Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 6, 1981, 369—398

MEROMORPHIC SOLUTIONS OF THE RICCATI DIFFERENTIAL EQUATION

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§ 1. Introduction

The special position of the Riccati differential equation

$$w' = a(z)+b(z)w+c(z)w^2 \tag{1.1}$$

with rational coefficients within the collection of birational differential equations $w'=R(z, w)$ was first established by J. Malmquist in his classical paper [11]. It has been more then twenty years since the appearance of the fundamental work concerning the Riccati differential equation in the complex domain by H. Wittich in his four articles [13], [14], [15] and [17] and the respective parts of his monograph [16]. This work done by H. Wittich was preceded by work of K. Yosida (see, e.g., [19]) and followed by work of E. Hille [5]—[8], C.-C. Yang [18] and K. Yosida [20], among others (see, e.g., [9], [10]).

Of course, many important problems concerning the Riccati differential equation (1.1) and its solutions still remain open. For instance, it has been largely open, under what conditions a Riccati equation (1.1) with meromorphic coefficients which are not all entire, actually admits meromorphic solutions in the complex plane. This article is mainly devoted to presenting some contributions to this problem. We should perhaps point out here that the problem of whether such meromorphic solutions are rational or transcendental in the case when the coefficients are rational, has been treated earlier by H. Wittich [13] and [17].

In this article, the term "meromorphic function" means meromorphic in the whole complex plane, unless otherwise explicitly stated. Almost all of our treatment is concentrated in the special case of (1.1):

$$u' = A(z)+u^2 \tag{1.2}$$

[1] The work of the first and the third author was supported in part by the National Science Foundation (MCS 78-02188 and MCS 80-02269). The work of the third author was also supported in part by a research grant from the Finnish Academy.

doi:10.5186/aasfm.1981.0604

with $A(z)$ meromorphic. This can be motivated in the following way. If $c(z)\not\equiv 0$, then the transformation

$$w = \frac{1}{c(z)}u - \frac{b(z)}{2c(z)} - \frac{c'(z)}{2c(z)^2} \tag{1.3}$$

transforms the general equation (1.1) into the special case (1.2) with

$$A = ac - \frac{b^2}{4} + \frac{b'}{2} - \frac{3}{4}\left(\frac{c'}{c}\right)^2 - \frac{b}{2}\frac{c'}{c} + \frac{1}{2}\frac{c''}{c}, \tag{1.4}$$

see [16], p. 77. The special case (1.2) may be considered as a normal form of the Riccati differential equation (1.1) with $c(z)\not\equiv 0$. Because of our main problem, this specialization means no loss of generality. In fact, a solution w of the Riccati differential equation (1.1) is meromorphic (resp. rational) if and only if the corresponding solution of the differential equation (1.2) is meromorphic (resp. rational, if all coefficients of (1.1) are supposed to be rational).

In § 2, we give some criteria for the existence of meromorphic solutions of (1.2). This section is mainly of a technical nature, to be applied in §§ 3—6.

In §§ 3—5, we assume that $A(z)$ is entire. It is well-known that all solutions of (1.2) are meromorphic functions in this case ([4], Satz 4.5). The main results in this part are as follows. In § 3, we are dealing with the general case (1.1) with constant coefficients. Using the Schwarzian derivative, we prove that all nonconstant solutions of the Riccati differential equation (1.1) with $c(z)\equiv c\neq 0$ are either transcendental meromorphic functions or Möbius transformations. In § 4, we assume $A(z)$ to be a nonconstant polynomial of degree n. We prove that if n is odd, all solutions of (1.2) are transcendental (see also [13], p. 285), while if n is even, then (1.2) admits at most one rational solution. Moreover, any finite set of distinct points in the complex plane can be the pole set of a rational function satisfying a differential equation of the form (1.2) with $A(z)$ a nonconstant polynomial. In § 5, $A(z)$ is assumed to be transcendental entire. In this case, the relation $A=u'-u^2$ shows that all solutions of (1.2) are transcendental, and that the order of any meromorphic solution is at least the order of A. We prove that if $A(z)$ is a transcendental entire function, then for any positive function $\varphi(r)$ on $(0, +\infty)$ satisfying the condition that $\limsup_{r\to+\infty}(\log\log\varphi(r)/\log r)<1$, the equation (1.2) possesses at most two distinct meromorphic solutions u_1, u_2 which satisfy the condition $T(r, u_j)=o(\varphi(T(r, A)))$ for $j=1, 2$, as $r\to+\infty$ outside a possible exceptional set of finite linear measure. If the order of $A(z)$ is finite, then the equation (1.2) admits at most two distinct meromorphic solutions of finite order. In addition, when $A(z)$ is of finite order, the number of entire solutions of (1.2) is at most two.

Our final section (§ 6) comprises a major part of this article. Supposing $A(z)$ in the differential equation (1.2) to be non-entire, it appears that the maximum number of distinct meromorphic solutions of (1.2) depends on the highest of the multiplicities of the poles of $A(z)$. More precisely, if all poles of $A(z)$ are simple, then

(1.2) admits at most one meromorphic solution, and if $A(z)$ possesses at least one pole of multiplicity $\geqq 3$, then (1.2) admits at most two distinct meromorphic solutions. The case where $A(z)$ admits at least one double pole and no poles of higher multiplicity, appears to be the most complicated. If $A(z)$ has a double pole at z_0 with the Laurent expansion

$$A(z) = \beta(z-z_0)^{-2}+\dots, \quad \beta \neq 0,$$

and if

$$B = \{1-n^2 | n \text{ is an integer } \geqq 2\},$$

then the existence of a double pole of $A(z)$ with $4\beta \notin B$ indicates that the Riccati differential equation (1.2) admits at most two distinct meromorphic solutions. The maximum number of distinct meromorphic solutions reduces to one, if there exists a double pole of $A(z)$ with $4\beta=1$. The case where all double poles satisfy $4\beta \in B$ remains somewhat problematic. A series of examples, beginning from Example 6.6, illustrate the various possibilities which may occur in the framework of § 6, including the possibility of non-existence of meromorphic solutions. The authors would like to thank here their colleague Matti Jutila, University of Turku, who pointed out some number theoretic information needed to discover Example 6.6.c.

§ 2. Existence of meromorphic solutions

In this section we shall consider the special case

$$u' = A(z)+u^2 \tag{1.2}$$

only, for reasons described in § 1. We should perhaps point out that Proposition 2.1 will be improved in Theorem 2.5.

Proposition 2.1. *If the Riccati differential equation* (1.2) *with* $A(z)$ *meromorphic possesses at least three distinct meromorphic solutions* u_1, u_2, u_3, *then the equation* (1.2) *possesses a one-parameter family* $(u_C)_{C\in \mathbf{C}}$ *of distinct meromorphic solutions with the property that any meromorphic solution* $u \neq u_1$ *of* (1.2) *satisfies* $u=u_C$ *for some* $C \in \mathbf{C}$.

Proof. Suppose that u_1, u_2, u_3 are three distinct meromorphic solutions of (1.2) and denote

$$w_1 = (u_1-u_2)^{-1}, \quad w_2 = (u_1-u_3)^{-1}.$$

Clearly w_1, w_2 both satisfy the linear differential equation

$$w'+2u_1 w = 1. \tag{2.1}$$

Denoting further

$$v_0 = w_1-w_2 \neq 0$$

we get

$$v_0' + 2u_1 v_0 = 0. \tag{2.2}$$

Let us consider the family of distinct meromorphic functions,

$$w_C = w_1 + Cv_0, \quad C \in \mathbf{C}.$$

For any $C \in \mathbf{C}$, the function w_C is a solution of the differential equation (2.1). Therefore $w_C \not\equiv 0$. Hence

$$u_C = u_1 - (w_C)^{-1}$$

is a meromorphic function. An elementary calculation shows that u_C satisfies the differential equation (1.2). Finally, the meromorphic functions $(u_C)_{C\in \mathbf{C}}$ are distinct, since the functions $(w_C)_{C\in \mathbf{C}}$ are distinct.

On the other hand, let $u \not\equiv u_1$ be any meromorphic solution of (1.2). Clearly $w=(u_1-u)^{-1}$ satisfies (2.1) and

$$(w/v_0)' = 1/v_0, \quad (w_1/v_0)' = 1/v_0.$$

Therefore

$$(w/v_0) = C + (w_1/v_0)$$

for some complex constant C. Hence

$$w = w_1 + Cv_0 = w_C$$

resulting in $u = u_C$.

Remark 2.2. By the above proof, all solutions of (1.2) are rational as soon as (1.2) possesses three distinct rational solutions. This is a well-known result due to H. Wittich ([17], p. 284). One should perhaps note already here, that (1.2) may possess a one-parameter family of meromorphic solutions such that these solutions are transcendental except for two, one or no exceptions. This follows by the subsequent results and examples.

Proposition 2.3. *Let u_1 and u_2 be two distinct meromorphic solutions of the differential equation* (1.2) *with $A(z)$ meromorphic. If all poles of u_1 and u_2 are simple and the residues of $2u_1$ (resp. $2u_2$) are integers at all poles of u_1 (resp. u_2), then the equation* (1.2) *possesses a one-parameter family $(u_C)_{C\in \mathbf{C}}$ of distinct meromorphic solutions with the property that any meromorphic solution $u \not\equiv u_1$ satisfies $u = u_C$ for some $C \in \mathbf{C}$.*

Proof. Clearly $w_0 = (u_1-u_2)^{-1}$ satisfies the linear differential equation

$$w' + 2u_1 w = 1. \tag{2.1}$$

Since $2u_1$ has only integer residues, there is a meromorphic function y in the complex plane such that $2u_1 = y'/y$, see, e.g., [12], p. 193. Therefore (2.1) may be written as

$$w' + (y'/y) w = 1. \tag{2.3}$$

All solutions of (2.3) are meromorphic. In fact, w_0 is a particular solution of (2.3) and the solutions of the corresponding homogeneous linear differential equation $w'+(y'/y)w=0$ are plainly meromorphic. The family of distinct solutions of (2.3) may be written

$$w_C = w_0+Cy^{-1},$$

where $C\in\mathbf{C}$. A straightforward calculation proves that

$$u_C = u_1-(w_0+Cy^{-1})^{-1}$$

satisfies (1.2) for all complex constants C. Finally, the meromorphic functions $(u_C)_{C\in\mathbf{C}}$ are distinct, since the functions $(w_C)_{C\in\mathbf{C}}$ are distinct.

On the other hand, let $u\neq u_1$ be any meromorphic solution of (1.2). Clearly $w=(u_1-u)^{-1}$ satisfies (2.3). Hence

$$(u_1-u)^{-1} = w_0+Cy^{-1}$$

for some $C\in\mathbf{C}$, resulting $u=u_C$.

Proposition 2.4. *Suppose that the differential equation* (1.2) *with* $A(z)$ *meromorphic admits a meromorphic solution* u_0 *such that at some pole of* u_0 *the residue of* $2u_0$ *is not an integer. Then* (1.2) *possesses at most two distinct meromorphic solutions.*

Proof. Suppose (1.2) possesses three distinct meromorphic solutions $u_1=u_0$, u_2 and u_3. Following the proof of Proposition 2.1 we may write the formula (2.2) in the form

$$v_0'/v_0 = -2u_0.$$

Therefore the residues of $2u_0$ would all be integers and we have a contradiction.

The following theorem may be considered as a summary of the preceding propositions. One should note that its complete proof will be postponed until § 6.3.

Theorem 2.5. a) *If the Riccati differential equation* (1.2) *with* $A(z)$ *meromorphic admits an entire solution* u_1, *then all solutions of* (1.2) *are meromorphic functions. The collection of these meromorphic solutions has the form* $\mathcal{U}=u_1\cup(u_C)_{C\in\mathbf{C}}$, *where* $(u_C)_{C\in\mathbf{C}}$ *is a one-parameter family of distinct solutions of* (1.2) *with the property that* $u_1\neq u_C$ *for all* $C\in\mathbf{C}$.

b) *If all meromorphic solutions of the Riccati differential equation* (1.2) *with* $A(z)$ *meromorphic are non-entire, then the following possibilities may occur:*

If (1.2) *admits a meromorphic solution* u_1 *such that either*

(i) u_1 *has a pole of multiple order* $\geqq 2$ *or*

(ii) u_1 *has at least one (simple) pole such that the residue of* $2u_1$ *at this pole is not an integer, then* (1.2) *has at most two distinct meromorphic solutions.*

Finally, if all meromorphic solutions u of (1.2) *have just simple poles such that all residues of* $2u$ *are integers, then*
(iii) *the equation* (1.2) *has at most one meromorphic solution or*
(iv) *all solutions of* (1.2) *are meromorphic functions and, supposing u_1 to be a particular (non-entire) solution of* (1.2), *the collection of all meromorphic solutions has the same form as in part* a).

Proof. a) Since u_1 is entire, $A=u_1'-u_1^2$ is an entire function. Therefore all solutions of (1.2) are meromorphic functions ([4], Satz 4.5). To prove the second assertion, we may solve the equation (1.2) explicitly. In fact, let v be a primitive of $2u_1$ and F be a primitive of e^v. Then all solutions of (2.1) are meromorphic and they can be represented in the form

$$w = e^{-v}(C+F),$$

where C is a complex constant. Immediately we may verify that all meromorphic functions of the form

$$u_C = u_1 - e^v(C+F)^{-1} \tag{2.4}$$

satisfy the equation (1.2). Conversely, any meromorphic solution $u \neq u_1$ of (1.2) is of the form (2.4) for some complex constant C. b) Proposition 2.3 proves the cases (iii) and (iv) immediately. Similarly, Proposition 2.4 proves the case (ii). Finally, the case (i) will be postponed until Corollary 6.13. See also Remark 6.14.

§ 3. The Riccati differential equation with constant coefficients

The Riccati differential equation (1.1) with constant coefficients $a(z)\equiv a$, $b(z)\equiv b$, $c(z)\equiv c\neq 0$, is particularly simple, elementary and well-known. Therefore the following proposition actually contains nothing new. However, we have not found in the literature the following simple idea about using the Schwarzian derivative in this connection.

Proposition 3.1. *The Schwarzian derivative*

$$S(w) = (w''/w')' - \tfrac{1}{2}(w''/w')^2$$

of all nonconstant solutions w of the equation (1.1) *with constant coefficients such that $c\neq 0$ is a constant:*

$$S(w) = (4ac-b^2)/2.$$

Therefore, all nonconstant solutions of (1.1) *are either transcendental meromorphic functions, if $S(w)\neq 0$, or Möbius transformations, if $S(w)=0$.*

Proof. The first assertion follows by a straightforward calculation. For the second assertion, we have to prove that the Schwarzian derivative $S(R)$ of a rational function R is a constant if and only if R is a Möbius transformation. To this end, let R be a nonconstant rational function such that $S(R)$ is a non-zero constant, say α. Then $f=R''/R'$ is a rational solution of the differential equation

(3.1) $$f' = \alpha+\tfrac{1}{2}f^2.$$

This implies an immediate contradiction. In fact, the constant solutions of (3.1) result in

$$R(z) = C_1+C_2e^{\pm(z\sqrt{-2\alpha})}$$

and this is never a nonconstant rational function. On the other hand, the nonconstant solutions of (3.1) are never rational.

§ 4. The equation $u'=A(z)+u^2$, $A(z)$ a nonconstant polynomial

It is well-known that all solutions of the differential equation

(1.2) $$u' = A(z)+u^2$$

are meromorphic, as soon as $A(z)$ is a polynomial ([4], Satz 4.5). The following theorem can be considered as a completion of the earlier results due to H. Wittich, see [13] and [16].

Theorem 4.1. *Let $A(z)$ be a nonconstant polynomial of degree n. If n is odd, then all solutions of the differential equation* (1.2) *are transcendental meromorphic functions, and if n is even, then* (1.2) *admits at most one rational solution. Moreover, given any finite set E of distinct points in the complex plane, there exist a nonconstant polynomial $A(z)$ and a rational solution u of the corresponding differential equation* (1.2) *such that the set of finite poles of u coincides with E.*

Proof. Let u be any solution of (1.2). Then u is a meromorphic function such that all poles of u are simple and the residue of u at all poles is -1. Therefore there exists an entire function g such that $u=-g'/g$. Clearly g satisfies the linear differential equation

(4.1) $$g''+A(z)g = 0.$$

If the degree n of $A(z)$ is odd, then g has infinitely many zeros, since the order of g is $(n+2)/2$ by the Wiman—Valiron theory. Therefore its logarithmic derivative cannot be rational, hence u must be transcendental (see [13], p. 285, where this result was already mentioned). Let then n be even and let u_1, u_2 be two distinct solutions of (1.2). Then $u=-g_1'/g_1$ and $u_2=-g_2'/g_2$ for some entire functions g_1, g_2. Since $u_1\neq u_2$, the entire functions g_1, g_2 must be linearly independent. By [1], Theorem 1, at least one of g_1, g_2 possesses infinitely many zeros. Thus the corresponding solution of (1.2) is transcendental.

To prove the second assertion, suppose first $E=\emptyset$. Then we have only to take any nonconstant polynomial $P(z)$. The polynomial $P(z)$ satisfies (1.2) with $A(z)=P'(z)-P(z)^2$. Suppose now that E consists of one point only, say α_1. Then, for any nonconstant polynomial $P(z)$ possessing a zero at $z=\alpha_1$, the rational function $u(z)=-(z-\alpha_1)^{-1}+P(z)$ satisfies (1.2) with $A(z)=P'(z)-P(z)^2+2P(z)/(z-\alpha_1)$. Clearly $A(z)$ will be a nonconstant polynomial.

Before going to prove the general case, we first derive a necessary condition to be satisfied before a rational function

$$u(z) = -\sum_{i=1}^{n}\frac{1}{z-\alpha_i}+P(z),\ P(z) \quad \text{polynomial,} \tag{4.2}$$

can be a solution of (1.2) with a polynomial coefficient $A(z)$. Clearly the degree of $A(z)$ must be at least 2 and the degree of $P(z)$ at least 1. A straightforward calculation gives

$$A(z) = u'(z)-u(z)^2 = P'(z)-P(z)^2+Q(z),$$

where

$$Q(z) = 2P(z)\sum_{i=1}^{n}\frac{1}{z-\alpha_i}-2\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{1}{z-\alpha_i}\frac{1}{z-\alpha_j},$$

is to be a polynomial. Evaluating $Q(z)$ we obtain

$$Q(z) = 2P(z)\sum_{i=1}^{n}\frac{1}{z-\alpha_i}+2\sum_{\substack{i,j=1\\ i<j}}^{n}\left(\frac{(\alpha_j-\alpha_i)^{-1}}{z-\alpha_i}+\frac{(\alpha_i-\alpha_j)^{-1}}{z-\alpha_j}\right)$$

$$= \sum_{k=1}^{n}\frac{2}{z-\alpha_k}\left(P(z)+\frac{1}{\alpha_1-\alpha_k}+\dots+\frac{1}{\alpha_{k-1}-\alpha_k}+\frac{1}{\alpha_{k+1}-\alpha_k}+\dots+\frac{1}{\alpha_n-\alpha_k}\right).$$

Hence the necessary conditions are

$$P(\alpha_k) = \frac{1}{\alpha_k-\alpha_1}+\dots+\frac{1}{\alpha_k-\alpha_{k-1}}+\frac{1}{\alpha_k-\alpha_{k+1}}+\dots+\frac{1}{\alpha_k-\alpha_n} \tag{4.3}$$

for $k=1, \dots, n$.

To prove the second assertion in the general case, let E contain at least two points, say $E=\{\alpha_1, \dots, \alpha_n\}$ with $n\geqq 2$. We may determine the coefficients $\beta_0, \dots, \beta_{n-1}$ of the polynomial

$$P(z) = \sum_{i=0}^{n-1}\beta_i z^i \tag{4.4}$$

in such a way that the rational function (4.2) using $\alpha_1, \dots, \alpha_n$ from the given set E and the polynomial (4.4) is the rational solution we are looking for. In fact, an application of (4.3) results in a linear system of equations

$$\beta_0+\beta_1\alpha_i+\dots+\beta_{n-1}(\alpha_i)^{n-1} = P(\alpha_i), \quad i = 1, \dots, n. \tag{4.5}$$

The linear system of equations (4.5) has a unique non-trivial solution, since its determinant is non-vanishing being the well-known Vandermonde determinant. Therefore (4.5) defines a rational function

$$u(z) = -\sum_{i=1}^{n} \frac{1}{z-\alpha_i} + \sum_{i=0}^{n-1} \beta_i z^i$$

solving the equation (1.2), where

$$A(z) = \frac{d}{dz}\left\{\sum_{i=0}^{n-1} \beta_i z^i\right\} - \left(\sum_{i=0}^{n-1} \beta_i z^i\right)^2 + Q(z).$$

Example 4.2. If $E=\{-1, 0, +1\}$, the above procedure results in

$$u(z) = -\frac{1}{z-1} - \frac{1}{z} - \frac{1}{z+1} + \frac{3}{2} z.$$

This rational function satisfies the Riccati differential equation

$$u' = \frac{21}{2} - \frac{9}{4} z^2 + u^2.$$

§ 5. The equation $u' = A(z) + u^2$, $A(z)$ transcendental entire

In this section we apply the standard notations and results of the Nevanlinna theory, see, e.g., [3]. Specially, the abbreviation n.e. means "everywhere in $(0, +\infty)$ outside a possible exceptional set of finite linear measure".

Again, all solutions of the differential equation

$$u' = A(z) + u^2 \tag{1.2}$$

are meromorphic functions. Because of the relation $A = u' - u^2$ they must be transcendental. Let σ denote the order of $A(z)$. From the same relation we observe that the order $\sigma(u)$ of any solution u of (1.2) satisfies $\sigma(u) \geqq \sigma$. We now obtain the following

Theorem 5.1. *Let $\varphi(r)$ be any positive function on $(0, +\infty)$ satisfying the condition*

$$\limsup_{r\to\infty} \frac{\log\log\varphi(r)}{\log r} < 1. \tag{5.1}$$

If $A(z)$ is a transcendental entire function, then the equation (1.2) *admits at most two distinct meromorphic solutions u_1, u_2 that satisfy the condition*

$$T(r, u_i) = o(\varphi(T(r, A))) \quad \text{n.e. as} \quad r \to \infty \tag{5.2}$$

for $i = 1, 2$.

Proof. Let u_1, u_2, u_3 be three distinct meromorphic solutions of (1.2) that satisfy the condition (5.2). All poles of u_1, if there are any of them, are simple and the residue of u_1 at all poles is -1. Therefore there is an entire function g such that $u_1=-g'/g$. Since g satisfies the linear differential equation

$$g''+A(z)g=0, \tag{4.1}$$

with $A(z)$ transcendental entire, we conclude that

$$T(r, A)=m\left(r, \frac{g''}{g}\right)=O(\log r+\log T(r, g)) \quad \text{n.e. as} \quad r\to\infty. \tag{5.3}$$

Using the notation of Proposition 2.1 we get the equation (2.2) which may be written as

$$v_0'-2(g'/g)v_0=0. \tag{5.4}$$

By (5.4), v_0 must be of the form

$$v_0=\frac{1}{u_1-u_2}-\frac{1}{u_1-u_3}=Cg^2 \tag{5.5}$$

for some complex constant $C\neq 0$. From our hypothesis on φ, there exist positive numbers $\alpha<1$ and $R_1(\alpha)$ such that $\varphi(r)\leqq\exp(r^\alpha)$ for all $r\geqq R_1(\alpha)$. Applying (5.5) we get therefore n.e. as $r\to\infty$ the following inequalities for some real number $M>0$:

$$\begin{aligned}T(r, g)=o(\varphi(T(r, A)))&\leqq o(\exp(T(r, A)^\alpha))\\&\leqq o(\exp[(M\log T(r, g)+M\log r)^\alpha])\\&\leqq o(\exp[M^\alpha(\log T(r, g))^\alpha+M\log r])\\&\leqq o(\exp[\log(T(r, g))^\alpha+M\log r])\\&\leqq o(r^M(T(r, g))^\alpha).\end{aligned}$$

This yields immediately

$$T(r, g)=o(r^{M/(1-\alpha)}) \quad \text{n.e. as} \quad r\to\infty.$$

Because of (5.3) we then obtain

$$T(r, A)=O(\log r) \quad \text{n.e. as} \quad r\to\infty.$$

This is impossible $A(z)$ being a transcendental function.

Corollary 5.2. *If $A(z)$ is a transcendental entire function of finite order, then the equation* (1.2) *admits at most two distinct meromorphic solutions of finite order.*

Proof. Let u_1, u_2, u_3 be three distinct meromorphic solutions of (1.2) of finite order. We may follow the proof of Theorem 5.1 up to the formula (5.5). Therefore g must be of finite order. But then (5.3) implies $T(r, A)=O(\log r)$ which is impossible.

The following two examples (together with Example 5.6) show that Theorem 5.1 is essentially the best possible. Example 5.3 shows that in the condition $\limsup_{r\to\infty} (\log\log\varphi(r)/\log r)<1$ the bound 1 cannot be increased, while Example 5.4 shows that Theorem 5.1 does not hold in general, if $A(z)$ is a non-entire meromorphic function.

Example 5.3. We consider the differential equation

$$u' = A(z)+u^2 \quad \text{with} \quad A(z) = -\frac{1}{4}-\frac{\exp(2z)}{4}. \tag{5.6}$$

Then $A(z)$ is a transcendental entire function such that $T(r, A)=2r/\pi+O(1)$ as $r\to\infty$. Clearly the equation (5.6) is satisfied by

$$u_1(z) = \tfrac{1}{2}(1+e^z) \quad \text{and} \quad u_2(z) = \tfrac{1}{2}(1-e^z).$$

Applying the proof of Theorem 2.5.a) we find that all solutions $u\neq u_1$ of (5.6) are

$$u(z) = \tfrac{1}{2}(1+e^z)-e^z[1+C\exp(-e^z)]^{-1}, \quad C\in\mathbf{C}. \tag{5.7}$$

Consider now the function $\varphi(r)=\exp(2r^\lambda)$. Then

$$\lim_{r\to\infty}\frac{\log\log\varphi(r)}{\log r} = \lambda.$$

Assuming $0<\lambda<1$ the hypothesis of Theorem 5.1 will be satisfied. As $r\to\infty$ we obtain

$$\varphi(T(r, A)) = \exp\left[2\left(\frac{2r}{\pi}\right)^\lambda(1+o(1))\right]$$

and

$$T(r, u_i) = \frac{r}{\pi}+O(1)$$

for $i=1, 2$. Therefore

$$T(r, u_i) = o(\varphi(T(r, A))) \quad \text{as} \quad r\to\infty$$

for $i=1, 2$. Let now $u\neq u_i$, $i=1, 2$, be a meromorphic solution of (5.6). Then u is of the form (5.7) for some $C\neq 0$. Then

$$T(r, u) = e^r(2\pi^3 r)^{-1/2}(1+o(1)) \quad \text{as} \quad r\to\infty,$$

see [3], p. 7. Hence we obtain in this case

$$\lim_{r\to\infty} T(r, u)/\varphi(T(r, A)) = +\infty.$$

On the other hand, if $\lambda\geqq 1$, then all meromorphic solutions of (5.6) satisfy the condition

$$T(r, u) = o(\varphi(T(r, A))) \quad \text{as} \quad r\to\infty.$$

Example 5.4. We consider the Riccati differential equation

$$u' = A(z)+u^2 \quad \text{with} \quad A(z) = -\frac{e^z+1}{(e^z-1)^2}. \tag{5.8}$$

Then

$$u_1(z) = (e^z-1)^{-1}$$

is a meromorphic solution of (5.8). All other meromorphic solutions of (5.8) can be written in the form

$$u(z) = \frac{1}{e^z-1}+\frac{(1-e^{-z})^2}{C-z-2e^{-z}+\frac{1}{2}e^{-2z}}, \quad C\in \mathbf{C}.$$

Choosing $\varphi(r)=\exp(r^\lambda)$ with $0<\lambda<1$, we have

$$\lim_{r\to\infty} \frac{\log\log\varphi(r)}{\log r} = \lambda < 1$$

and

$$\varphi(T(r, A)) = \exp\left[\left(\frac{2r}{\pi}\right)^\lambda (1+o(1))\right] \quad \text{as} \quad r\to\infty.$$

For any meromorphic solution u of (5.8) we have

$$T(r, u) \leqq \frac{3r}{\pi}(1+o(1)) \quad \text{as} \quad r\to\infty$$

and therefore

$$T(r, u) = o(\varphi(T(r, A))) \quad \text{as} \quad r\to\infty.$$

Theorem 5.5. *If $A(z)$ is a transcendental entire function of finite order σ, then the equation* (1.2) *admits at most two distinct entire solutions.*

Proof. Because of Corollary 5.2 it is sufficient to prove that the order $\sigma(u)$ of all entire solutions u of (1.2) satisfies $\sigma(u)=\sigma$. To prove this assertion, let u be an entire solution of (1.2) such that $\sigma(u)>\sigma$. Writing the equation (1.2) in the form

$$u^2 = u'-A(z) \tag{5.9}$$

we may apply a variant of the Tumura—Clunie theory to (5.9), see [1]. If $\sigma(u)<+\infty$, we infer

$$T(r, u) = m(r, u) = O(r^{\sigma+\varepsilon})$$

for all $\varepsilon>0$ by [1], Sect. 7. Therefore we get $\sigma(u)\leqq\sigma$, a contradiction. Finally, if $\sigma(u)=+\infty$, then [1], Sect. 7, may be applied yielding

$$T(r, u) = m(r, u) = O(r^{\sigma+\varepsilon}+\log T(r, u)) \quad \text{n.e. as} \quad r\to\infty.$$

Therefore

$$T(r, u) = O(r^{\sigma+\varepsilon}) \quad \text{n.e. as} \quad r\to\infty. \tag{5.10}$$

It is now easy to prove that, for some r_0, (5.10) holds for all $r > r_0$. Hence $\sigma(u) \leqq \sigma$, a contradiction also in this case.

Example 5.6. Let φ be a nonconstant entire function. Especially φ may be taken to be a polynomial. Let further h be a primitive of the entire function e^{φ} and define $g = -(\varphi + h)/2$. An elementary calculation proves that $u_1 = -g'$ and $u_2 = -g' - h'$ are two entire solutions of the differential equation

$$u' = -\frac{1}{4}(h'^2 + \varphi'^2 - 2\varphi'') + u^2$$

of the form (1.2), where the coefficient standing for $A(z)$ is a transcendental entire function. Clearly $\sigma(u_1) = \sigma(u_2) = \sigma(h) = \sigma(h'^2 + \varphi'^2 - 2\varphi'')$. Therefore the number of exceptional solutions in Theorem 5.1, Corollary 5.2 and Theorem 5.5 is the best possible, as shown also by Example 5.3.

Remark 5.7. Concerning the distribution of poles of meromorphic solutions of (1.2) when $A(z)$ is entire, we have observed already that for a solution u of (1.2) we have $u = -g'/g$, where g is an entire solution of the differential equation $g'' + A(z)g = 0$. Hence the sequence of zeros of g coincides with the sequence of poles of u, and so we can immediately translate the results in [1] concerning the distribution of zeros of solutions of $g'' + A(z)g = 0$ into results about the distribution of poles of solutions of (1.2).

To this end, let $\lambda(f, a)$ denote the exponent of convergence of the sequence of a-points of a meromorphic function f. We have the following results from [1]:

(a) If $A(z)$ is a nonconstant polynomial of degree n, then there is at most one meromorphic solution u of (1.2) for which $\lambda(u, \infty) \neq (n+2)/2$. If n is odd, then all solutions of (1.2) satisfy $\lambda(u, \infty) = (n+2)/2$.

(b) If $A(z)$ is an entire transcendental function whose order $\sigma(A)$ is a finite number which is not an integer, then there is at most one meromorphic solution u of (1.2) with $\lambda(u, \infty) < \sigma(A)$. The same conclusion holds (regardless of the order of $A(z)$) if $A(z)$ has the property that its sequence of distinct zeros has exponent of convergence less than $\sigma(A)$.

(c) In the case where $\lambda(A, 0) < \sigma(A)$ the inequality $\lambda(u, \infty) \geqq \sigma(A)$ holds for all solutions u of (1.2).

(d) For any σ, where $0 \leqq \sigma \leqq \infty$, there is an entire transcendental function $A(z)$ of order σ, such that the equation (1.2) admits an entire solution.

§ 6. The equation $u' = A(z) + u^2$, $A(z)$ meromorphic non-entire

The existence of meromorphic solutions of

$$u' = A(z) + u^2 \tag{1.2}$$

with $A(z)$ meromorphic and non-entire, has been somewhat problematic, even if $A(z)$ is rational. It appears that criteria for the maximum number of distinct meromorphic solutions can be organized according to the highest of the multiplicities of the poles of $A(z)$. Also the subsequent examples have been organized in this way.

6.1. All poles of $A(z)$ are simple.

Theorem 6.1. *If all poles of $A(z)$ are simple, then the equation* (1.2) *admits at most one meromorphic solution.*

Proof. Let u be a meromorphic solution of (1.2). Since $A(z)$ is non-entire, the same conclusion holds for u. Clearly, all poles of u are simple and the residue of u at all poles is -1. Thus there is an entire function g such that $u = -g'/g$. As stated earlier, g satisfies the linear differential equation

$$g'' + A(z)g = 0. \tag{4.1}$$

Let g_1 and g_2 be two linearly independent meromorphic solutions of (4.1) and let h denote the meromorphic function g_2/g_1. A simple application of Abel's identity results in $h' = c(g_1)^{-2}$ for some complex constant $c \neq 0$. Let z_0 be a pole of $A(z)$. By (4.1), g_1 and g_2 both must have a zero at z_0. Suppose the zero of g_1 at z_0 is of multiplicity $\mu \geqq 1$. Then $(g_1)^{-2}$ (resp. h) has a pole of multiplicity 2μ (resp. $2\mu - 1$) at z_0. If $\mu > 1$, then g_2 would have a pole of order $\mu - 1 > 0$ at z_0 by the definition of h. If then $\mu = 1$, then g_2 would have a regular point at z_0 such that $g_2(z_0) \neq 0$, again by the definition of h. Thus we get a contradiction for all $\mu \geqq 1$. Therefore all meromorphic solutions of (4.1) must be linearly dependent, hence their logarithmic derivatives coincide. The assertion follows immediately.

Example 6.2. If $A(z)$ has exactly one simple pole and no poles of higher multiplicity, then the equation (1.2) admits exactly one meromorphic solution. In fact, we may apply the Frobenius method (see, e.g., [6], p. 157) at the pole z_0 of $A(z)$ to conclude that the linear differential equation $g'' + A(z)g = 0$ admits a solution of the form $g_1(z) = (z - z_0)\varphi(z)$, where $\varphi(z)$ is an entire function. This follows from the indical equation $r(r-1) = 0$. Therefore $u_1(z) = -g_1'(z)/g_1(z)$ is a meromorphic function satisfying (1.2). The solution may be either rational or transcendental. In fact, the differential equation

$$u' = \frac{2}{z} - 1 + u^2$$

admits a rational solution

$$u(z) = -\frac{1}{z}+1,$$

while the differential equation

$$u' = \frac{1}{z}+u^2 \tag{6.1}$$

admits a transcendental solution. To see the latter assertion, we consider an entire function g solving the corresponding linear differential equation

$$g''+\frac{1}{z}g = 0.$$

By the Wiman—Valiron theory (see, e.g., [16]) we may determine the order of g resulting $\sigma(g)=1/2$. Therefore g possesses an infinite number of zeros and $u=-g'/g$ is a transcendental meromorphic solution of (6.1).

Example 6.3. The differential equation

$$u' = \frac{1}{z(z+1)}+u^2 \tag{6.2}$$

admits no meromorphic solutions. In fact, let u be a meromorphic solution of (6.2). Then there is an entire function g such that $u=-g'/g$ and that g satisfies the linear differential equation

$$g''+\frac{1}{z(z+1)}g = 0. \tag{6.3}$$

By the Wiman—Valiron theory, g must be a polynomial, say

$$g(z) = c_n z^n+c_{n-1}z^{n-1}+\ldots+c_0, \quad c_n \neq 0.$$

Substituting g into (6.3) we get

$$(z^2+z)[n(n-1)c_n z^{n-2}+(n-1)(n-2)c_{n-1}z^{n-3}+\ldots]+c_n z^n+\ldots+c_0 \equiv 0.$$

Collecting terms of degree n we obtain

$$n(n-1)+1 = 0,$$

contradicting the fact that n is an integer.

6.2. *$A(z)$ admits at least one double pole.*

At the double poles z_0 of $A(z)$ we consider the Laurent expansion

$$A(z) = \beta(z-z_0)^{-2}+\ldots, \quad \beta \neq 0,$$

of $A(z)$. We denote further

$$B = \{1-n^2 | n \text{ is an integer } \geqq 2\}.$$

Theorem 6.4. *Suppose $A(z)$ admits at least one double pole. If there is a double pole of $A(z)$ such that $4\beta \notin B$, then the differential equation*

$$u' = A(z)+u^2 \tag{1.2}$$

admits at most two distinct meromorphic solutions. Moreover, if there is a double pole of $A(z)$ such that $4\beta=1$, then the equation (1.2) *admits at most one meromorphic solution.*

Proof. Let u be a meromorphic solution of (1.2). Clearly it must have a simple pole at z_0. Let the Laurent expansion of u at z_0 be

$$u(z) = \alpha(z-z_0)^{-1}+\dots, \quad \alpha \neq 0. \tag{6.4}$$

Substituting (6.4) into (1.2) we obtain

$$\alpha^2+\alpha+\beta = 0$$

and therefore

$$2\alpha = -1 \pm \sqrt{1-4\beta}. \tag{6.5}$$

Clearly 2α is an integer if and only if $4\beta \in B \cup \{1\}$. If $4\beta \notin B \cup \{1\}$, then (1.2) admits at most two distinct meromorphic solutions by Proposition 2.4.

Finally, suppose that $A(z)$ has a double pole at z_0 such that $4\beta=1$. Let now u_1, u_2 be two distinct meromorphic solutions of (1.2). By (6.4) and (6.5) both of them have at z_0 the Laurent expansion of the form

$$u_i(z) = -\tfrac{1}{2}(z-z_0)^{-1}+\dots, \quad i = 1, 2.$$

Therefore u_1-u_2 is analytic at z_0, and we have

$$w(z_0) \neq 0, \quad \text{where} \quad w = (u_1-u_2)^{-1}.$$

By the equation

$$w'+2u_1(z)w = 1, \tag{2.1}$$

w must have a pole of multiplicity $\mu \geqq 1$ at z_0. Substituting the Laurent expansions

$$w(z) = \gamma(z-z_0)^{-\mu}+\dots, \quad \gamma \neq 0,$$

and

$$w'(z) = -\mu\gamma(z-z_0)^{-\mu-1}+\dots$$

into (2.1) we obtain $\mu=-1$ which is impossible.

Remark 6.5. The case, where $4\beta \in B$ at all double poles of $A(z)$, remains somewhat problematic, even if $A(z)$ is a rational function. Also, if there is a double pole of $A(z)$ such that $4\beta \notin B$, the framework given by Theorem 6.4 contains several possibilities. The remainder of this section, consisting mostly of concrete examples in both of the above cases, serves to illustrate some of these possibilities that may occur.

Example 6.6. The Riccati differential equation

$$u' = \frac{\beta_0}{z^2} + \frac{\beta_1}{(z+1)^2} + u^2, \quad \text{where} \quad \beta_0 \neq 0,\ \beta_1 \neq 0, \tag{6.6}$$

does not admit meromorphic solutions, if the coefficients β_0, β_1 are chosen conveniently.

Suppose u is a meromorphic solution of (6.6). Clearly u has simple poles at $z=0$ and $z=-1$. The corresponding residues $r_0 \neq 0$ and $r_1 \neq 0$ satisfy the quadratic equations

$$r_i^2 + r_i + \beta_i = 0, \quad i = 0, 1. \tag{6.7}$$

The function

$$w(z) = u(z) - \frac{r_0}{z} - \frac{r_1}{z+1}$$

does not vanish identically, because $u(z) = r_0/z + r_1/(z+1)$ does not solve the equation (6.6). Moreover, w has only simple poles with residue -1. Therefore there exists an entire function g such that $w = -g'/g$. Substituting

$$u(z) = \frac{r_0}{z} + \frac{r_1}{z+1} - \frac{g'}{g}$$

into (6.6) and applying (6.7) we obtain after some calculation

$$z(z+1)g'' - 2(r_0 + (r_0+r_1)z)g' + 2r_0 r_1 g = 0. \tag{6.8}$$

In both possible cases, $r_0 + r_1 \neq 0$ or $r_0 + r_1 = 0$, we see by the Wiman—Valiron theory that g must be a polynomial, say

$$g(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0, \quad c_n \neq 0.$$

Substituting into (6.8) and collecting all terms of degree n we get a quadratic equation

$$n^2 - (2(r_0+r_1)+1)n + 2r_0 r_1 = 0. \tag{6.9}$$

The selection of the coefficients β_0, β_1 will be done in four different ways.

a) Suppose first that $0 < \beta_i \leqq 1/4$ for $i = 0, 1$ and $\beta_0 + \beta_1 > 1/4$. Then the residues r_0, r_1 are both real and the discriminant Δ of the quadratic equation (6.9) satisfies

$$\Delta = 1 - 4(\beta_0 + \beta_1) < 0.$$

Therefore n is not real, which is absurd. Hence the equation (6.6) does not admit meromorphic solutions in this case. Note finally that $4\beta_i \notin B$ for $i = 0, 1$. We may have $4\beta = 1$ at both poles of $A(z) = \beta_0 z^{-2} + \beta_1 (z+1)^{-2}$, resp. at one of the two poles or resp. at neither of the two poles, just choosing, say, $\beta_0 = \beta_1 = 1/4$, resp. $\beta_0 = 1/4$, $\beta_1 = 1/5$ or resp. $\beta_0 = \beta_1 = 1/5$.

b) Supposing $4\beta_0=1-n_0^2$, $4\beta_1=1-n_1^2$ for some integers $n_i\geqq 2$, $i=0, 1$, we have $4\beta_0\in B$ and $4\beta_1\in B$. By (6.7) there are two possibilities for both of the residues r_0, r_1:

$$2r_i+1=\pm n_i, \quad i=0, 1.$$

From (6.9) we obtain

$$2n+1=\pm n_0\pm n_1\pm\sqrt{n_0^2+n_1^2-1},$$

with some combination of the three $\pm$ signs. Choosing now n_0, n_1 such that $n_0^2+n_1^2-1\notin\{k^2|k\in N\}$ we infer that n cannot be an integer, a contradiction.

c) The special case of (6.6) with $4\beta_0=-48=1-7^2$ and $4\beta_1=-15=1-4^2$ is perhaps of some interest. We may apply the preceding case b) to conclude that all possible meromorphic solutions of (6.6) are of the type

$$u(z)=\frac{r_0}{z}+\frac{r_1}{z+1}-\frac{g'}{g},$$

where g is a polynomial. The possible residues r_0, r_1 of $u(z)$ at $z=0$, $z=-1$ will be defined by

$$2r_0+1=\pm 7, \quad 2r_1+1=\pm 4,$$

and all possible degrees n of the polynomial g arise from

$$2n+1=\pm 7\pm 4\pm\sqrt{7^2+4^2-1}=\pm 7\pm 4\pm 8$$

with convenient combinations of the $\pm$ signs. Since n is to be a positive integer, four combinations are actually possible, determining then also r_0 and r_1. These combinations yield

$$r_0=3, \quad r_1=3/2, \quad n=1;$$
$$r_0=3, \quad r_1=3/2, \quad n=9;$$
$$r_0=3, \quad r_1=-5/2, \quad n=5;$$
$$r_0=-4, \quad r_1=3/2, \quad n=2.$$

Determining the coefficients of the polynomial g, we obtain the corresponding solutions of the equation $u'=-12z^{-2}-(15/4)(z+1)^{-2}+u^2$:

$$u_1(z)=\frac{3}{z}+\frac{3}{2(z+1)}-\frac{1}{z+(2/3)};$$

$$u_C(z)=\frac{3}{z}+\frac{3}{2(z+1)}-\frac{63z^8+144z^7+84z^6+C}{7z^9+18z^8+12z^7+C(z+(2/3))}, \quad C\in \mathbf{C};$$

$$u_2(z)=\frac{3}{z}-\frac{5}{2(z+1)}-\frac{35z^4-40z^3+30z^2-16z+5}{7z^5-10z^4+10z^3-8z^2+5z-2};$$

$$u_3(z)=-\frac{4}{z}+\frac{3}{2(z+1)}-\frac{14z+18}{7z^2+18z+12}.$$

Referring to Proposition 2.3 we remark that $u_2(z)$ arises from $u_C(z)$ with $C=-3$ and $u_3(z)$ with $C=0$.

d) Finally, if $4\beta_0=1$ and $4\beta_1=1-n_1^2 \in B$, the discriminant Δ of the quadratic equation (6.9) must be of the form

$$\Delta = n_1^2-1.$$

Therefore n determined from (6.9) cannot be a positive integer, a contradiction. Thus the equation (6.6) cannot admit meromorphic solutions.

Example 6.7. Let $g(z)$ be an entire function and $\beta \neq 0$ be a constant. We consider the Riccati differential equation

$$u' = \frac{\beta}{z^2}+g(z)+u^2 \tag{6.10}$$

dividing our considerations into several subcases.

a) If $4\beta=1$ and $g(z)\equiv 0$, the equation (6.10) admits exactly one meromorphic solution, namely the rational function

$$u(z) = -\frac{1}{2z}.$$

b) Suppose then $4\beta=1$ and $g(z)\not\equiv 0$. By the Frobenius method, the linear differential equation

$$\varphi''+\frac{1}{z}\varphi'+g(z)\varphi = 0 \tag{6.11}$$

possesses an entire solution $\varphi \neq 0$. We observe immediately that

$$u(z) = -\frac{1}{2z}-\frac{\varphi'}{\varphi}$$

is the only meromorphic solution of (6.10); see also Theorem 6.4. This solution may be rational or transcendental, depending on $g(z)$. For instance, the equation

$$u' = \frac{1}{4z^2}-z^2+2+u^2$$

admits a rational solution

$$u(z) = -\frac{1}{2z}+z,$$

while the equation

$$u' = \frac{1}{4z^2}+1+u^2$$

admits a transcendental meromorphic solution. In the latter case, the linear differential equation (6.11) takes the form

$$z^2\varphi''+z\varphi'+z^2\varphi = 0.$$

This will be solved by the Bessel function

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}(z/2)^{2m}$$

of order zero. Since $J_0(z)$ is an even entire function of order one, it has infinitely many zeros. Therefore

$$u(z) = -\frac{1}{2z} - \frac{J_0'(z)}{J_0(z)}$$

is a transcendental function.

c) Let us consider now the case $4\beta \notin B \cup \{1\}$ assuming that $g(z) \equiv 0$. By Theorem 6.4, there exist at most two distinct meromorphic solutions of (6.10). Let α_1, α_2 be the (distinct) roots of the quadratic equation

$$\alpha^2 + \alpha + \beta = 0. \tag{6.12}$$

Then we observe that the equation (6.10) admits exactly two distinct rational solutions, namely

$$u_1(z) = \frac{\alpha_1}{z} \quad \text{and} \quad u_2(z) = \frac{\alpha_2}{z}.$$

d) Suppose now that $4\beta \notin B \cup \{1\}$ and that $g(z) \not\equiv 0$. Let again α_1, α_2 be the (distinct) roots of (6.12). By the Frobenius method, the linear differential equations

$$\varphi'' - \frac{2\alpha_i}{z}\varphi' + g(z)\varphi = 0, \quad i = 1, 2,$$

both admit an entire solution, say φ_1, φ_2, respectively, such that $\varphi_1(0) = \varphi_2(0) = 1$. It is again an immediate observation that

$$u_1(z) = \frac{\alpha_1}{z} - \frac{\varphi_1'(z)}{\varphi_1(z)} \quad \text{and} \quad u_2(z) = \frac{\alpha_2}{z} - \frac{\varphi_2'(z)}{\varphi_2(z)}$$

both satisfy the equation (6.10). Since $\varphi_1(0) = \varphi_2(0) \neq 0$, u_1 and u_2 have different residues at $z = 0$. Therefore u_1 and u_2 are distinct meromorphic functions. Again rational and transcendental solutions may appear. To see this fact we consider some specific examples.

We first consider the differential equation

$$u' = \frac{3}{16z^2} + \frac{5}{2} - z^2 + u^2 \tag{6.13}$$

solved by the rational function

$$u_1(z) = -\frac{3}{4z} + z.$$

The second meromorphic solution u_2 of (6.13) must be of the form

$$u_2(z) = -\frac{1}{4z} - \frac{\varphi'(z)}{\varphi(z)},$$

where φ is an entire solution of the linear differential equation

$$\varphi'' + \frac{1}{2z}\varphi' + \left(\frac{5}{2} - z^2\right)\varphi = 0 \tag{6.14}$$

such that $\varphi(0)=1$. Now φ cannot be a polynomial, since otherwise φ''/φ and φ'/φ would tend to 0 as $z\to\infty$, and the differential equation (6.14) would give a contradiction. Therefore we may apply the Wiman—Valiron theory to determine the order of φ resulting $\sigma(\varphi)=2$. The equation (6.14) is also satisfied by $h(z)=\varphi(-z)$. If $\varphi(z)$ and $\varphi(-z)$ were linearly independent, then all solutions of (6.14) would be entire functions. This is impossible, since the indical equation $r^2-r/2=0$ for (6.14) at $z=0$ implies that (6.14) admits a non-entire solution of the form $z^{1/2}\omega(z)$, where $\omega\neq 0$ is entire (see, e.g., [6], p. 157). Therefore we must have $\varphi(z)=C\varphi(-z)$ for some complex constant C. Since $\varphi(0)=1$, we find $C=1$ and φ must be an even function of order $\sigma(\varphi)=2$. Suppose now φ has only finitely many zeros. Therefore $\varphi(z)=P(z)\exp(az^2)$, where $P(z)$ is a polynomial of degree n. Since $\varphi(z)=\exp(az^2)$ does not satisfy (6.14), we have $n\geqq 1$. Substitution into the equation (6.14) results in

$$\frac{P''(z)}{P(z)} + 4a^2z^2 + 4az\frac{P'(z)}{P(z)} + \frac{1}{2z}\frac{P'(z)}{P(z)} + 3a + \frac{5}{2} \equiv z^2. \tag{6.15}$$

Letting $z\to\infty$ we conclude from (6.15) that

$$4a^2 = 1 \quad \text{and} \quad a(4n+3) = -5/2.$$

This is impossible because n is a positive integer. Therefore φ must have infinitely many zeros and the solution $u_2(z)$ of (6.13) must be transcendental.

Our second specific example is the differential equation

$$u' = \frac{3}{16z^2} + 1 + u^2 \tag{6.16}$$

admitting two distinct transcendental meromorphic solutions. In fact, the linear differential equations given by the Frobenius method

$$\varphi'' + \frac{3}{2z}\varphi' + \varphi = 0 \tag{6.17}$$

and

$$\psi'' + \frac{1}{2z}\psi' + \psi = 0 \tag{6.18}$$

both admit an entire solution $\not\equiv 0$ and the corresponding meromorphic solutions of (6.16) have the form

$$u_1(z) = -\frac{3}{4z} - \frac{\varphi'(z)}{\varphi(z)}$$

and

$$u_2(z) = -\frac{1}{4z} - \frac{\psi'(z)}{\psi(z)}.$$

Let us consider more closely the solution $u_1(z)$. We may again prove u_1 to be transcendental by showing that φ has infinitely many zeros. A similar reasoning as in the preceding example shows that φ must be an even function. Now φ cannot be a polynomial, since otherwise φ''/φ and φ'/φ would tend to 0 as $z\to\infty$, and the differential equation (6.17) would give a contradiction. An application of the Wiman—Valiron theory determines the order of φ resulting $\sigma(\varphi)=1$. If φ has only finitely many zeros, it must be of the form $\varphi(z)=P(z)\exp(az)$, where $P(z)$ is a polynomial and $a\neq 0$ is a constant. This is absurd, since φ is to be even. Finally, we may apply a similar reasoning to prove that u_2 is transcendental just using (6.18) instead of (6.17).

e) Suppose now that $4\beta\in B$ and that $g(z)\equiv 0$. Then the equation (6.10) has the form

$$u' = \frac{1-n^2}{4z^2} + u^2,$$

where n is an integer $\geqq 2$. This equation has rational solutions only, namely the function

$$u_1(z) = -\frac{n+1}{2z}$$

and the one-parameter family of rational functions

$$u(z) = -\frac{n+1}{2z} - \left[Cz^{n+1} - \frac{z}{n}\right]^{-1}, \quad C\in\mathbf{C}.$$

f) Finally, suppose that $4\beta\in B$ and that $g(z)\not\equiv 0$. In this case the equation (6.10) has either one meromorphic solution or a one-parameter family of meromorphic solutions. Both of these two cases may actually occur. The possible residues of any meromorphic solution u of (6.10) at $z=0$ are the distinct roots $\alpha_1=(n-1)/2$ and $\alpha_2=-(n+1)/2$ of the quadratic equation $\alpha^2+\alpha+\beta=0$. The linear differential equation corresponding to α_1 is

$$\varphi'' - \frac{n-1}{z}\varphi' + g(z)\varphi = 0 \tag{6.19}$$

whose indical equation has the roots $r=0$ and $r=n\geqq 2$. By the Frobenius method, the equation (6.19) admits at least one entire non-vanishing solution φ. We have two cases to consider.

1) The equation (6.19) possesses two linearly independent entire solutions φ_1, φ_2. In this case

$$u_1(z) = \frac{n-1}{2z} - \frac{\varphi_1'(z)}{\varphi_1(z)} \quad \text{and} \quad u_2(z) = \frac{n-1}{2z} - \frac{\varphi_2'(z)}{\varphi_2(z)}$$

are two distinct meromorphic solutions of (6.10) by the linear independence of φ_1 and φ_2. Therefore the equation (6.10) admits a one-parameter family of meromorphic solutions by Theorem 2.5.

As a specific example, let us consider the differential equation

$$u' = \frac{1-n^2}{4z^2} + n^2 z^{2n-2} + u^2,$$

where n is an integer $\geqq 2$. This equation has a one-parameter family of transcendental meromorphic solutions, namely

$$u(z) = \frac{n-1}{2z} + nz^{n-1} \tan(C+z^n), \quad C \in \mathbf{C}.$$

The corresponding linear differential equation (6.19) has now linearly independent entire solutions $\varphi_1(z) = \cos(z^n)$ and $\varphi_2(z) = \sin(z^n)$.

2) All entire solutions φ of the equation (6.19) are linearly dependent. In this case, the equation (6.10) has exactly one meromorphic solution. To prove this, let φ be an entire non-vanishing solution of (6.19). Then

$$u_1(z) = \frac{n-1}{2z} - \frac{\varphi'(z)}{\varphi(z)}$$

is a meromorphic solution of (6.10). If

$$u(z) = \frac{n-1}{2z} - \frac{\psi'(z)}{\psi(z)}$$

is another meromorphic solution of (6.10) with ψ an entire function, then ψ must satisfy (6.19). Therefore $\psi = C\varphi$ for some complex constant C and we get $u = u_1$. All other possible meromorphic solutions u of (6.10) must be of the form

$$u(z) = -\frac{n+1}{2z} - \frac{f'(z)}{f(z)}$$

where f is an entire function. Defining $\psi_0(z) = z^n f(z)$ we conclude that

$$u(z) = \frac{n-1}{2z} - \frac{\psi_0'(z)}{\psi_0(z)}.$$

Since ψ_0 is an entire function, we have $u = u_1$ applying the same reasoning as above. Therefore u_1 is the only meromorphic solution of (6.10).

To illustrate this case, we consider the equation

$$u' = -\frac{3}{4z^2}+1+u^2.$$

The equation (6.19) takes now the form

$$\varphi''-\frac{1}{z}\varphi'+\varphi = 0.$$

Under the transformation $f(z)=z^{-1}\varphi(z)$ we get

$$z^2f''+zf'+(z^2-1)f=0.$$

This is Bessel's equation of order one whose entire solutions all are linearly dependent. Therefore the same is true of (6.19) in this special case. Hence the equation $u'=-(3/4)z^{-2}+1+u^2$ has only one meromorphic solution, namely the function

$$u_1(z) = \frac{1}{2z}-\frac{(zJ_1(z))'}{zJ_1(z)}$$

where J_1 is the Bessel function of order one.

Example 6.8. A concrete example slightly different to Example 6.7 is given by the Riccati differential equation

$$u' = \frac{15}{64z^2}-\frac{3}{8z}-\frac{1}{4}+u^2 \tag{6.20}$$

which admits a rational solution

$$u_1(z) = -\frac{3}{8z}-\frac{1}{2}$$

and a transcendental meromorphic solution $u_2(z)$. Since $4\beta\notin B$, these are the only meromorphic solutions of (6.20) by Theorem 6.4. To find out the transcendental solution $u_2(z)$ we consider the linear differential equation

$$y''+\left(1+\frac{3}{4z}\right)y'=0, \tag{6.21}$$

which has a regular singular point at $z=0$ with the indicial equation $r(r-1)+3r/4=0$ whose roots are $r=1/4$ and $r=0$. By the Frobenius method, (6.21) has a solution of the form $y(z)=z^{1/4}\varphi(z)$, where φ is an entire function such that $\varphi(0)=1$. Substituting back into (6.21) results

$$\varphi''+\left(1+\frac{5}{4z}\right)\varphi'+\frac{1}{4z}\varphi = 0. \tag{6.22}$$

Therefore the meromorphic function

$$u_2(z) = -\frac{5}{8z}-\frac{1}{2}-\frac{\varphi'(z)}{\varphi(z)}$$

satisfies (6.20). It remains to show that u_2 is transcendental by showing that φ has infinitely many zeros. If φ were a (nonconstant) polynomial, then, by equating the leading coefficients in (6.22), we infer that the degree of φ would be $-1/4$, which is absurd. Therefore φ is transcendental and we may apply the Wiman—Valiron theory to conclude $\sigma(\varphi)=1/2$. A theorem of Borel (see, e.g., [2], Theorem 2.9.2) implies that φ must have infinitely many zeros.

Example 6.9. Let us consider the Riccati differential equation

$$w' = a(z)+b(z)w+c(z)w^2, \quad c \neq 0, \tag{1.1}$$

with polynomial coefficients. It is well-known that all solutions of (1.1) are meromorphic functions. Transforming (1.1) into the normal form (1.2) tells us that all solutions of (1.2) must be meromorphic in this case. Since $A(z)$ in (1.2) has the form

$$A = ac-\frac{b^2}{4}+\frac{b'}{2}-\frac{3}{4}\left(\frac{c'}{c}\right)^2-\frac{b}{2}\frac{c'}{c}+\frac{1}{2}\frac{c''}{c} \tag{1.4}$$

we see that all possible poles of $A(z)$ appear at the zeros of $c(z)$. Writing $c(z)=\gamma(z-z_0)^{n-1}(1+Q(z))$, where $Q(z)$ is a polynomial such that $Q(z_0)\neq -1$, n is an integer $\geqq 2$ and $\gamma\neq 0$ a constant, we see that $A(z)$ must have a double pole at z_0. The Laurent expansion of $A(z)$ at z_0 satisfies $4\beta=1-n^2\in B$ in this case, see Theorem 6.4.

The following Proposition describes a fairly general situation which turns out to be a special case of the preceding Example 6.9.

Proposition 6.10. (i) *Suppose that the differential equation* (1.2) *with a rational coefficient* $A(z)$ *admits two meromorphic solutions* u_1 *and* u_2 *such that at least one of them, say* u_1, *is rational. If* $w=(u_1-u_2)^{-1}$ *has a pole at all poles of* u_1, *then there exist two polynomials* $P\neq 0$ *and* Q *such that*

$$u_1(z) = \frac{1}{2}\left(\frac{P'(z)}{P(z)}+Q'(z)\right). \tag{6.23}$$

Further, all solutions $u\neq u_1$ *of* (1.2) *are meromorphic and can be represented in the form*

$$u(z) = u_1(z)-P(z)e^{Q(z)}[C+g(z)]^{-1}, \tag{6.24}$$

where C *is a complex constant and* g *is a primitive of the function* Pe^Q.

(ii) *Conversely, if a rational function* u_1 *is of the form* (6.23) *and a meromorphic function* u *is of the form* (6.24), *then both of them are solutions of a differential equation* (1.2), *where the coefficient* $A(z)=u_1'(z)-u_1(z)^2$ *is a rational function. In the* *particular case when* $Q'(z)\equiv\alpha\in\boldsymbol{C}$, (6.24) *reduces into the form*

$$u(z)=u_1(z)-\frac{S'(z)+\alpha S(z)}{S(z)+Ce^{-\alpha z}},$$

where C *is a complex constant and* S *is a polynomial.*

Remark 6.11. 1) Proposition 6.10 is a special case of Example 6.9. In fact, by a straightforward computation we observe that the equation (1.2) with $A(z)=u_1'(z)-u_1(z)^2$, $u_1(z)$ given by (6.23), arises from the differential equation

$$w'=Q'(z)w+P(z)w^2$$

via the transformation (1.3).

2) We note that the family of solutions given by (6.23) and (6.24) contains exactly two rational functions, if $Q'(z)\equiv\alpha\neq 0$. All solutions in this family are rational functions, if $Q'(z)\equiv 0$.

Proof of Proposition 6.10. (i) Clearly w satisfies the linear differential equation

$$w'+2u_1w=1. \tag{2.1}$$

Since w has a pole whenever u_1 has a pole, all poles of u_1 must be simple and the residue of $2u_1$ at any pole must be a positive integer by inspection of (2.1). Therefore, there is an entire function h such that $2u_1=h'/h$. Since u_1 is rational, h must be of the form $h=Pe^Q$, where $P\neq 0$ is a polynomial and Q is an entire function. By differentiating we get $2u_1=Q'+P'/P$. Again, since u_1 is rational, Q must be a polynomial and we have (6.23).

Let now g be a primitive of h. Substituting (6.23) into (2.1) we observe that $w'+(h'/h)w=1$. Hence $hw=C_1+g$ for some complex constant C_1. Now we may apply Proposition 2.3 and its proof to deduce that all solutions $u\neq u_1$ of (1.2) are meromorphic and can be represented in the form

$$u(z)=u_1(z)-h(z)[C_1+g(z)+C_2]^{-1},$$

where C_2 is a complex constant. Denoting $C=C_1+C_2$ we obtain (6.24).

(ii) The first assertion in (ii) is a straightforward computation which may be omitted. To prove the second assertion, suppose that $Q'(z)\equiv\alpha\in\boldsymbol{C}$. It is easy to see that there is no loss in generality if we assume that $Q(z)\equiv\alpha z$ in (6.24). By the method of undetermined coefficients, we can find a polynomial S such that $S'+\alpha S=P$. Then $v=Se^Q$ satisfies $v'=Pe^Q$. Therefore $v=g+\gamma$, where γ is a constant. Hence $g=Se^{\alpha z}-\gamma$. From (6.24) we obtain the second assertion in (ii).

6.3. $A(z)$ *admits at least one pole of multiplicity* $\geqq 3$.

Theorem 6.12. *Suppose* $A(z)$ *admits at least one pole of multiplicity* $m\geqq 3$. *If there exists at least one pole of odd multiplicity* $m\geqq 3$, *then there exist no solutions* *of the differential equation*

$$u' = A(z)+u^2 \tag{1.2}$$

meromorphic in the complex plane. If all poles of $A(z)$ *with multiplicity* $m\geqq 3$ *are of even multiplicity, then there exist at most two distinct meromorphic solutions of* (1.2). *Moreover, if* $A(z)$ *has a pole of even multiplicity* $\geqq 4$ *at* z_0, *and* u_1, u_2 *are two distinct meromorphic solutions of* (1.2), *then* u_1+u_2 *has a simple pole at* z_0 *with residue* $-m/2$.

Proof. Suppose u_1 is a meromorphic solution of (1.2) and consider a pole of $A(z)$ of multiplicity $m\geqq 3$ at z_0. Clearly u_1 must have a pole of multiplicity $\alpha=m/2$ at z_0, since $m\geqq 3$. If m is odd, the assertion follows immediately. Therefore, we may assume that $m=2\alpha\geqq 4$ is even. Suppose that the Laurent expansions of $A(z)$ and $u_1(z)$ at z_0 are

$$\begin{cases} A(z) = d_m(z-z_0)^{-m}+\dots \\ u_1(z) = c_\alpha(z-z_0)^{-\alpha}+\dots \end{cases}$$

Substituting these expansions into (1.2) we obtain $c_\alpha^2=-d_m$. Let u_2 be any other meromorphic solution of (1.2) with the Laurent expansion

$$u_2(z) = e_\alpha(z-z_0)^{-\alpha}+\dots$$

at z_0. Then similarly $e_\alpha^2=-d_m$ and therefore $e_\alpha^2=c_\alpha^2$. But $w=(u_1-u_2)^{-1}$ satisfies

$$w'+2u_1w = 1,$$

and so w must have a zero of multiplicity α at z_0, because of $\alpha\geqq 2$. Therefore we must necessarily have $e_\alpha=-c_\alpha$. Finally, let u_3 be a third meromorphic solution of (1.2) and suppose that the solutions u_1, u_2 and u_3 are distinct. If u_3 has the Laurent expansion

$$u_3(z) = f_\alpha(z-z_0)^{-\alpha}+\dots$$

at z_0, then, by repeating the above reasoning, we see that $f_\alpha=-c_\alpha$, and therefore $e_\alpha=f_\alpha$. On the other hand, beginning the above reasoning with u_2 instead of u_1 and setting $w=(u_2-u_3)^{-1}$ we infer $e_\alpha=-f_\alpha$, a contradiction.

To prove the last assertion, we observe that $w=(u_1-u_2)^{-1}$ satisfies

$$\begin{cases} w'+2u_1w = 1 \\ w'+2u_2w = -1 \end{cases}$$

and therefore

$$\frac{w'}{w} = -(u_1+u_2).$$

Since w has a zero of multiplicity α at z_0, the function u_1+u_2 must have a simple pole (with the residue $-\alpha$) at z_0.

Corollary 6.13. *If the Riccati differential equation* (1.2) *admits a meromorphic solution which has at least one pole of multiplicity* $\geqq 2$, *then there are at most two distinct meromorphic solutions of* (1.2).

Proof. Let u be a meromorphic solution of (1.2) with a pole of multiplicity $\alpha \geqq 2$ at z_0. Then $A(z)=u'(z)-u(z)^2$ has a pole of multiplicity $2\alpha \geqq 4$ at z_0.

Remark 6.14. The proof of Theorem 2.5 is now complete.

Example 6.15. Let h be a nonconstant entire function and let $\alpha \geqq 2$ be an integer. Then the Riccati differential equation

$$n' = \frac{h''(z)}{2} - \frac{(h'(z))^2}{4} + \frac{\alpha h'(z)}{2z} + \frac{2\alpha-\alpha^2}{4z^2} - \frac{e^{2h(z)}}{4z^{2\alpha}} + u^2$$

admits exactly two transcendental meromorphic solutions

$$u(z) = \frac{h'(z)}{2} - \frac{\alpha}{2z} \pm \frac{e^{h(z)}}{2z^{\alpha}}.$$

Example 6.16. Choosing $h=0$ in Example 6.15 we get the Riccati differential equation

$$u' = \frac{2\alpha-\alpha^2}{4z^2} - \frac{1}{4z^{2\alpha}} + u^2,$$

which admits exactly two rational solutions

$$u(z) = -\frac{\alpha}{2z} \pm \frac{1}{2z^{\alpha}}.$$

Example 6.17. The Riccati differential equation

$$u' = 3z^{-4} - z^{-6} + u^2 \tag{6.27}$$

admits exactly one meromorphic solution, namely the rational function

$$u_1(z) = -z^{-3}.$$

To prove this assertion, let u_2 be another meromorphic solution of (6.27). By a theorem of Wittich ([17], p. 283—284), u_2 must be rational. Then $w=(u_1-u_2)^{-1}$ is a rational solution of the linear differential equation

$$w' - 2z^{-3}w = 1. \tag{6.28}$$

It is easy to see that w has no poles and it must have a zero of multiplicity three at $z=0$. Hence w is a polynomial of the form

$$w(z) = c_3 z^3 + \ldots + c_n z^n$$

with $c_3 \neq 0$, $c_n \neq 0$ and $n \geqq 3$. Therefore

$$\lim_{z\to\infty} [w'(z) - 2z^{-3}w(z)] = +\infty.$$

This contradiction with (6.28) proves the assertion.

Example 6.18. There are no meromorphic solutions of the differential equation

$$u' = cz^{-4}+dz^{-2}+u^2, \tag{6.29}$$

where $c\neq 0$ is a complex constant and $d\neq 0$ is a real constant such that the equation $\chi^2-\chi+d=0$ does not have any positive integer roots. Suppose u would be a meromorphic solution of (6.29). Then u must have a double pole at $z=0$. Substituting the Laurent expansion

$$u(z) = az^{-2}+bz^{-1}+\alpha_0+\alpha_1 z+\dots \tag{6.30}$$

into (6.29) we conclude that

$$a^2+c = 0 \quad \text{and} \quad 2ab = -2a.$$

Since $a\neq 0$, we get $b=-1$. Therefore the expansion (6.30) must have the form

$$u(z) = az^{-2}-z^{-1}+\alpha_0+\alpha_1 z+\dots$$

Hence

$$w(z) = u(z)-az^{-2}$$

has only simple poles and the residue at each pole is -1. Since the function az^{-2} does not satisfy (6.29), $w\neq 0$ and there exists an entire function g such that $w=-g'/g$. Substituting

$$u(z) = az^{-2}-\frac{g'}{g}$$

into (6.29) and taking into account $a^2+c=0$ we obtain

$$z^3 g''-2azg'+(dz+2a)g = 0. \tag{6.31}$$

By the Wiman—Valiron theory, g must be a polynomial, say

$$g(z) = c_n z^n+c_{n-1}z^{n-1}+\dots+c_0$$

with $c_n\neq 0$. Substituting this into (6.31) and collecting terms of degree $n+1$ we get

$$n^2-n+d = 0.$$

By our condition imposed upon d, n cannot be a positive integer and we have a contradiction.

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Received 28 May 1981