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Commentationes in memoriam
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ON NEVANLINNA'S PROXIMITY FUNCTION

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1. On the growth of and $T(r,f)$ and $T(r,f')$

Let f be a transcendental meromorphic function in the plane. We denote by $T=T(r,f)$ and $T'=T(r,f')$ the characteristic functions of f and f'. Nevanlinna [4, p. 104, and 5, p. 236] conjectured that

$$1+o(1) \leqq \frac{T'}{T} \leqq 2+o(1)$$

as $r\to\infty$ outside an exceptional set E of values of r. This conjecture holds in the following form.

Theorem 1. *Let f be a transcendental meromorphic function and let $\varphi(r)$ be any positive and increasing function of r such that $\varphi(r)\to\infty$ as $r\to\infty$. Then*

$$\limsup_{r\to\infty} \frac{T(r,f')}{T(r+1/r,f)} \leqq 2, \tag{1}$$

$$\liminf_{r\to\infty} \frac{T(r\varphi(r),f')}{T(r,f)} \geqq \frac{1}{2}, \tag{2}$$

and

$$\limsup_{r\to\infty} \frac{T(Kr,f')}{T(r,f)} \geqq 1 \tag{3}$$

for some $K\geqq 1$, and if f is an entire transcendental function, then

$$\limsup_{r\to\infty} \frac{T(r,f')}{T(r+1/r,f)} \leqq 1 \tag{4}$$

and

$$\liminf_{r\to\infty} \frac{T(r\varphi(r),f')}{T(r,f)} \geqq 1. \tag{5}$$

The inequalities (1) and (4) follow from Lemma 1 of Nevanlinna [5, p. 244], (2) and (5) follow from Lemma 1 of Hayman [3, p. 99], and (3) is a consequence of the following result of [7].

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Theorem 2. *Let f be a transcendental meromorphic function of lower order zero. Then*

(6) $$\limsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \geqq 1$$

and there exists a sequence r_n, $r_n \to \infty$ as $n \to \infty$, such that

(7) $$m(r_n, f) = m(r_n, f') + o(T(r_n, f)) \quad as \quad n \to \infty.$$

Theorem 2 shows that we may take $K=1$ in (3) for functions of order zero. If the order of f is infinite, it is not difficult to see that (3) holds for all $K>1$. In the other direction, we have

Theorem 3. *Given any ε, $0<\varepsilon<\infty$, there exist a meromorphic function f of order ε and $K>1$ such that*

(8) $$\limsup_{r\to\infty} \frac{T(Kr,f')}{T(r,f)} < 1$$

and that for some $\delta>0$,

(9) $$m(r,f) \geqq m(r,f') + \delta T(r,f)$$

for all large values of r.

Proof. Such a function f is constructed in the proof of Theorem 2 of [7].

If f is an entire function, the following a little stronger result than (3) holds (unpublished).

Theorem 4. *There exists an absolute constant $Q>1$ such that*

(10) $$\limsup_{r\to\infty} \frac{T(Qr,f')}{T(r,f)} \geqq 1$$

for any transcendental entire function f.

It is not possible to take $Q=1$ in Theorem 4, for in [6] an entire function f of order one is constructed such that f satisfies (8) for some $K>1$.

The following theorem shows that the constant 1/2 in (2) cannot be replaced by a larger one, and that (7) need not hold for all large values of r, not even for slowly growing functions.

Theorem 5. *Let $\varphi(r)$ be as in Theorem 1. There exists a transcendental meromorphic function f satisfying*

(11) $$T(r,f) = O(\varphi(r)\log r) \quad as \quad r\to\infty$$

such that for some sequences r_n, $r_n\to\infty$ as $n\to\infty$, and K_n, $K_n\to\infty$ as $n\to\infty$,

(12) $$\lim_{n\to\infty} \frac{T(K_n r_n, f')}{T(r_n,f)} = \frac{1}{2},$$

(13) $$m(r_n, f') = 0 \quad for\ any \quad n,$$

and

(14) $$m(r_n, f) = \left(\frac{1}{2}+o(1)\right) T(r_n, f) \quad as \quad n \to \infty.$$

Proof. Such a function f is constructed in the proof of Theorem 3 of [7].

For slowly growing functions, Hayman [3] proved the following result stronger than Theorem 1.

Theorem 6. *Suppose that f is meromorphic in the plane and not a linear polynomial, further that*

(15) $$T(r,f) = O((\log r)^2) \quad as \quad r \to \infty.$$

Then

(16) $$\frac{1}{2} \leqq \liminf_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqq \limsup_{r\to\infty} \frac{T(r,f')}{T(r,f)} \leqq 2.$$

If, further, f is a transcendental integral function, then

(17) $$\lim_{r\to\infty} \frac{T(r,f')}{T(r,f)} = 1.$$

In [7] it is proved that the growth condition (15) in Theorem 6 can be replaced by the smoothness condition

(18) $$\lim_{r\to\infty} \frac{T(2r,f)}{T(r,f)} = 1.$$

If f satisfies (15), then it satisfies (18), too. The following result of Hayman [3] shows that the conditions (15) and (18) are essentially the best possible for Theorem 6.

Theorem 7. *Let $\varphi(r)$ be as in Theorem* 1. *There exists an integral function f such that*

(19) $$T(r,f) = O(\varphi(r)(\log r)^2) \quad as \quad r \to \infty$$

and

(20) $$\frac{T(r,f')}{T(r,f)} \to 0 \quad as \quad r \to \infty$$

through a set of values E having infinite logarithmic measure.

If f has finite order, then

(21) $$m(r,f') \leqq m(r,f) + O(\log r) \quad \text{as} \quad r \to \infty.$$

This together with the following result of [7] describes the connection between $m(r,f)$ and $m(r,f')$ for all values of r.

Theorem 8. *If f is a transcendental meromorphic function satisfying* (18), *then*

(22) $$m(r,f) \leqq N(r,f) + m(r,f') + o(T(r,f)) \quad as \quad r \to \infty.$$

Theorem 5 shows that (22) is sharp, and from Theorem 7 we conclude that (15) and (18) are essentially the best possible conditions under which Theorem 8 holds.

Remark. The conditions (3), (6), (7) and (22) do not hold for polynomials. The function $f(z)=z+1/z$ does not satisfy (7). For rational functions other than polynomials, the conditions (6) and (22) hold.

2. On the deficiencies of f and f'

From the proof of the second main theorem of Nevanlinna [5, pp. 238—247] we get

Theorem 9. *Suppose that f is meromorphic in the plane and not a linear polynomial. If the order of f is finite, then*

$$\delta(\infty, f') \leqq \delta(\infty, f), \tag{23}$$

$$\Delta(\infty, f') \leqq \Delta(\infty, f), \tag{24}$$

$$\sum_{a \neq \infty} \delta(a, f) \leqq 2\delta(0, f'), \tag{25}$$

and for any finite a

$$\Delta(a, f) \leqq 2\Delta(0, f'). \tag{26}$$

If f has infinite or finite order, then

$$\delta(\infty, f') \leqq \Delta(\infty, f) \tag{27}$$

and

$$\sum_{a \neq \infty} \delta(a, f) \leqq 2\Delta(0, f'). \tag{28}$$

In the other direction, the following theorem is proved in [8].

Theorem 10. *Let f be a transcendental meromorphic function satisfying* (18). *Then*

$$\delta(\infty, f') \geqq 2\delta(\infty, f) - 1, \tag{29}$$

$$\Delta(\infty, f') \geqq \frac{\Delta(\infty, f)}{2 - \Delta(\infty f)}, \tag{30}$$

with equality in (30) *if f has only simple poles, and, furthermore, there exists a finite value a such that*

$$\delta(a, f) \geqq \delta(0, f'). \tag{31}$$

From Theorem 2 we get

Theorem 11. *If f is a transcendental meromorphic function of order zero, then*

$$\Delta(\infty, f') \geqq \frac{\delta(\infty, f)}{2-\delta(\infty, f)}. \tag{32}$$

The condition (29) holds for those rational functions which are not linear polynomials. If we take $f(z)=z+1/z$, then

$$\Delta(\infty, f) = \delta(\infty, f) = 1/2$$

and

$$\Delta(\infty, f') = \delta(\infty, f') = 0.$$

For this rational function f the conditions (30) and (32) do not hold. The function

$$f(z) = \frac{z^2+1}{z^3}$$

does not satisfy (31). Modifying a little the function f constructed in [6] we get a meromorphic function of order one which does not satisfy (32). The following result of [8] shows that (29) is sharp.

Theorem 12. *Let $\varphi(r)$ be as in Theorem* 1. *For any δ, $1/2 \leqq \delta < 1$, there exists a transcendental meromorphic function f satisfying* (11) *such that $\delta(\infty, f)=\delta$ and $\delta(\infty, f')=2\delta-1$.*

Especially, if $\delta=1/2$, then we have $\delta(\infty, f)=1/2$ and $\delta(\infty, f')=0$ in Theorem 12. The following theorem of [8] shows that the conditions (15) and (18) are essentially the best possible for Theorem 10.

Theorem 13. *Let $\varphi(r)$ be as in Theorem* 1. *There exist transcendental meromorphic functions f, g and h satisfying* (19) *such that*

$$\delta(\infty, f') = 0 \quad \textit{but} \quad \delta(\infty, f) = 1,$$

$$\Delta(\infty, g') = 0 \quad \textit{but} \quad \Delta(\infty, g) = 1,$$

and

$$\delta(0, h') > 0 \quad \textit{but} \quad \delta(a, h) = 0$$

for all values a.

Let g be an entire transcendental function with simple zeros satisfying (15). Then we see from Theorem 6 that

$$T(r, g') = (1+o(1))T(r, g) \quad \text{as} \quad r \to \infty.$$

Using Theorem 1 of Hayman [2] we conclude that

$$\begin{aligned} N(r, 0, g') &= (1+o(1)T(r, g') \\ &= (1+o(1))T(r, g) \\ &= (1+o(1))N(r, 0, g) \quad \text{as} \quad r \to \infty, \end{aligned}$$

and that the function $f=1/g$ satisfies

$$T(r,f') = (2+o(1))N(r,0,g) \quad \text{as } r \to \infty.$$

Since $N(r,0,f')=N(r,0,g')$, we conclude that $\delta(0,f')=\Delta(0,f')=1/2$. Clearly

$$\delta(0,f) = \Delta(0,f) = 1.$$

This example shows that the constant 2 in the inequalities (25), (26) and (28) cannot be replaced by a smaller one, not even for slowly growing functions.

The conditions (23), (24), (25) and (26) need not hold for functions of infinite order. In fact, there exist meromorphic functions f, g and h of infinite order such that

$$\delta(\infty,f) = 0 \quad \text{but} \quad \delta(\infty,f') = 1,$$

$$\Delta(\infty,g) = 0 \quad \text{but} \quad \Delta(\infty,g') = 1,$$

$$\delta(0,g) = 1 \quad \text{but} \quad \delta(0,g') = 0,$$

and

$$\Delta(0,h) = 1 \quad \text{but} \quad \Delta(0,h') = 0.$$

For a proof, we refer to [8] and [9].

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