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POSITIVE BIHARMONIC FUNCTIONS

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Let Q be the class of quasiharmonic functions q, defined by $\Delta q=1$, $\Delta=d\delta+\delta d$, on a Riemannian manifold. The best source for counterexamples in the Q-classification of Riemannian manifolds has been the Poincaré N-ball B_α^N, that is, the unit N-ball $\{x=(x^1, \ldots, x^N) | |x|<1\}$ endowed with the Riemannian metric $ds_\alpha=(1-|x|^2)^\alpha |dx|$, $\alpha\in\mathbf{R}$. In Sario-Wang [3] it was shown that for each of the classes QP, QB, QD, and QC of Q-functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively, values α can be found for which these classes are void. By contrast, the class QN of negative Q-functions on B_α^N is not void for any α. Even the Euclidean plane $\mathbf{R}^2$, which is void of any other functions considered in classification theory, trivially carries QN-functions.

It was, therefore, long thought that there may exist no Riemannian manifolds which carry no QN-functions. That this is, however, not the case, was shown in Nakai-Sario [1].

Trivially, the existence of QN-functions implies that the class H^2P of positive nonharmonic biharmonic functions is nonvoid. The natural question thus arose, and has remained unsolved thus far: do there exist Riemannian manifolds which do not carry H^2P-functions? In the present paper we shall answer this question in the affirmative for any dimension $N\geqq 2$: the class $O_{H^2P}^N$ of Riemannian N-manifolds which carry no H^2P-functions is nonvoid. We also show that the inclusion $O_{H^2P}^N\subset O_{QN}^N$ is strict by constructing a manifold in the delicately "small" class of Riemannian N-manifolds which carry H^2P-functions but nevertheless do not carry QN-functions.

1. Let S be the complex plane with the conformal metric

$$ds^2 = \lambda(x)\,dr^2+\lambda(x)\,r^2\,d\theta^2,$$

where $x=re^{i\theta}$ and

$$\lambda(x) = 1+(3+\varepsilon)(5+\varepsilon)(1-\sin\theta)\,r^{2+\varepsilon}$$

(cf. Nakai—Sario [1]). Denote by Δ the Laplace—Beltrami operator on S and by Δ_e the Euclidean Laplacian. It is clear that $\Delta=\lambda^{-1}\Delta_e$. Therefore, harmonic functions on S are the same as harmonic functions on the Euclidean plane, viz. [2],

$$h(x) = h(re^{i\theta}) = \sum_{n=0}^{\infty} r^n (a_n \sin n\theta + b_n \cos n\theta).$$

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An arbitrary biharmonic function w on S must satisfy $\Delta w = h$. Let H^2 be the class of nonharmonic biharmonic functions. By a straightforward but somewhat tedious computation we obtain:

Lemma. *Every* $w(x) \in H^2(S)$ *can be written as*

$$w(x) = h(x) + cq(x) + \sum_{n=1}^{\infty} c_n u_n(x) + \sum_{n=1}^{\infty} d_n v_n(x)$$

with $\Delta h(x) = 0$, $\Delta q(x) = 1$, $\Delta u_n(x) = r^n \sin n\theta$, *and* $\Delta v_n(x) = r^n \cos n\theta$. *Here*

$$h(x) = \sum_{n=0}^{\infty} r^n (a_n \sin n\theta + b_n \cos n\theta),$$

$$q(x) = -\frac{1}{4} r^2 - [(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2} - \sin\theta] r^{4+\varepsilon},$$

$$u_n(x) = -(4n+4)^{-1} r^{n+2} \sin n\theta - (3+\varepsilon)(5+\varepsilon)[(n+4+\varepsilon)^2 - n^2]^{-1} r^{n+4+\varepsilon} \sin n\theta$$
$$-(3+\varepsilon)(5+\varepsilon) r^{n+4+\varepsilon} (\alpha_n \sin\theta \sin n\theta + \beta_n \cos\theta \cos n\theta),$$

where α_n, β_n *satisfy*

$$\begin{cases} [(n+4+\varepsilon)^2 - n^2 - 1]\alpha_n + 2n\beta_n = -1, \\ 2n\alpha_n + [(n+4+\varepsilon)^2 - n^2 - 1]\beta_n = 0, \end{cases}$$

and

$$v_n(x) = -(4n+4)^{-1} r^{n+2} \cos n\theta - (3+\varepsilon)(5+\varepsilon)[(n+4+\varepsilon)^2 - n^2]^{-1} r^{n+4+\varepsilon} \cos n\theta$$
$$-(3+\varepsilon)(5+\varepsilon) r^{n+4+\varepsilon} (\gamma_n \sin\theta \cos n\theta + \delta_n \cos\theta \sin n\theta),$$

where γ_n, δ_n *satisfy*

$$\begin{cases} [(n+4+\varepsilon)^2 - n^2 - 1]\gamma_n - 2n\delta_n = -1, \\ 2n\gamma_n - [(n+4+\varepsilon)^2 - n^2 - 1]\delta_n = 0. \end{cases}$$

2. We shall show that the manifold S carries no positive nonharmonic biharmonic functions:

Lemma. $S \in O^2_{H^2P}$.

Proof. Suppose there exists an H^2P-function $w(x)$ on S. Clearly

$$\frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta})(1 \pm \sin n\theta)\, d\theta \geqq 0,$$

where the integral is taken along the circle $|x| = r$. For $n \geqq 2$,

$$\frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta})(1 + \sin n\theta)\, d\theta = b_0 - \frac{1}{4} cr^2 - (3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2} r^{4+\varepsilon} + \frac{1}{2} a_n r^n$$
$$-\frac{1}{2} c_n \{(4n+4)^{-1} r^{n+2} + (3+\varepsilon)(5+\varepsilon)[(n+4+\varepsilon)^2 - n^2]^{-1} r^{n+4+\varepsilon}\} \geqq 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})(1-\sin n\theta)d\theta = b_0-\frac{1}{4}cr^2-(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}r^{4+\varepsilon}-\frac{1}{2}a_n r^n$$

$$+\frac{1}{2}c_n\{(4n+4)^{-1}r^{n+2}+(3+\varepsilon)(5+\varepsilon)[(n+4+\varepsilon)^2-n^2]^{-1}r^{n+4+\varepsilon}\} \geqq 0.$$

On letting $r\to\infty$ we conclude from these two inequalities that $c_n=0$ for $n\geqq 2$, and consequently $a_n=0$ for $n\geqq 5$. For $n=1$, we repeat the argument and obtain $c_1=0$ and therefore $c_n=0$ for all n.

Similarly, an integration of $w(re^{i\theta})(1\pm\cos n\theta)$ gives the corresponding results for the cosine terms: $d_n=0$, or all n and $b_n=0$, or $n\geqq 5$.

We have reduced the expansion of w to

$$w(x) = \sum_{n=0}^{4} r^n(a_n\sin n\theta+b_n\cos n\theta)-c\left\{\frac{1}{4}r^2+[(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}-\sin\theta]r^{4+\varepsilon}\right\}.$$

As $r\to\infty$,

$$w(x) = -c[(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}-\sin\theta]r^{4+\varepsilon}+O(r^4).$$

Since

$$(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}-\sin\pi/2 < 0,$$

$$(3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}-\sin(-\pi/2) > 0,$$

and $w(re^{i\theta})\geqq 0$ for all r, θ, we must have $c=0$. Thus

$$w(x) = \sum_{n=0}^{4} r^n(a_n\sin n\theta+b_n\cos n\theta),$$

a harmonic function. Furthermore, w reduces to a constant since a parabolic manifold S does not carry nonconstant positive harmonic functions.

3. From the above case of dimension 2 we proceed to the construction of an N-dimensional, $N\geqq 2$, manifold which carries no positive nonharmonic biharmonic functions (cf. Nakai—Sario [1]).

Theorem. $O^N_{H^2P}\neq\emptyset$, $N\geqq 2$.

Proof. We may assume $N>2$. Let S be as in Section 1, and take any compact bordered $(N-2)$-manifold T, with local coordinates $y=(y^1, \dots, y^{N-2})$, say, and with $\partial T=\emptyset$. For our purposes it suffices to consider the special case of the torus $|y^i|<1$, $i=1, \dots, N-2$, with each $\{y^i=1\}$ identified with $\{y^i=-1\}$. Its border $\partial T=\emptyset$ can be reviewed as an oriented compact $(N-3)$-manifold traced in opposite directions. Endow the product manifold $S\times T$ with the metric

$$ds^2 = \lambda(x)dx^2 + dy^2.$$

The Laplace—Beltrami operator Δ on $S\times T$ is then

$$\Delta = \Delta_x + \Delta_y,$$

where Δ_x and Δ_y are the Laplace—Beltrami operators on S and T, respectively. We shall show that $S\times T$ carries no H^2P-functions whenever S carries none. Suppose $S\times T \notin O^N_{H^2P}$ and let $w(x, y) \in H^2P(S\times T)$. Define

$$w_0(x) = \int_T w(x, y)\,dy$$

on S. In view of $\partial T = \emptyset$, we have

$$\Delta_x w_0(x) = \int_T \Delta_x w(x, y)dy = \int_T (\Delta - \Delta_y)w(x, y)dy = \int_T \Delta w(x, y)dy - \int_T \Delta_y w(x, y)dy$$

$$= \int_T \Delta w(x, y)dy + \int_{\partial T} *_y d_y w(x, y) = \int_T \Delta w(x, y)dy.$$

By $\Delta^2 w = 0$,

$$\Delta_x^2 w_0(x) = \int_T \Delta_x(\Delta w(x, y))dy = \int (\Delta - \Delta_y)(\Delta w(x, y))dy$$

$$= -\int_T \Delta_y(\Delta w(x, y))dy = \int_{\partial T} *_y d_y(\Delta w(x, y)) = 0.$$

Thus $w_0(x) \in H^2(S)$. Obviously, $w_0(x)$ is positive whenever $w(x, y)$ is. Therefore, $S\times T \notin O^N_{H^2P}$ implies $S \notin O^2_{H^2P}$, a contradiction.

4. Next we assert:

Theorem. *The strict inclusion*

$$O^N_{H^2P} < O^N_{QN}$$

holds for all $N \geqq 2$.

Proof. Remove from the complex plane the closed unit disk and consider the remaining manifold

$$S_1 = \{(r, \theta) | 1 < r < \infty, -\pi \leqq \theta \leqq \pi\}$$

endowed with the conformal metric $ds^2 = \lambda(x)dx^2$ as in Section 1. We shall first show that the manifold S_1 carries H^2P-functions but no QN-functions.

Every $w \in H^2(S_1)$ has an expansion

$$w(x) = h(x) + cq(x) + dp(x) + \sum_{n\neq 0} [c_n u_n(x) + d_n v_n(x)],$$

where $q(x)$, $u_n(x)$, $v_n(x)$ are as in Section 1 for $n \neq -1$, and

$$h(x) = a \log r + \sum_{n=-\infty}^{\infty} r^n (a_n \sin n\theta + b_n \cos n\theta),$$

$$p(x) = -\frac{1}{4} r^2(\log r - 1) + (3+\varepsilon)(5+\varepsilon)(4+\varepsilon)^{-2}[2(4+\varepsilon)^{-1} - \log r] r^{4+\varepsilon}$$
$$+ \{\log r - 2(4+\varepsilon)[(4+\varepsilon)^2 - 1]^{-1}\} \sin \theta r^{4+\varepsilon},$$

$$u_{-1}(x) = -\frac{1}{2} r \log r \sin \theta + (3+\varepsilon)(5+\varepsilon)[(3+\varepsilon)^2 - 1]^{-1} r^{3+\varepsilon} \sin \theta$$
$$-(3+\varepsilon)(5+\varepsilon) r^{3+\varepsilon}(\alpha_{-1} \sin^2 \theta + \beta_{-1} \cos^2 \theta),$$

$$v_{-1}(x) = -\frac{1}{2} r \log r \cos \theta - (3+\alpha)(5+\alpha)[(3+\varepsilon)^2 - 1]^{-1} r^{3+\varepsilon} \cos \theta$$
$$-(3+\varepsilon)(5+\varepsilon) r^{3+\varepsilon}(\gamma_{-1} - \delta_{-1}) \sin \theta \cos \theta,$$

where α_{-1}, β_{-1}, γ_{-1}, and δ_{-1} are as in Section 1 for $n=-1$. In view of Section 1, $\Delta u_n(x) = r^n \sin n\theta$ and $\Delta v_n(x) = r^n \cos n\theta$. Clearly $u_n(x)$ and $v_n(x)$ are biharmonic. By virtue of $u_n(x) = O(r^{n+4+\varepsilon})$ and $v_n(x) = O(r^{n+4+\varepsilon})$, $u_n(x)$ and $v_n(x)$ are bounded if $n \leqq -5$. By adding an appropriate constant c, we obtain a function $w_0(x) = u_n(x) + c$ or $v_n(x) + c$ which belongs to $H^2P(S_1)$, so that $S_1 \in \tilde{O}^2_{H^2P}$; here and later $\tilde{O}$ stands for the complement of O.

In order to show that $S_1 \times T \in \tilde{O}^N_{H^2P}$, define

$$w_1(x, y) = w_0(x), \quad (x, y) \in S_1 \times T.$$

Since

$$\Delta^2 w_1(x, y) = (\Delta_x + \Delta_y)^2 w_0(x) = \Delta_x^2 w_0(x) = 0,$$

and w_1 is obviously positive, $w_1 \in H^2P(S_1 \times T)$.

To show that $S_1 \in O^2_{QN}$, we reason as in Section 2. The conclusion $S_1 \times T \in O^N_{QN}$ then follows as in Section 3. Thus we have $S_1 \times T \in O^N_{QN} \cap \tilde{O}^N_{H^2P}$, hence the Theorem.

5. We have seen that the manifold $S = \{|x| < \infty\}$ with the Riemannian metric $ds^2 = \lambda(x) dx^2$, $\lambda(x) = 1 + (3+\varepsilon)(5+\varepsilon)(1 - \sin \theta) r^{2+\varepsilon}$, carries no H^2P-functions. On the other hand, the manifold S_1 obtained by deleting the unit disk from S does carry H^2P-functions. It is perhaps of some interest to investigate the intermediate manifold $S_0 = \{0 < |x| < \infty\}$ with the same Riemannian metric as S and S_1. Even though every $w \in H^2(S_0)$ has the same expansion as $w \in H^2(S_1)$, repeating the reasoning in Section 2 leads to the conclusion that, contrary to the case of S_1, the manifold S_0 carries no H^2P-functions. Thus S_0 is another counterexample to show the nonvoidness of $O^2_{H^2P}$, and hence also of O^2_{QN}. An analogue of the argument in Section 3 extends the conclusion to an arbitrary dimension.

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