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# ON SOME EXTENSIONS OF A THEOREM OF HARDY AND LITTLEWOOD

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**1. Introduction.** Let $G\subseteqq \mathbf{C}$ be a domain with $\partial G\neq\emptyset$ and $\omega(t)$ a modulus of continuity, that is, a continuous and increasing function $\omega(t)$ $(t\geqq 0)$ with

$$(1)\qquad \begin{cases} \text{(i)} & \omega(t)>0 \quad \text{for} \quad t>0,\\ \text{(ii)} & \lim_{t\to 0^+}\omega(t)=0 \quad \text{and}\\ \text{(iii)} & \omega(t_1+t_2)\leqq \omega(t_1)+\omega(t_2).\end{cases}$$

With $G$ and $\omega$ we associate the following class of functions:
$A_\omega(G)=\{f\colon G\to\mathbf{C};\ f$ regular in $G$ and continuous in $\bar G$ with

$$|f(z)-f(w)|\leqq\omega(\delta)\,\forall\, z,\ w\in\bar G \quad \text{with} \quad |z-w|\leqq\delta\}.$$

If $\omega(t)=O(t^\alpha)$ $(0<\alpha\leqq 1)$ (that is, $f\in A_\omega$ is Lipschitz-continuous in $\bar G$), we will write $A_\alpha(G)$. For $G=U=\{z\colon |z|<1\}$ we have the following result due to Hardy and Littlewood [1]:

$f\in A_\alpha(\bar U)$ if and only if for all $z\in U$ with $|z|=r$

$$(2)\qquad |f'(z)|\leqq C\cdot(1-r)^{\alpha-1}=C\cdot(\operatorname{dist}(z,\partial U))^{\alpha-1}.$$

In [7] it was shown that (2) has a natural extension to so-called uniform domains (see [5]). This result contains the previous generalizations [3] and [9] as special cases. We will show that (2) remains valid if we replace $U$ by a uniform domain $D$ and $\alpha$ by any modulus of continuity $\omega(t)$. Hence our theorems will contain the result of [7] as a special case.

The main idea in what follows is to sharpen the following known necessary condition (see Theorem 1 below):

*For* $f\in A_\omega(\bar G)$, $z\in G$ *and* $\delta=\operatorname{dist}(z,\partial G)$ *we have*

$$(3)\qquad |f'(z)|\leqq\omega(\delta)/\delta.$$

*Proof.* We write $B=B_\delta(z)=\{w\in G\colon |w-z|\leqq\delta\}$. Since $f\in A_\omega(\bar G)$ we have

$$|f'(z)|\leqq\left|\frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta\right|=\left|\frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)-f(z)}{(\zeta-z)^2}d\zeta\right|\leqq\frac{1}{2\pi}\int_{\partial B}\omega(\delta)/\delta^2 d\zeta=\omega(\delta)/\delta.$$

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It seems that (3) is too weak to show that this condition is also sufficient for $f$ to belong to $A_\omega(\bar{G})$ (even in the case $G=U$; see [3]).

**2. Uniform domains and moduli of continuity.** As in [6] a domain $G\subseteqq\mathbf{C}$ is called an $(\alpha,\beta)$-John domain $0<\alpha\leqq\beta<\infty)$ if there is a point $z_0\in G$ such that each point $z\in G$ can be joined with $z_0$ by means of a rectifiable path $\gamma\colon [0,d]\to G$ (arc length as parameter) with

$$(4)\qquad \begin{cases} \text{(i)} & \gamma(0)=z, \quad \gamma(d)=z_0,\\ \text{(ii)} & d\leqq\beta \quad \text{and}\\ \text{(iii)} & \operatorname{dist}(\gamma(s),\partial G)\geqq\alpha\cdot s/d \quad (0\leqq s\leqq d).\end{cases}$$

A domain $D\subseteqq\mathbf{C}$ is called an $(\alpha,\beta)$-uniform domain $(0<\alpha\leqq\beta<\infty)$ if for all $z_1, z_2\in D$, $z_1\neq z_2$, there is an $(\alpha|z_1-z_2|, \beta|z_1-z_2|)$-John domain $G$ in $D$ containing $z_1$ and $z_2$.

At first sight the definition for a uniform domain looks complicated, but it turns out that a simply connected domain $D\neq\mathbf{C}$ is uniform if and only if it is a quasiconformal disc (see [6]). Because of this the boundary $\partial D$ need not be lipschitzian and its Hausdorff dimension may be arbitrary near 2 (see [5]). There is also an interesting conformally invariant condition (see [5]).

In [4; Ch. 3] Lorentz shows that if $\omega$ is a modulus of continuity as in (1), then there is a concave modulus of continuity $\omega^*$ with

$$\omega(t)\leqq\omega^*(t)\leqq 2\cdot\omega(t)$$

for all $t\geqq 0$. Hence in the rest of this note all moduli of continuity $\omega$ will be assumed concave.

If $\omega(t)$ $(t\geqq 0)$ is concave, then $\omega(t)$ is continuous for $t\geqq 0$, has a right hand derivative $D^+\omega(t)$ at each $t\geqq 0$ (with, possibly, $D^+\omega(0)=\infty$) and a left hand derivative $D^-\omega(t)$ at each $t>0$. For $0\leqq t_1\leqq t_2$, we have

$$(5)\qquad D^+\omega(t_1)\geqq D^-\omega(t_2)\geqq D^+\omega(t_2).$$

Hence $\omega'(t)$ exists and is continuous except for at most countably many $t$, and we will also have (see [2; Theorem 18.14])

$$(6)\qquad \int_0^\delta D^+\omega(t)\,dt=\int_0^\delta \omega'(t)\,dt\leqq\omega(\delta)-\omega(0)=\omega(\delta).$$

From (iii) of (1) one can deduce the inequality

$$(7)\qquad \omega(\lambda\cdot t)\leqq(\lambda+1)\cdot\omega(t)\;\forall\lambda>0.$$

Finally we have for $0<t_1<t_2$

$$(8)\qquad D^+\omega(t_1)\geqq\frac{\omega(t_2)-\omega(t_1)}{t_2-t_1}.$$

*Proof.* Since $\omega(t)$ is concave we have for $0 \leqq \lambda \leqq 1$

$$\omega(t_1+\lambda(t_2-t_1)) \geqq \omega(t_1)+\lambda\cdot(\omega(t_2)-\omega(t_1))$$

and hence

$$D^+\omega(t_1) = \lim_{\substack{\lambda\downarrow 0\\ \lambda\leqq 1}} \frac{\omega(t_1+\lambda\cdot(t_2-t_1))-\omega(t_1)}{\lambda\cdot(t_2-t_1)}$$

$$\geqq \lim_{\lambda\downarrow 0} \frac{\omega(t_1)+\lambda\cdot(\omega(t_2)-\omega(t_1))-\omega(t_1)}{\lambda\cdot(t_2-t_1)} = \frac{\omega(t_2)-\omega(t_1)}{t_2-t_1}.$$

**3. The main results.** First we will improve the estimate (3).

Theorem 1. *Let $G\subseteqq \mathbf{C}$ be a domain with $\partial G\neq\emptyset$, $\omega(t)$ any modulus of continuity and $f\in A_\omega(\bar{G})$. Then we have for all $z\in G$ with $d_z=\text{dist}(z, \partial G)$*

(9) $$|f'(z)| \leqq C\cdot D^+\omega(d_z),$$

*where $C$ is a constant independent of $z$.*

*Proof.* Suppose there is a sequence $\{z_n\}$, $n=1, 2, \dots$ of points in $G$ with $d_{z_n}=\text{dist}(z_n, \partial G)$ and

$$\overline{\lim_{n\to\infty}} \frac{|f'(z_n)|}{D^+\omega(d_{z_n})} = +\infty.$$

Without loss of generality we may assume that $d_{z_n} \geqq d_{z_{n+1}} > 0$ for $n\geqq 1$. Let $Q_n = |f'(z_n)|/D^+\omega(d_{z_n})$, choose $n(0)\geqq 1$ such that $Q_{n(0)}\geqq 1$, and let $\delta_0=d_{z_{n(0)}}>0$. Suppose that $\delta_0, \delta_1, \dots, \delta_k$ and $n(0), \dots, n(k)$ have already been defined. Let $\delta^*_{k+1}$ be the unique number in $(0, \delta_k)$ with

(10) $$\omega(\delta^*_{k+1}) = \frac{1}{2}\cdot\omega(\delta_k)$$

and look for $n=n(k+1)\geqq n(k)$ with

(11) $$0 < d_{z_{n(k+1)}} \leqq \delta^*_{k+1} \quad \text{and}$$

(12) $$Q_{n(k+1)} \geqq 2^{k+1}.$$

Because of $\lim_{n\to\infty} d_{z_n}=0$ and $\overline{\lim}_{n\to\infty} Q_n=+\infty$ this is always possible. Now define $\delta_{k+1}=d_{z_{n(k+1)}}$.

By (10), (11) and (12) we therefore have

(13) $$|f'(z_{n(k)})| \geqq 2^k\cdot D^+\omega(\delta_k) \quad \text{and}$$

(14) $$0<\omega(\delta_{k+1}) \leqq \frac{1}{2}\cdot\omega(\delta_k), \quad k=1, 2, \dots.$$

For the sake of simplicity let us write $z_k$ instead of $z_{n(k)}$. Now let $k\geqq 1$, $\delta=\delta_k/2$, and $B=B_\delta(z_k)$. Because of $\operatorname{dist}(z_k, \partial G)=\delta_k>\delta$ we have

$$|f'(z_k)| = \left|\frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)-f(z_k)}{(\zeta-z_k)^2}\,d\zeta\right|$$

$$\leqq \max_{\zeta\in\partial B}|f(\zeta)-f(z_k)|\cdot\frac{1}{2\pi}\int_{\partial B} d\zeta/\delta^2 = 2\cdot|f(\zeta_0)-f(z_k)|/\delta_k$$

with $|\zeta_0-z_k|=\delta_k/2$. Hence with (13), (8) and (14) we have

$$|f(\zeta_0)-f(z_k)| \geqq \frac{1}{2}\cdot\delta_k|f'(z_k)| \geqq 2^{k-1}\cdot\delta_k\cdot D^+\omega(\delta_k)$$

$$\geqq 2^{k-1}\cdot\delta_k\cdot\frac{\omega(\delta_k)-\omega(\delta_{k+1})}{\delta_k-\delta_{k+1}} \geqq 2^{k-1}(\omega(\delta_k)-\omega(\delta_{k+1}))$$

$$\geqq 2^{k-1}\left(\omega(\delta_k)-\frac{1}{2}\cdot\omega(\delta_k)\right) = 2^{k-2}\cdot\omega(\delta_k). \tag{15}$$

On the other hand we have by (7)

$$|f(\zeta_0)-f(z_k)| \leqq \sup_{|z-w|\leqq\delta}|f(z)-f(w)| \leqq \omega(\delta) = \omega\left(\frac{1}{2}\cdot\delta_k\right) \leqq \frac{3}{2}\cdot\omega(\delta_k),$$

which is a contradiction to (15) if $k\geqq 4$.

Remark. If $\omega(t)=C\cdot t^\alpha$ $(0<\alpha\leqq 1)$, then (3) and (9) give the same bound (up to a constant). But, if $\lim_{t\downarrow 0}\omega(t)/t^\alpha=+\infty$ for every $\alpha>0$, then (9) is much stronger than (3).

Theorem 2. *Suppose that $D\subseteqq \mathbf{C}$, $D\neq\emptyset$, is an $(\alpha, \beta)$-uniform domain and that $f\colon D\to\mathbf{C}$ is regular in $D$. If, for any $z\in D$ and $d_z=\operatorname{dist}(z, \partial D)$*

$$|f'(z)| \leqq C\cdot D^+\omega(d_z),$$

*then $f\in A_\omega(\bar{D})$.*

*Proof.* The proof is similar to that of [7]. If $z_1, z_2\in D$, $z_1\neq z_2$, then there is an $(\alpha|z_1-z_2|, \beta|z_1-z_2|)$-John domain $G\subseteqq D$ containing $z_1$ and $z_2$. Let $z_0$ be the point as in the definition of a John domain and let $\gamma_k\colon [0, d_k]\to G$ be the corresponding paths joining $z_k$ to $z_0$ $(k=1, 2)$. Then

$$I_k = \left|\int_{\gamma_k} f'(\zeta)\,d\zeta\right| \leqq \int_0^{d_k}|f'(\gamma_k(s))|\,ds \leqq C\cdot\int_0^{d_k} D^+\omega(\operatorname{dist}(\gamma_k(s), \partial G))\,ds.$$

Hence by (iii) of (4), (5), (6) and (ii) of (4) we have

$$I_k \leqq C\cdot\int_0^{d_k} D^+\omega(\alpha|z_1-z_2|\cdot s/d_k)\,ds = C\cdot\int_0^{d_k}\omega'(\alpha|z_1-z_2|\cdot s/d_k)\,ds$$

$$= \frac{C\cdot d_k}{\alpha\cdot|z_1-z_2|}\int_0^{\alpha|z_1-z_2|}\omega'(t)\,dt \leqq \frac{C\cdot d_k}{\alpha\cdot|z_1-z_2|}\cdot\omega(\alpha|z_1-z_2|) \leqq \frac{C(\alpha+1)}{\alpha}\cdot\omega(|z_1-z_2|).$$

Finally we obtain from this last estimate uniformly in $D$

$$|f(z_1)-f(z_2)| = \left|\int_{z_1}^{z_2} f'(\zeta)\,d\zeta\right| \leqq I_1+I_2 \leqq \frac{2C(\alpha+1)}{\alpha}\cdot\omega(|z_1-z_2|).$$

Remark. Theorem 4 of [7] shows that an $(\alpha, \beta)$-uniform domain is fat in the sense of [8] if its complement $\mathbf{C}\setminus D$ only has a finite number of components. Hence by [7; Theorem 2.6] we also have

Theorem 3. *Let $D$ be an $(\alpha, \beta)$-uniform domain so that $\mathbf{C}\setminus D$ has a finite number of components. Let $f\colon D\to\mathbf{C}$ be regular in $D$ and continuous in $\bar{D}$. Then*

$$\sup_{\substack{|\zeta_1-\zeta_2|\leqq\delta\\ \zeta_1,\zeta_2\in\partial D}} |f(\zeta_1)-f(\zeta_2)| \leqq C_1\cdot\omega(\delta)$$

*if and only if* (9) *holds in $D$.*

Note that even in the Lipschitz case of the unit disc Theorem 3 is not true any longer if "regular" is replaced by "harmonic".

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