows by Lemma 6.5.

Next assume $x \in E$. Let $r > 0$ and let $|y - x| = r$. If $y \in D_1$, $|f(y) - x| \leq Hl(x, f, r) \leq Hr$. Since $k_{D_2}(F(y), f(y)) \leq 1$, [GP, (2.2)] implies $|F(y) - f(y)| \leq (e-1)d(f(y), E) \leq (e-1)|f(y) - x| \leq (e-1)Hr$. Hence $|F(y) - x| \leq eHr$, which implies the first inequality of (7.16).

Similarly, $|f(y) - x| \geq L(x, f, r)/H \geq r/H$, and $|F(y) - f(y)| \leq (e-1)d(F(y), E) \leq (e-1)|F(y) - x|$, which implies $|F(y) - x| \geq |f(y) - x|/e \geq r/He$, and we obtain the second inequality of (7.16). \square

7.17. Approximation by LIP homeomorphisms. In Theorem 7.4 we approximated a solid homeomorphism $f: D \rightarrow D'$ by a homeomorphism F which was bilipschitz in the quasihyperbolic metric. We shall next give a related result which is weaker in two respects: (1) f is supposed to be K -QC and not only solid, (2) F is only locally bilipschitz. On the other hand, it is stronger in the following respects: F is K_1 -QC with $K_1 = K_1(K, n)$, (2) the approximation is in the majorant topology and not in the uniform topology of the quasihyperbolic metric, (3) D can be the whole space R^n , and (4) it is strongly relative. Since the proof is a modification of the proof of Theorem 7.4, we shall omit some details. The case $n \leq 3$ was proved by M. Kiiikka [Kk] who showed that in this case F can be chosen to be PL.

7.18. Theorem. *Let D be a domain in R^n , $n \neq 4$, let U be an open set in D , and let $\varepsilon: U \rightarrow R^1$ be continuous and positive. Then for every K -QC homeomorphism $f: D \rightarrow D'$ there is a homeomorphism $F: D \rightarrow D'$ such that*

- (1) $|F(x) - f(x)| \leq \varepsilon(x)$ for every $x \in U$,
- (2) $F = f$ in $D \setminus U$,
- (3) $F|U$ is a LIP embedding,
- (4) F is K_1 -QC with K_1 depending only on K and n .

Proof. The theorem reduces easily to the absolute case $U=D$. Indeed, in the general case we first replace $\varepsilon(x)$ by $\min(\varepsilon(x), d(x, \partial U))$ and then apply the special case in each component of U . We obtain a K_1 -QC LIP homeomorphism $F_1: U \rightarrow fU$ which can be extended by $f|_{D \setminus U}$ to a homeomorphism $F: D \rightarrow D'$, which is K_2 -QC with $K_2 = \max(K, K_1)$ [Vä₂, Theorem 2].

From now on, we suppose that $U=D$ and that $f: D \rightarrow D'$ is a K -QC homeomorphism. We may assume that $\varepsilon(x) \leq d(f(x), \partial D')$ if $D' \neq R^n$ and that $\varepsilon(x) \leq (1 + |f(x)|)^{-1}$. We replace the cube decomposition $\mathcal{H}(D)$ of 7.2 by another decomposition $\mathcal{H}(D)$ which also depends on f and ε . Let \mathcal{L} be as in 7.2, and let $\mathcal{H}^1(D)$ be the family of all maximal cubes of \mathcal{L} satisfying the conditions $\bar{Q}(3) \subset D$ and $d(f\bar{Q}(3)) \leq \min \varepsilon \bar{Q}(3)$. Dividing each cube of $\mathcal{H}^1(D)$ into $N=2^n$ subcubes by bisecting the sides we obtain the family $\mathcal{H}(D)$. It has the following properties:

(1) $\cup \mathcal{H}(D) = D$.

(2) If $Q, R \in \mathcal{H}(D)$, $Q \neq R$ and $Q \cap R \neq \emptyset$, then $\text{int } Q \cap \text{int } R = \emptyset$, and $\lambda_Q/\lambda_R \in \{1/2, 1, 2\}$.

(3) If $Q \in \mathcal{H}(D)$, $\bar{Q}(5) \subset D$.

(4) If $x \in Q \in \mathcal{H}(D)$, $d(fQ) \leq \varepsilon(x)$.

We divide $\mathcal{H}(D)$ into subfamilies $\mathcal{H}_1(D), \dots, \mathcal{H}_N(D)$ as in 7.2, and set $\mathcal{H}_j^*(D) = \mathcal{H}_1(D) \cup \dots \cup \mathcal{H}_j(D)$.

We next show that there is $M = M(K, n)$ such that

$$(7.19) \quad \frac{1}{M} \leq \frac{d(fQ)}{d(fR)} \leq M$$

whenever $Q, R \in \mathcal{H}(D)$ with $Q \cap R \neq \emptyset$. We may assume that $Q \neq R$. By symmetry, it suffices to prove the second inequality. Since $Q(5) \subset D$, it follows from [Vä₄, 2.4] that $f|_{\bar{Q}(3)}$ is η -quasisymmetric with some η depending only on K and n . Let $z \in Q$. Then $|z - z_Q| \leq |z_R - z_Q| \sqrt{n}$. Since $z_R \in \bar{Q}(3)$, this implies $|f(z) - f(z_Q)| \leq \eta(\sqrt{n})|f(z_R) - f(z_Q)|$, and hence

$$d(fQ) \leq 2\eta(\sqrt{n})|f(z_R) - f(z_Q)|.$$

Let y be a vertex of R . Then $|z_Q - z_R| \leq 3|y - z_R|$, which implies

$$|f(z_Q) - f(z_R)| \leq \eta(3)d(fR),$$

and we obtain (7.19) with $M = 2\eta(\sqrt{n})\eta(3)$.

We let α_Q be as in 6.13 but replace β_Q^f by the affine map

$$\gamma_Q^f(z) = \frac{z - f(z_Q)}{d(fQ)},$$

and set $f_Q = \gamma_Q^f f \alpha_Q$. Let \mathcal{G}_K be the family of all K -QC embeddings $g: I^n(2) \rightarrow R^n$ such that $g(0) = 0$ and $d(gI^n) = 1$. The compactness properties of QC maps imply that \mathcal{G}_K is solid. Let $\mathcal{F}_K = \{g|_{I^n(3/2)}: g \in \mathcal{G}_K\}$. Using the notation of 7.7 we apply Lemma 3.9 and obtain the following variation of Lemma 7.8:

7.20. **Lemma.** *Let $\varepsilon' > 0, K \geq 1$ and $\bar{K} \geq 1$. Then there are $\delta = \delta(\varepsilon', K, n) > 0$ and $K' = K'(\varepsilon', K, n, \bar{K}) \geq \bar{K}$ with the following properties:*

Let $Q \in \mathcal{H}(D)$, let $h: V_Q(D, 3/2) \rightarrow \mathbb{R}^n$ be a \bar{K} -QC LIP embedding, and let $g \in \mathcal{F}_K$ such that $d(g, h; V_Q(D, 3/2)) \leq \delta$. Then there is a LIP embedding $h': I^n(3/2) \rightarrow \mathbb{R}^n$ such that

- (1) $d(h', g; I^n(4/3)) \leq \varepsilon'$,
- (2) $h' = h$ in $W_Q(4/3)$,
- (3) $h'|I^n(4/3)$ is K' -QC. \square

We introduce the numbers q and $\delta_N, \dots, \delta_0$ as in 7.9, where now $\delta_N = \min(1, q/(M+2))$ and M is the constant of (7.19). Moreover, we set $K_N = 1$ and $K_j = K'(\delta_j, K, n, K_{j-1})$ for $1 \leq j \leq N$. Corresponding to Lemma 7.10_j we prove for every $j \in [0, N]$:

7.21_j. **Lemma.** *There is a LIP embedding $F_j: V_j(D) \rightarrow D'$ with the following properties:*

- (1) $d(F_j, f; Q(1+2^{-j-1})) \leq \delta_j d(fQ)$ for every $Q \in \mathcal{H}_j^*(D)$.
- (2) $F_j Q(1+2^{-j-1}) \subset fQ(3/2)$ for every $Q \in \mathcal{H}_j^*(D)$.
- (3) F_j is K_j -QC.

Proof. Assume that 7.21_{j-1} is true. Define $F_j(x) = F_{j-1}(x)$ for $x \in W_{j-1}(D)$. Let $Q \in \mathcal{H}_j(D)$ and set $h_Q = \gamma_Q^f F_{j-1} \alpha_Q |V_Q(D, 3/2)$. By (7.19) we obtain

$$d(h_Q, f_Q; V_Q(D, 3/2)) \leq \delta(\delta_j, K, n).$$

Hence we can apply 7.20 with the substitution $\varepsilon' \mapsto \delta_j, K \mapsto K, \bar{K} \mapsto K_{j-1}, h \mapsto h_Q, g \mapsto f_Q |I^n(3/2)$. We obtain a LIP embedding $h'_Q: I^n(3/2) \rightarrow \mathbb{R}^n$ and define $F_j = (\gamma_Q^f)^{-1} h'_Q \alpha_Q^{-1}$ in $Q(1+2^{-j-1})$. As in 7.10, we can verify that $F_j: V_j(D) \rightarrow D'$ satisfies the conditions (1), (2), (3) and is a LIP embedding. \square

The proof of Theorem 7.18 can now be completed as in 7.11. The desired map F is F_N , which is K_N -QC with K_N depending only on K and n . Indeed, if $x \in Q \in \mathcal{H}(D)$, it follows from the definition of $\mathcal{H}(D)$ that

$$|F_N(x) - f(x)| \leq \delta_N d(fQ) \leq d(fQ) \leq \varepsilon(x). \quad \square$$

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