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ON THE GROWTH OF THE SPHERICAL DERIVATIVE OF A MEROMORPHIC FUNCTION

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1. Introduction

Let f be meromorphic in the plane. We denote

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2},$$

$$\mu(r,f) = \sup\{\varrho(f(z)): \ |z| = r\}$$

and

$$\lambda(r,f) = \inf\{\varrho(f(z)): \ |z| = r\}.$$

In this paper, we shall give some estimates on the growth of $\mu(r,f)$ and $\lambda(r,f)$.

2. On the growth of $\lambda(r, f)$

We shall employ the usual notation of the Nevanlinna theory. First we shall estimate the growth of $\lambda(r, f)$ from below.

Theorem 1. *Let f be a transcendental meromorphic function of finite lower order. Then*

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} > -\infty. \tag{2.1}$$

This result need not hold for functions of infinite lower order. If we take $f(z) = \exp\{e^z\}$, then

$$\log \lambda(r, f) \leqq \log \varrho(f(r)) \leqq -(1+o(1))e^r$$

and $T(r, f) = o(e^r)$ as $r \to \infty$. These estimates imply that

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} = -\infty. \tag{2.2}$$

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If f is a meromorphic function of lower order zero, then f satisfies

$$\liminf_{r\to\infty} \frac{T(2r, f)}{T(r, f)} = 1. \tag{2.3}$$

For this class of functions we prove

Theorem 2. *Let f be a transcendental meromorphic function satisfying* (2.3). *Then*

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} \geqq -1. \tag{2.4}$$

This does not hold for all functions of positive order. If we take $f(z)=e^z$, then

$$\lambda(r,f) \leqq \varrho(f(r)) \leqq e^{-r},$$

and since

$$T(r, f) = (1+o(1))\frac{r}{\pi}$$

we deduce that

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} \leqq -\pi < -1 \tag{2.5}$$

for this function f.

In the other direction we have

Theorem 3. *Let f be a transcendental meromorphic function of finite order. Then*

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} \leqq -\delta(\infty, f). \tag{2.6}$$

Theorems 2 and 3 together show that if f is an entire transcendental function of finite order satisfying (2.3), then

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} = -1. \tag{2.7}$$

The condition (2.6) does not hold for all meromorphic functions of infinite order.

Theorem 4. *There exists an entire function of infinite order and of lower order zero such that*

$$\limsup_{r\to\infty} \frac{\log \lambda(r, f)}{T(r, f)} = \infty. \tag{2.8}$$

For meromorphic functions without Nevanlinna deficient values the following theorem gives in some cases a sharper estimate than (2.6).

Theorem 5. *Let f be a transcendental meromorphic function. Then*

$$\limsup_{r\to\infty}\frac{r\lambda(r, f)}{T(2r, f)}<\infty. \tag{2.9}$$

On the other hand, we have

Theorem 6. *Given any increasing and positive function $\varphi(r)$ such that $\varphi(r)\to\infty$ as $r\to\infty$, there exists a transcendental meromorphic function f satisfying*

$$T(r, f)=O(\varphi(r)(\log r)^2) \quad as \quad r\to\infty \tag{2.10}$$

such that

$$\limsup_{r\to\infty}\frac{r\lambda(r, f)}{T(2r, f)}>0 \tag{2.11}$$

and that

$$\limsup_{r\to\infty}\frac{r\lambda(r, f)}{T(r, f)}=\infty. \tag{2.12}$$

The function $\varphi(r)$ in (2.10) cannot be replaced by a positive constant. We have

Theorem 7. *If f is a transcendental meromorphic function satisfying*

$$T(r, f)=O((\log r)^2) \quad as \quad r\to\infty, \tag{2.13}$$

then

$$\limsup_{r\to\infty}\frac{r\lambda(r, f)}{T(r, f)}=0. \tag{2.14}$$

3. On the growth of $\mu(r, f)$

The following theorem gives a lower bound for the growth of $\mu(r,f)$.

Theorem 8. *Let f be a transcendental meromorphic function. Then*

$$\liminf_{r\to\infty}\frac{\log\mu(r, f)}{T(r, f)}\geqq -1. \tag{3.1}$$

In the other direction, we have

Theorem 9. *Let f be a transcendental meromorphic function satisfying* (2.3). *Then*

$$\liminf_{r\to\infty}\frac{\log\mu(r, f)}{T(r, f)}\leqq -\delta(\infty, f). \tag{3.2}$$

Combining Theorems 8 and 9, we deduce that

$$\liminf_{r\to\infty} \frac{\log \mu(r, f)}{T(r, f)} = -1 \tag{3.3}$$

for transcendental entire functions satisfying (2.3).

The following theorem shows that (3.2) need not hold if the order of f is positive.

Theorem 10. *Given* $d, 0<d<1$, *there exists a meromorphic function* f *of order* d *with* $\delta(\infty, f)>0$ *such that*

$$\liminf_{r\to\infty} \frac{r\mu(r, f)}{T(r, f)} > 0. \tag{3.4}$$

On the other hand, the following theorem shows that a transcendental meromorphic function of lower order zero cannot satisfy (3.4).

Theorem 11. *Let* f *be a transcendental meromorphic function. Then*

$$\liminf_{r\to\infty} \frac{r\mu(r, f)}{T(2r, f)} < \infty, \tag{3.5}$$

and if the lower order of f *is finite, then*

$$\liminf_{r\to\infty} \frac{r\mu(r, f)}{T(r, f)} < \infty. \tag{3.6}$$

If, further, the lower order of f *is zero, then*

$$\liminf_{r\to\infty} \frac{r\mu(r, f)}{T(r, f)} = 0. \tag{3.7}$$

If the order of f is infinite, then (3.6) need not hold. We take

$$f(z) = \exp\{ie^z\}.$$

Then $f'(z)=ie^z f(z)$, and since $|f(z)|=1$ on the positive real axis, we get

$$\mu(r, f) \geqq \frac{1}{2} e^r. \tag{3.8}$$

Since

$$T(r,f) = o(e^r) \quad \text{as} \quad r \to \infty,$$

we get from (3.8)

$$\liminf_{r\to\infty} \frac{\mu(r, f)}{T(r+K, f)} = \infty$$

for any positive constant K.

4. Proof of Theorem 1

We use Lemma 3 of Hayman [4] in the following form.

Lemma A. *Let* $a_n, n=1, \ldots, p$, *lie in* $0<|z|<\infty$. *For any* $B\geqq 9$, *there exists a set E which is a countable union of discs* $|z-c_k|<d_k$ *such that*

$$\sum \frac{d_k}{|c_k|} < 4000e^{-B} \tag{4.1}$$

and that

$$\sum_{|z|/2<|a_n|<2|z|} \log \frac{|z|+|a_n|}{|z-a_n|} \leqq pB \tag{4.2}$$

when $z\neq 0$ *and z lies outside E.*

Let f be meromorphic in the plane and let a_n be the a-points of f. We write

$$S(z, a) = \sum_{|z|/2<|a_n|<2|z|} \log \frac{|z|+|a_n|}{|z-a_n|}.$$

Lemma 1. *Let f be a non-constant meromorphic function. Then for any complex value a and any* $B\geqq 12$ *there exists a set E which is a countable union of discs* $|z-c_k|<d_k$ *such that*

$$\sum_{r<|c_k|<2r} \frac{d_k}{|c_k|} < 16000e^{-B} \tag{4.3}$$

for all positive r and that

$$S(z, a) \leqq n(4|z|, a, f)B \tag{4.4}$$

when $z\neq 0$ *and z lies outside E.*

Proof. It follows from Lemma A that

$$S(z, a) \leqq n(2^{k+1}, a, f)B \leqq n(4|z|, a, f)B \tag{4.5}$$

if $2^{k-1}<|z|\leqq 2^k$ and z lies outside a set E_k satisfying (4.1). We select from each E_k those discs which have at least one common point with the annulus $2^{k-1}<|z|<2^k$ and denote the union of these discs by E. It follows from (4.5) that $S(z, a)$ satisfies (4.4) outside E when $z\neq 0$. Since $B\geqq 12$, it follows from (4.1) that all discs which are selected from E_k are contained in $2^{k-2}<|z|<2^{k+1}$, and (4.3) follows from (4.1). Lemma 1 is proved.

Let f be as in Lemma 1. We choose $B=20$ and $a=\infty$ in Lemma 1, and denote the corresponding exceptional set E by E.

Lemma 2. *With the above notation we have*

$$\log|f(z)| \leqq 37T(8|z|, f) \tag{4.6}$$

when $|z|\geqq 1$ *and z lies outside E, where E satisfies* (4.3) *with* $B=20$.

Proof. Let b_k be the poles of f and $|z|\geqq 1$. Applying the Poisson—Jensen formula with $R=2|z|$, we get

$$\log|f(z)| \leqq 3m(R,f)+\sum_{|b_k|<R}\log\left|\frac{R^2-\bar{b}_k z}{R(z-b_k)}\right|. \tag{4.7}$$

If $R/4<|b_k|<R$, then

$$\log\left|\frac{R^2-\bar{b}_k z}{R(z-b_k)}\right| \leqq \log\frac{|z|+|b_k|}{|z-b_k|}+\log 8,$$

and for other terms in the sum of (4.7) we get the upper bound $\log 8<3$. These estimates together with (4.7) imply that

$$\log|f(z)| \leqq 3T(2|z|,f)+3n(2|z|,\infty,f)+S(z,\infty),$$

and we deduce from Lemma 1 that

$$\log|f(z)| \leqq 3T(2|z|,f)+23n(4|z|,\infty,f) \tag{4.8}$$

when $|z|\geqq 1$ and z lies outside E. Since

$$n(r,\infty,f)\log 2 \leqq \int_r^{2r} n(t,\infty,f)t^{-1}dt = N(2r,\infty,f)-N(r,\infty,f) \leqq T(2r,f)$$

for $r\geqq 1$, (4.6) follows from (4.8). Lemma 2 is proved.

Now we prove Theorem 1. Let f be a transcendental meromorphic function of finite lower order. Then there exist $K_1>0$ and a sequence r_n, $r_n\to\infty$ as $n\to\infty$, such that

$$T(32r_n,f) < K_1 T(r_n,f) \tag{4.9}$$

for all n. Since

$$m(r,f') \leqq m(r,f)+m(r,f'/f),$$

we deduce from the lemma on the logarithmic derivative [Nevanlinna 7 p. 245] that

$$m(16r_n,f') \leqq m(16r_n,f)+o(T(32r_n,f)) \quad \text{as} \quad n\to\infty,$$

which together with the fact that $N(r,f')\leqq 2N(r,f)$ implies that

$$T(16r_n,f') \leqq (2+o(1))T(32r_n,f) \quad \text{as} \quad n\to\infty. \tag{4.10}$$

Using Lemma 2, we deduce that there exists t_n, $r_n\leqq t_n<2r_n$, such that

$$\log|f(z)| \leqq O(T(8|z|,f)) = O(T(16r_n,f))$$

and

$$-\log|f'(z)| \leqq O(T(8|z|,1/f')) = O(T(16r_n,f'))$$

for all z lying on $|z|=t_n$, and we deduce from (4.9) and (4.10) that

$$-\log\lambda(t_n,f) \leqq O(T(32r_n,f)) = O(T(t_n,f)) \quad \text{as} \quad n\to\infty,$$

which proves Theorem 1.

5. Some properties of functions satisfying (2.3)

We denote by $f^{(k)}$ the k-th derivative of f.

Lemma 3. *Let f be a transcendental meromorphic function satisfying* (2.3), *and let k be a positive integer. Then there exists a sequence r_p, $r_p \to \infty$ as $p \to \infty$,* such that

(5.1) $$T(p^2 r_p, f^{(k)}) = (1+o(1))T(r_p, f) \quad as \quad p \to \infty$$

and

(5.2) $$T(p^2 r_p, f^{(k)}) = T(r_p, f^{(k)}) + o(T(r_p, f))$$

as $p \to \infty$, and that for any complex value a

(5.3) $$n(p^2 r_p, a, f) = o(T(r_p, f)),$$

(5.4) $$N(p^2 r_p, a, f) = N(r_p, a, f) + o(T(r_p, f)),$$

(5.5) $$n(p^2 r_p, a, f^{(k)}) = o(T(r_p, f))$$

and

(5.6) $$N(p^2 r_p, a, f^{(k)}) = N(r_p, a, f^{(k)}) + o(T(r_p, f))$$

as $p \to \infty$.

Proof. It follows from (2.3) that there exists a sequence t_n, $t_n \to \infty$ as $n \to \infty$, such that

(5.7) $$T(2t_n, f) = (1+o(1))T(t_n, f) \quad \text{as} \quad n \to \infty.$$

We choose $r_1 = 2$. Let $p \geqq 2$ be a positive integer and

$$x = \frac{1}{p^4 e^{2p}}.$$

Since $T(r, f)$ is an increasing and convex function of $\log r$, we get

$$T(t_n, f) - T(xt_n, f) \leqq \frac{\log(1/x)}{\log 2}(T(2t_n, f) - T(t_n, f)),$$

and we deduce from (5.7) that

$$T(t_n, f) = (1+o(1))T(xt_n, f) \quad \text{as} \quad n \to \infty.$$

We choose n so large that $xt_n > p^2 r_{p-1}$ and

(5.8) $$T(t_n, f) < 2T(xt_n, f)$$

and we set $r_p = xt_n$.

Let a be a complex value. We get from (5.8)

(5.9)
$$n(e^p r_p, a, f)p \leqq \int_{e^p r_p}^{e^{2p} r_p} n(t, a, f)t^{-1}dt \leqq (1+o(1))T(e^{2p} r_p, f) \leqq (2+o(1))T(r_p, f),$$

which proves (5.3). From (5.9) we get

(5.10)

$$N(p^2 r_p, a, f) - N(r_p, a, f) \leqq 2n(p^2 r_p, a, f) \log p \leqq \frac{1}{p}(4+o(1))T(r_p, f) \log p,$$

which proves (5.4). If we choose a such that $\Delta(a, f)=0$, we get from (5.4)

$$(5.11) \qquad T(p^2 r_p, f) - T(r_p, f) = N(p^2 r_p, a, f) - N(r_p, a, f) + o(T(p^2 r_p, f)) = o(T(p^2 r_p, f)),$$

which proves (5.1).

It follows from the lemma on the logarithmic derivative that

$$T(t_n/2, f^{(k)}) \leqq (k+1)N(t_n/2, \infty, f) + m(t_n/2, \infty, f) + m(t_n/2, \infty, f^{(k)}/f) \leqq (k+1+o(1))T(t_n, f),$$

and we deduce from (5.8) that

$$(5.12) \qquad T(t_n/2, f^{(k)}) \leqq (2k+2+o(1))T(r_p, f)$$

as $p \to \infty$. Just as in the proof of (5.9), we deduce from (5.12) that

(5.13)

$$n(e^p r_p, a, f^{(k)}) \leqq \frac{1}{p}(1+o(1))T(e^{2p} r_p, f^{(k)}) \leqq \frac{1}{p}(2k+2+o(1))T(r_p, f),$$

which proves (5.5), and just as in the proof of (5.10) and (5.11), we deduce that (5.6) and (5.2) follow from (5.13). Lemma 3 is proved.

We choose $B=20$ in Lemma 1, and deduce that there exist sets E_1, E_2, E_3 and E_4 each of them satisfying (4.3) such that

$$(5.14) \qquad S(z, \infty, f) = S(z, \infty) \leqq 20n(4|z|, \infty, f)$$

outside E_1, and that corresponding estimates hold for $S(z, 0, f)$, $S(z, \infty, f^{(k)})$ and $S(z, 0, f^{(k)})$ outside the union of E_2, E_3 and E_4. We write

$$E = \bigcup_{k=1}^{4} E_k.$$

Let $z=re^{i\varphi}$ lie in $r_p \leqq |z| \leqq pr_p$. Let b_s be the poles of f. Applying the Poisson-Jensen formula with $R=p^2 r_p$, we get

$$(5.15) \qquad \log|f(z)| \leqq \frac{p+1}{p-1} m(R, \infty, f) - \frac{p-1}{p+1} m(R, 0, f) + \sum_{|b_s|<R} \log\left|\frac{R^2 - \bar{b}_s z}{R(z-b_s)}\right|.$$

If $|z|/2<|b_s|<2|z|$, then

$$\log\left|\frac{R^2-\bar{b}_s z}{R(z-b_s)}\right| \leqq \log\frac{|z|+|b_s|}{|z-b_s|}+\log(2p^2),$$

and for other terms in the sum of (5.15) we get the upper bound $\log(4p^2)$. These estimates together with (5.15) imply that

$$\log|f(z)| \leqq \frac{p+1}{p-1}m(R,\infty,f)-\frac{p-1}{p+1}m(R,0,f)$$
$$+S(z,\infty)+n(R,\infty,f)\log(4p^2),$$

and we deduce from (5.14) that

$$\log|f(z)| \leqq N(R,0,f)-N(R,\infty,f)$$
$$+2\left(\frac{p+1}{p-1}-1+o(1)\right)T(R,f)$$
$$+n(R,\infty,f)\big(20+\log(4p^2)\big)$$

if z lies outside E. Therefore we get from (5.1), (5.4) and (5.9)

$$\log|f(z)| \leqq N(r_p,0,f)-N(r_p,\infty,f)+o\big(T(r_p,f)\big) \tag{5.16}$$

for all z lying in $r_p\leqq|z|\leqq pr_p$ outside E. A similar estimate holds for $1/f$, and using (5.13) and Lemma 3 we get a similar estimate for $1/f^{(k)}$ and $f^{(k)}$. We have proved the following result.

Lemma 4. *Let f and k be as in Lemma 3. The sequence r_p in Lemma* 3 *can be chosen such that*

$$\log|f(z)| = N(r_p,0,f)-N(r_p,\infty,f)+o\big(T(r_p,f)\big) \tag{5.17}$$

and

$$\log|f^{(k)}(z)| = N(r_p,0,f^{(k)})-N(r_p,\infty,f^{(k)})+o\big(T(r_p,f)\big) \tag{5.18}$$

as $z\to\infty$ through the union of the annuli $r_p\leqq|z|\leqq pr_p$ outside a set E which is the union of four sets satisfying (4.3) *with* $B=20$.

6. Proof of Theorem 2

Let f be as in Theorem 2. We may suppose that $f(0)=0$, because in other cases we can consider $1/f$ or the function

$$\frac{f(z)-f(0)}{1+\overline{f(0)}f(z)}.$$

Let

$$f(z) = c_k z^k+c_{k+1}z^{k+1}+\dots \tag{6.1}$$

be the Laurent expansion of f at the origin. Then $k \geqq 1$, and from (6.1) we deduce that there exists t, $0<t<1$, such that if $0<|a|<t$, then

$$n\big((|a|/(2|c_k|))^{1/k}, a, f\big) = 0$$

and

$$n\big((2|a|/|c_k|)^{1/k}, a, f\big) = k.$$

These estimates imply that if $0<|a|<t$ and

$$r > 9(1+1/|c_k|) = s_0,$$

then

$$(6.2) \qquad \big|\log|a|\big| - N(r, a) \leqq \big|\log|a|\big| - k\log\big(r(|c_k|/(2|a|))^{1/k}\big) \leqq \log\frac{2|a|}{r^k|c_k||a|} \leqq 0.$$

Let $0<|a|<9$. It follows from the first main theorem of the Nevanlinna theory that

$$m(r, a) \leqq T(r, f) - N(r, a) + \big|\log|a|\big| + \log^+|a| + \log 2,$$

and we deduce from (6.2) that

(6.3)
$$m(r, a) \leqq T(r, f) + \log 2 + \log(1/t) + 2\log 9 \leqq T(r, f) + 6 + \log(1/t)$$

if $0<|a|<9$ and $r>s_0$.

We write

$$M(r, f) = \sup\{|f(z)| : |z| = r\}.$$

We apply Lemma 4 with $f^{(k)}=f'$, and choose a sequence r_p as in Lemmas 3 and 4. We may assume that the circle $|z|=r_p$ lies outside the exceptional set E of Lemma 4, because in other cases we may choose r_p', $r_p<r_p'<2r_p$, such that the circle $|z|=r_p'$ lies outside E, and consider $\lambda(r_p', f)$ instead of $\lambda(r_p, f)$.

Let z_p, $f(z_p)\neq 0$, lie on $|z|=r_p$. We get for any z lying on $|z|=r_p$

$$(6.4) \qquad |f(z)-f(z_p)| = \Big|\int_{z_p}^{z} f'(w)\,dw\Big| \leqq \pi r_p M(r_p, f').$$

It follows from (5.18) that

$$(6.5) \qquad \log|f'(z)| = \log M(r_p, f') + o\big(T(r_p, f)\big)$$

on $|z|=r_p$, and from (5.17) we get

$$(6.6) \qquad \log|f(z)| = \log|f(z_p)| + o\big(T(r_p, f)\big)$$

for all z lying on $|z|=r_p$.

We consider first those values of p for which $|f(z_p)|<4$. From (6.4) we deduce that

$$m\big(r_p, f(z_p), f\big) \geqq -\log M(r_p, f') - \log(\pi r_p),$$

which implies together with (6.3) that

(6.7) $$\log M(r_p, f') \geqq -T(r_p, f)+o(T(r_p, f)).$$

Combining this with (6.5) and (6.6) we get

$$\log \lambda(r_p, f) \geqq \log M(r_p, f')+o(T(r_p, f)) \geqq -T(r_p, f)+o(T(r_p, f)),$$

which proves Theorem 2 in the case where $|f(z_p)|<4$ for an infinite number of values of p.

Let us suppose that $|f(z_p)|\geqq 4$. Applying the Jensen formula we get

$$\log|f(0)-f(z_p)| \leqq \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_p)-f(r_p e^{i\alpha})|d\alpha+N(r_p, \infty, f),$$

and since $f(0)=0$, it follows from (6.4) that

$$\log |f(z_p)| \leqq \log M(r_p, f')+N(r_p, \infty, f)+o(T(r_p, f)).$$

This implies together with (6.5) and (6.6) that

$$\log \lambda(r_p, f) \geqq \log M(r_p, f')-2\log M(r_p, f)+o(T(r_p, f))$$
$$\geqq -N(r_p, \infty, f)-\log |f(z_p)|+o(T(r_p, f)),$$

and using (5.17) we deduce that

$$\log \lambda(r_p, f) \geqq -T(r_p, f)+o(T(r_p, f)).$$

This completes the proof of Theorem 2.

7. Proof of Theorem 3

Let f be as in Theorem 3. Let z lie on the circle $|z|=r$. Since

$$\log^+|f(z)| \leqq \log\left(\left|\frac{f'(z)}{f(z)}\right|\frac{1+|f(z)|^2}{|f'(z)|}\right) \leqq \log^+\left|\frac{f'(z)}{f(z)}\right|-\log \lambda(r, f),$$

we get

(7.1) $$m(r, f) \leqq m(r, f'/f)-\log \lambda(r, f).$$

Since f has finite order, it follows from the lemma on the logarithmic derivative that

$$m(r, f'/f) = o(T(r, f)) \quad \text{as} \quad r \to \infty,$$

and we deduce from (7.1) that

$$\log \lambda(r, f) \leqq -m(r, f)+o(T(r, f)) \leqq (-\delta(\infty, f)+o(1))T(r, f) \quad \text{as} \quad r \to \infty.$$

This proves Theorem 3.

8. Proof of Theorem 4

We set

$$f(z)=\sum_{n=1}^{\infty}(z/r_n)^{s_n},$$

where $r_n=\log s_n$, $s_1=100$, and for $n>1$, s_n is a positive integer such that

(8.1) $$\log\log s_n > s_{n-1}.$$

Then

$$f'(z)=\sum_{n=1}^{\infty}(s_n/r_n)(z/r_n)^{s_n-1}.$$

It follows from (8.1) that

(8.2) $$\log M(r_n,f)\leqq s_{n-1}\log r_n\leqq(\log r_n)^2,$$

which implies that the lower order of f is zero. For $|z|=r_n$ it follows from (8.1) that

$$\log|f'(z)|\geqq\log\left(\frac{s_n}{r_n}-\sum_{k=1}^{n-1}s_k r_n^{s_k}\right)\geqq(1+o(1))\log s_n=(1+o(1))r_n,$$

and we deduce from (8.2) that

$$\frac{\log\lambda(r_n,f)}{\log M(r_n,f)}\geqq\frac{(1+o(1))r_n}{(\log r_n)^2}\to\infty\quad\text{as}\quad n\to\infty.$$

This proves Theorem 4.

9. Proof of Theorem 5

Let f be a transcendental meromorphic function. We write

(9.1) $$n(r)=n(r,0,f)+n(r,1,f)+n(r,\infty,f).$$

We have

$$n(3r/2)\log(4/3)\leqq\int_{3r/2}^{2r}n(t)t^{-1}dt\leqq(3+o(1))T(2r,f)\leqq 4T(2r,f)$$

for $r\geqq r_0$. This implies that for all large values of r we may choose z_r lying on $|z|=r$ such that f does not take any of the values 0, 1 and ∞ in the disc

(9.2) $$|z-z_r|<\frac{r}{16T(2r,f)}.$$

It follows from Schottky's theorem that there exists an absolute constant K_1 such that if $|f(z_r)|\leqq 1$, then

(9.3) $$|f(z)|<K_1$$

in

$$D_r = \left\{z : |z - z_r| < \frac{r}{32T(2, f)}\right\},$$

and if $|f(z_r)| > 1$, then $1/f$ satisfies (9.3) in D_r.

We write $g(z) = f(z)$ if $|f(z_r)| \leqq 1$ and $g(z) = 1/f(z)$ if $|f(z_r)| > 1$. Integrating along the boundary of D_r we get from (9.3)

$$|g'(z_r)| = \left|\frac{1}{2\pi}\int \frac{g(w)}{(z_r - w)^2}\, dw\right| \leqq \frac{32T(2r, f)K_1}{r},$$

which implies that

$$\frac{|f'(z_r)|}{1+|f(z_r)|^2} = \frac{|g'(z_r)|}{1+|g(z_r)|^2} \leqq \frac{32K_1 T(2r, f)}{r}, \tag{9.4}$$

and we deduce that

$$\frac{r\lambda(r, f)}{T(2r, f)} \leqq 32K_1$$

for all large values of r. This proves Theorem 5.

10. Proof of Theorem 6

It does not mean any restriction to assume that the function $\varphi(r)$ given in Theorem 6 satisfies the condition

$$\varphi(r) = o(\log\log r) \quad \text{as} \quad r \to \infty. \tag{10.1}$$

We choose $s_1 = 8$, $r_1 = 100$, and for $n \geqq 2$, s_n and r_n are chosen such that s_n is a positive integer,

$$\log\log r_n > r_{n-1}, \tag{10.2}$$

$$s_n > 8ns_{n-1}\log r_n \tag{10.3}$$

and

$$s_n < \varphi(\sqrt{r_n})\log r_n. \tag{10.4}$$

We set

$$f_n(z) = \frac{(-1)^n (z/r_n)^{s_n}}{1+(z/r_n)^{s_n}}$$

and

$$f(z) = \sum_{n=1}^{\infty} f_n(z).$$

Then

$$f_n'(z) = \frac{(-1)^n (s_n/r_n)(z/r_n)^{s_n-1}}{(1+(z/r_n)^{s_n})^2}$$

and

$$f'(z) = \sum_{n=1}^{\infty} f_n'(z).$$

We choose t_n such that

(10.5) $$(t_n/r_n)^{s_n} = 2.$$

Suppose that $t_n \leqq |z| \leqq \sqrt{r_{n+1}}$. For $k<n$ we get from (10.2) and (10.3)

(10.6) $$|f_k(z)-(-1)^k| = \left|\frac{1}{1+(z/r_k)^{s_k}}\right| \leqq 2(r_k/r_n)^{s_k} < e^{-k}.$$

For $k>n$ we get from (10.2) and (10.3)

(10.7) $$|f_k(z)| \leqq 2\left(\sqrt{r_{n+1}}/r_k\right)^{s_k} < e^{-k}.$$

It follows from (10.5) that $|f_n(z)| \leqq 2$, which implies together with (10.6) and (10.7) that

(10.8) $$|f(z)| \leqq 4$$

for all z lying in $t_n \leqq |z| \leqq \sqrt{r_{n+1}}$.

Let $t_n \leqq r \leqq \sqrt{r_{n+1}}$. We get from (10.3)

(10.9) $$N(r, f) \leqq s_n \log(r/r_n) + \log r \sum_{k=1}^{n-1} s_k \leqq s_n \log(r/r_n) + 2s_{n-1} \log r.$$

This implies together with (10.8) and (10.4) that

(10.10) $$T(r,f) \leqq 2s_n \log r + \log 4 \leqq 2\varphi\left(\sqrt{r_n}\right)(\log r)^2 + \log 4$$

for $t_n \leqq r \leqq \sqrt{r_{n+1}}$. Since $T(r, f)$ is an increasing function of r, we get for $\sqrt{r_n} < r < t_n$

$$T(r, f) \leqq T(2r_n, f) \leqq 2\varphi(r)\left(\log(2r^2)\right)^2 + \log 4,$$

which together with (10.10) proves (2.10).

From (10.9) and (10.5) we deduce that

$$N(t_n, f) \leqq \log 2 + 2s_{n-1} \log t_n$$

and

$$N(2t_n, f) \leqq s_n \log 3 + 3s_{n-1} \log r_n.$$

These estimates combined with (10.3) and (10.8) yield

(10.11) $$T(t_n, f) \leqq 3s_{n-1} \log r_n \leqq s_n/n$$

and

(10.12) $$T(2t_n, f) \leqq 4s_n.$$

Suppose that $|z|=t_n$. We get from (10.5)

$$|f'(z)| \geqq \frac{s_n}{9r_n} - 2\sum_{k<n}(s_k/r_k)(r_k/r_n)^{s_k+1} - 2\sum_{k>n}(s_k/r_k)\big((2r_n)/r_k\big)^{s_k-1}$$

$$\geqq \frac{s_n}{9r_n} - \frac{2}{r_n}\sum_{k<n} s_k - 2\sum_{k>n} s_k/r_k,$$

and we deduce from (10.3) and (10.2) that

$$|f'(z)| \geqq \frac{s_n}{9r_n} - \frac{4s_{n-1}}{r_n} - \frac{4}{\sqrt{r_{n+1}}} \geqq \frac{s_n}{18r_n}.$$

This implies together with (10.8) that

$$t_n\lambda(t_n, f) \geqq \frac{s_n}{18\cdot 17}.$$

This together with (10.11) and (10.12) proves (2.12) and (2.11). Theorem 6 is proved.

11. Proof of Theorem 7

Let f be as in Theorem 7. Let $n(r)$ be defined by (9.1). We have

$$n(r)\log r \leqq \int_r^{r^2} n(t)t^{-1}dt \leqq (3+o(1))T(r^2, f),$$

and we deduce from (2.13) that there exists $K>0$ such that

$$n(r) < K\log r$$

for all large values of r. This implies that for any large r, there exists z_r, $|z_r|=r$, such that f does not take any of the values 0, 1 and ∞ in

$$|z-z_r| < \frac{r}{K\log r}, \tag{11.1}$$

and just as in the proof of (9.4), we deduce from (11.1) that

$$\lambda(r, f) = O\left(\frac{\log r}{r}\right) \quad \text{as} \quad r\to\infty.$$

Since f is transcendental, we get

$$r\lambda(r, f) = O(\log r) = o\big(T(r, f)\big) \quad \text{as} \quad r\to\infty,$$

which proves Theorem 7.

12. Proof of Theorem 8

Contrary to the assertion of Theorem 8, let us suppose that there exists a transcendental meromorphic function f such that

$$\liminf_{r\to\infty} \frac{\log \mu(r, f)}{T(r, f)} \leqq -1-9d \tag{12.1}$$

for some $d>0$. Just as in the proof of Theorem 2, we may assume that $f(0)=0$, and deduce (6.3) so that there exist $t>0$ and $s_0>0$ such that

$$m(r, a) \leqq T(r, f)+6+\log(1/t) \tag{12.2}$$

if $0<|a|<9$ and $r>s_0$.

It follows from (12.1) that there exists an increasing sequence $r_n, r_n\to\infty$ as $n\to\infty$, such that

$$\log \varrho(f(z)) < -(1+8d)T(r_n, f) \tag{12.3}$$

on $|z|=r_n$ for all n. Since $m(r, \infty, f)\leqq T(r, f)$ and

$$m(r, 0, f) \leqq (1+o(1))T(r, f) \quad \text{as} \quad r\to\infty,$$

we may choose a point z_n lying on $|z|=r_n$ such that

$$\big|\log|f(z_n)|\big| \leqq (1+d)T(r_n, f) \tag{12.4}$$

if n is large enough.

Let us suppose that

$$\big|\log|f(w_1)|\big| > (1+2d)T(r_n, f) \tag{12.5}$$

for some w_1 lying on $|z|=r_n$. Then we may choose an arc J_n contained in $|z|=r_n$ such that $z_n\in J_n$,

$$\big|\log|f(w)|\big| \leqq (1+2d)T(r_n, f) \tag{12.6}$$

for all $w\in J_n$, and that

$$\big|\log|f(w_k)|\big| = (1+2d)T(r_n, f) \tag{12.7}$$

$(k=2, 3)$ for the end points w_2 and w_3 of the arc J_n. For $z\in J_n$ we get from (12.3) and (12.6)

$$\log|f'(z)/f(z)| = \log(\varrho f(z))+\log(|f(z)|+1/|f(z)|) \leqq -(6d+o(1))T(r_n, f),$$

and integrating along J_n we deduce that

$$\big|\log|f(z)/f(z_n)|\big| \leqq \left|\int_{z_n}^{z} \frac{f'(w)}{f(w)}\,dw\right| \leqq \exp\{-(6d+o(1))T(r_n, f)\} = o(1) \tag{12.8}$$

for all $z\in J_n$. This implies together with (12.4) that

$$\big|\log|f(w_2)|\big| \leqq (1+d+o(1))T(r_n, f).$$

This is a contradiction with (12.7) if n is large, and we conclude that

(12.9) $$\left|\log|f(z)|\right| \leqq \left(1+2d+o(1)\right)T(r_n, f)$$

for all z lying on $|z|=r_n$, and that (12.8) holds for all z lying on $|z|=r_n$.

Let us suppose that there exist large values of n such that

(12.10) $$|f(z_n)| \leqq 4.$$

Then it follows from (12.8) that $|f(z)|\leqq 8$ on $|z|=r_n$, and from (12.3) we get

(12.11) $$\log|f'(z)| \leqq -(1+8d)T(r_n, f)+\log 65$$

on $|z|=r_n$. Integrating along the circle $|z|=r_n$, we deduce from (12.11) that

$$\log|f(z)-f(z_n)| \leqq -(1+8d)T(r_n, f)+\log 65+2\log r_n$$

on $|z|=r_n$, which implies that

$$m\left(r, f(z_n), f\right) \geqq \left(1+8d+o(1)\right)T(r_n, f).$$

This contradicts (12.2) and we deduce that (12.10) is not possible if n is large. Therefore we have

(12.12) $$|f(z_n)| \geqq 4$$

for all large n.

It follows from (12.8) that

$$|f(z)| = \left(1+o(1)\right)|f(z_n)|$$

for all z lying on $|z|=r_n$, and we get from (12.12)

(12.13) $$\log|f(z)| = \left(1+o(1)\right)m(r_n, \infty, f)$$

for all z lying on $|z|=r_n$. This implies together with (12.3) that

$$\log|f'(z)| \leqq -\left(1+8d+o(1)\right)T(r_n, f)+2m(r_n, \infty, f)$$

on $|z|=r_n$, and integrating along the circle $|z|=r_n$, we get

(12.14) $$\log|f(z)-f(z_n)| \leqq -\left(1+8d+o(1)\right)T(r_n, f)+2m(r_n, \infty, f)$$

on $|z|=r_n$. Applying the Jensen formula to the function $f(z)-f(z_n)$, we get from (12.14), since $f(0)=0$,

$$\log|f(z_n)| \leqq \frac{1}{2\pi}\int_0^{2\pi} \log|f(r_n e^{i\alpha})-f(z_n)|\,d\alpha+N(r_n, \infty, f)$$

$$\leqq -\left(1+8d+o(1)\right)T(r_n, f)+N(r_n, \infty, f)+2m(r_n, \infty, f),$$

which together with (12.13) implies that

$$\left(1+8d+o(1)\right)T(r_n, f) \leqq T(r_n, f) \quad \text{as} \quad n\to\infty.$$

This is a contradiction, and we deduce that (3.1) holds for all transcendental meromorphic functions. Theorem 8 is proved.

13. Proof of Theorem 9

Let f be as in Theorem 9. Since

$$m(r, f') \leqq m(r, f) + o(T(r, f))$$

outside a set of finite linear measure, it follows from Lemma 4 that we may choose a sequence r_n, $r_n \to \infty$ as $n \to \infty$, such that

$$\log |f(z)| = m(r_n, \infty, f) - m(r_n, 0, f) + o(T(r_n, f)) \tag{13.1}$$

and

$$\begin{aligned} \log |f'(z)| &= m(r_n, \infty, f') - m(r_n, 0, f') + o(T(r_n, f)) \\ &\leqq m(r_n, \infty, f) + o(T(r_n, f)) \end{aligned} \tag{13.2}$$

on $|z| = r_n$.

If $m(r_n, 0, f) > 0$, we deduce from (13.1) that

$$m(r_n, \infty, f) = o(T(r_n, f)),$$

and if $m(r_n, \infty, f) > 0$, then

$$m(r_n, 0, f) = o(T(r_n, f)).$$

These estimates imply together with (13.1) that

$$\log (1 + |f(z)|^2) = 2m(r_n, f) + o(T(r_n, f)) \tag{13.3}$$

on $|z| = r_n$ for all n. From (13.2) and (13.3) we get

$$\log \mu(r_n, f) \leqq -m(r_n, \infty, f) + o(T(r_n, f)) \leqq -(\delta(\infty, f) + o(1))T(r_n, f) \quad \text{as} \quad n \to \infty,$$

which proves Theorem 9.

14. Proof of Theorem 10

Let d satisfy $0 < d < 1$. We set

$$f(z) = \prod_{n=1}^{\infty} \frac{r_n + z}{r_n - z},$$

where $r_n = n^{1/d}$. We have

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{2r_n}{r_n^2 - z^2},$$

and since $|f(ir)| = 1$ for any real r, we get

$$\mu(r, f) \geqq \frac{|f'(ir)|}{1 + |f(ir)|^2} \geqq \sum_{n=1}^{\infty} \frac{r_n}{r_n^2 + r^2} \tag{14.1}$$

for any $r > 0$.

From the choice of r_n we deduce that

$$n(r, 0, f) = (1+o(1))r^d \quad \text{as} \quad r \to \infty,$$

which implies that

$$n(2r, 0, f) - n(r, 0, f) = (2^d - 1 + o(1))r^d \tag{14.2}$$

as $r \to \infty$. It follows from (14.1) and (14.2) that

$$\mu(r, f) \geqq \sum_{r \leqq r_n \leqq 2r} \frac{r_n}{r_n^2 + r^2} \geqq \big(n(2r, 0, f) - n(r, 0, f)\big)\frac{r}{5r^2} \tag{14.3}$$

$$\geqq (2^d - 1 + o(1))\frac{r^d}{5r} \quad \text{as} \quad r \to \infty.$$

If $|z| = r$ and $|\arg z| < \pi/6$, then

$$\log|f(z)| \geqq \sum_{r \leqq r_n \leqq 2r} \log\left|\frac{r_n + z}{r_n - z}\right|$$

$$\geqq \big(n(2r, 0, f) - n(r, 0, f)\big)\log(5/4) \geqq (2^d - 1 + o(1))r^d \log(5/4).$$

This implies that

$$m(r, \infty, f) \geqq \frac{1}{6}\log(5/4)(2^d - 1 + o(1))r^d. \tag{14.4}$$

On the other hand, since

$$n(r, 0, f) = n(r, \infty, f) = O(r^d) \quad \text{as} \quad r \to \infty,$$

we have (see e.g. Nevanlinna [7, p. 223])

$$T(r, f) = O(r^d) \quad \text{as} \quad r \to \infty. \tag{14.5}$$

From (14.4) and (14.5) we deduce that $\delta(\infty, f) > 0$ and that the order of f is d. From (14.3) and (14.5) it follows that f satisfies (3.4). Theorem 10 is proved.

15. Proof of Theorem 11

Let f be a transcendental meromorphic function. Let $n(r)$ be defined by (9.1). Just as in the proof of Theorem 5, we deduce that there exists r_0 such that

$$n(3r/2) \leqq \frac{4T(2r, f)}{\log(4/3)} \tag{15.1}$$

for $r \geqq r_0$.

If the lower order of f is infinite, we choose $r_n = e^n$ for any positive integer n. Let us suppose that

$$\liminf_{r \to \infty} \frac{T(2r, f)}{T(r, f)} = \infty. \tag{15.2}$$

Then we have for any $K>1$,

(15.3) $$T(2r,f)>KT(r,f)$$

for $r\geqq r_K$. This implies that

$$T(2^n r_K,f)>K^n T(r_K,f)$$

for all n, and we deduce that if $2^{n-1}r_K\leqq t\leqq 2^n r_K$, then

$$\frac{\log T(t,f)}{\log t}\geqq\frac{(n-1)\log K+\log T(r_K,f)}{n\log 2+\log r_K}.$$

This implies that

(15.4) $$\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}>\log K,$$

and we deduce that if (15.2) holds, then the lower order of f is infinite. Therefore, if the lower order of f is finite and positive, we may choose a sequence r_n, $r_n\to\infty$ as $n\to\infty$, such that

(15.5) $$\frac{T(2r_n,f)}{T(r_n,f)}<A$$

for all n, A being a constant.

If f satisfies (15.3) for some $K>1$, we deduce from (15.4) that the lower order of f is positive. Therefore, if the lower order of f is zero, then f satisfies (2.3), and it follows from Lemma 3 that there exists a sequence r_n, $r_n\to\infty$ as $n\to\infty$, such that

(15.6) $$n(3r_n/2)=o\big(T(r_n,f)\big)\quad\text{as}\quad n\to\infty.$$

In all cases, we write, if $n(r_n)\geqq 1$,

$$d_n=\frac{r_n}{4n(3r_n/2)}.$$

Then there exists t_n, $r_n\leqq t_n\leqq 3r_n/2$, such that f does not take any of the values 0, 1 and ∞ in the annulus

$$B_n=\{z:t_n-d_n<|z|<t_n+d_n\}.$$

Since the disc $|z-w|<d_n$ is contained in B_n for all w satisfying $|w|=t_n$, we get, just as in the proof of (9.4),

$$\varrho\big(f(w)\big)\leqq\frac{2K_1}{d_n}=\frac{8K_1n(3r_n/2)}{r_n}$$

for all w lying on $|z|=t_n$. This implies that

(15.7) $$t_n\mu(t_n,f)\leqq 16K_1n(3r_n/2).$$

Combining (15.1) and (15.7), we get

(15.8) $$t_n\mu(t_n,f)=O\big(T(2r_n,f)\big)\quad\text{as}\quad n\to\infty.$$

This proves (3.5), since $r_n \leqq t_n \leqq 2r_n$ and $T(r, f)$ is an increasing function of r. Similarly, combining (15.8) and (15.5), we get (3.6) for functions of finite positive lower order. If the lower order of f is zero, then (3.7) follows from (15.7) and (15.6). Theorem 11 is proved.

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