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ON THE SINGULARITIES OF CERTAIN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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0. Introduction. In this paper we shall study some aspects of the singularities of equations $-\Delta_p u = f(x, u)$ where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p>1)$ is the so-called p-Laplacian, and f a continuous function subject to certain growth restrictions. Equations of this type have been studied in connection with a variety of problems (cf. references in [10]).

Theorem 1 of this paper complements a recent series of works on removable singularities [3], [1], [12], [11]. In [11] the following is proved:

Theorem A. *Let Ω be an open set in $\boldsymbol{R}^n$, $q\in\Omega$, and $\Omega'=\Omega-\{q\}$. Suppose $1<p<n$, that f is a continuous real function on $\Omega\times\boldsymbol{R}$ satisfying*

$$(1)\qquad \liminf_{r\to\infty}\frac{f(x,r)}{r^{n(p-1)/(n-p)}}>0,\quad \limsup_{r\to-\infty}\frac{f(x,r)}{|r|^{n(p-1)/(n-p)}}<0$$

uniformly in Ω, that $u\in W^{1,p}_{\mathrm{loc}}(\Omega')\cap L^{\infty}_{\mathrm{loc}}(\Omega')$, and $\Delta_p u\in L^1_{\mathrm{loc}}(\Omega')$ (in the sense of distributions). Then if u is a solution of

$$(2)\qquad -\Delta_p u+f(x,u)=0$$

in $\mathscr{D}'(\Omega')$, there exists a locally Hölder continuous function $\tilde{u}$, defined in all of Ω, which coincides with u a.e. in Ω and satisfies (2) *in $\mathscr{D}'(\Omega)$.*

We have

Theorem 1. *If in Theorem* A *we take $p=n$ and replace condition* (2) *by*

$$(3)\qquad \liminf_{r\to\infty}\frac{f(x,r)}{e^{r^{\delta}}}>0,\quad \limsup_{r\to-\infty}\frac{f(x,r)}{e^{|r|^{\delta}}}<0$$

for some fixed $\delta>1$, then the conclusions of Theorem A *again hold.*

In §5 we shall discuss examples to show $\delta>1$ in (3) is essential.

In [10], interior estimates are derived for functions u satisfying

$$(4)\qquad -\Delta_p u+f(u)\leq 0 \qquad\text{(a.e.)}$$

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in the case of "weak diffusion for large intensities":

$$\int_0^\infty dr\Big(\int_0^r f(s)\,ds\Big)^{-1/p} < \infty. \tag{5}$$

For $p=2$, conditions of this type have been used in connection with nonexistence of entire solutions by numerous authors [14], [13], [6], [7], [2]. In our next result we study this effect for the p-Laplacian.

Theorem 2. *Let $f(s)$ be a positive nondecreasing locally Lipschitz function defined on $\boldsymbol{R}$ and satisfying* (5).

(A) *If* $1<p$ *then* (4) *has no subsolutions u with* $u\in W^p_{\mathrm{loc}}(\boldsymbol{R}^n)\cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n)$ and $\Delta_p u\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ (in the sense of distributions).

(B) *If* $1<p<n$ *and S is any compact subset of $\boldsymbol{R}^n$ then there are no subsolutions u of* (5) *with* $u\in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n-S)\cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n-S)$ *and* $\Delta_p u\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n-S)$ *(in the sense of distributions).*

I. Preliminary lemmas. There are general comparisons theorems which cover the p-Laplacian. We require only a very simple version (cf. [11; p. 5]).

Lemma A. *In a region* $\Omega\subseteqq\boldsymbol{R}^n$ *suppose* $u,v\in W^{1,p}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ $(1<p)$, $\Delta_p u$, $\Delta_p v\in L^1_{\mathrm{loc}}(\Omega)$ *(in the sense of distributions) and* $(u-v)^+\in W^{1,p}_0(\Omega)$. *If g is a nondecreasing function on $\boldsymbol{R}$ and*

$$-\Delta_p u+g(u)\leqq 0 \quad in\ \mathcal{D}'(\Omega)$$

$$-\Delta_p v+g(v)\geqq 0 \quad in\ \mathcal{D}'(\Omega),$$

then $u\leqq v$ *a.e. in* Ω.

Proof. Let $\Psi\in\mathcal{C}^1(\boldsymbol{R})$ be bounded, vanishing on $(-\infty,0]$, and strictly increasing on $[0,\infty)$. Then, since $\Psi(u-v)\in W^{1,p}_0(\Omega)$, (1.1) implies

$$\int_\Omega (|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot(\nabla u-\nabla v)\,\Psi'(u-v)\,dx \leqq \int_\Omega (g(v)-g(u))\,\Psi(u-v)\,dx.$$

Now, $p>1$ so $(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}v)\cdot(\nabla u-\nabla v)\geqq 0$, and $\Psi'\geqq 0$ as well. Thus, it follows from the Poincaré lemma that $u\leqq v$ a.e. on Ω.

Lemma 1. *Let* $\Omega\subseteqq\boldsymbol{R}^n$ *be a region and* $q\in\Omega$, $\Omega'=\Omega-\{q\}$. *Suppose* $u\in W^{1,n}_{\mathrm{loc}}(\Omega')\cap L^\infty_{\mathrm{loc}}(\Omega')$ *and* $\Delta_n u\in L^1_{\mathrm{loc}}(\Omega')$ *(in the sense of distributions in Ω'). If for some constants* $a>0$, $C\geqq 0$, $\delta>1$

$$-\Delta_n u+e^{au^\delta}\leqq C \tag{1.1}$$

a.e. on $\{x\in\Omega\colon u(x)\geqq 0\}$, *then* $u^+\in L^\infty_{\mathrm{loc}}(\Omega)$.

Proof. We may take $q=0$ and $\Omega=\{|x|<\varrho\}$ for some $0<\varrho<1$. Given x_0 such that $0<|x_0|<\varrho/2$ we define, for $\delta\geqq 1$

(1.2)
$$V(x)=V_{x_0}(x)=\left(\log\frac{1}{R^{n/(n-1)}-|x-x_0|^{n/(n-1)}}\right)^{1/\delta}\left(R=\frac{|x_0|}{2},\ |x-x_0|<R\right).$$

Now V as defined is radial about x_0, and hence writing $r=|x-x_0|$ and taking differentiations with respect to r we have $(n-1)\dot{V}^{n-2}(\ddot{V}+(1/r)\dot{V})=\Delta_n V$. Now,

$$\dot{V}=\frac{n}{\delta(n-1)}\left(\log\frac{1}{R^{n/(n-1)}-r^{n/(n-1)}}\right)^{(1/\delta)-1}\left(\frac{r^{1/(n-1)}}{R^{n/(n-1)}-r^{n/(n-1)}}\right)=$$

$$=\frac{n}{\delta(n-1)}V^{1-\delta}e^{V^\delta}r^{1/(n-1)}$$

$$\ddot{V}=\frac{n}{\delta(n-1)}\left[\frac{n(\delta^{-1}-1)}{(n-1)}\left(\log\frac{1}{R^{n/(n-1)}-r^{n/(n-1)}}\right)^{(1/\delta)-2}\left(\frac{r^{1/(n-1)}}{R^{n/(n-1)}-r^{n/(n-1)}}\right)^2\right.$$

$$+\left(\log\left(\frac{1}{R^{n/(n-1)}-r^{n/(n-1)}}\right)\right)^{(1/\delta)-1}\times$$

$$\left.\times\left(\frac{r^{(2-n)/(n-1)}}{(n-1)(R^{n/(n-1)}-r^{n/(n-1)})}+\frac{nr^{2/(n-1)}}{(n-1)(R^{n/(n-1)}-r^{n/(n-1)})^2}\right)\right]$$

$$\leqq\frac{n}{n-1}V^{1-\delta}\left(\frac{e^{V^\delta}r^{(2-n)/(n-1)}}{n-1}+\frac{ne^{2V^\delta}}{n-1}\right)\leqq 2\left(\frac{n}{n-1}\right)^2V^{1-\delta}e^{2V^\delta}r^{(2-n)/(n-1)}.$$

Thus,

$$\Delta_n V\leqq\frac{n^n}{(n-1)^{n-1}}V^{(1-\delta)(n-2)}e^{(n-2)V^\delta}r^{(n-2)/(n-1)}(2V^{1-\delta}e^{2V^\delta}r^{(2-n)/(n-1)}$$
$$+V^{(1-\delta)}e^{V^\delta}r^{(2-n)/(n-1)})\leqq\frac{4n^n}{(n-1)^{n-1}}V^{(1-\delta)(n-1)}e^{nV^\delta}. \tag{1.3}$$

Let $v=a^{-\delta^{-1}}nV$. Then, from (1.3) it follows that

$$\Delta_n v\leqq\frac{4n^{2n-1}a^{(1-n)/\delta}}{(n-1)^{n-1}}V^{(1-\delta)(n-1)}e^{nV^\delta}\leqq\frac{4n^{2n-1}a^{(1-n)/\delta}}{(n-1)^{n-1}}V^{(1-\delta)(n-1)}e^{av^\delta}. \tag{1.4}$$

From (1.2) we find that for x_0 and hence R sufficiently small, V and v can be made arbitrarily large. It then follows from (1.4) that there exists $R_0=R_0(n,\delta)$ such that for all x_0 with $0<|x_0|\leqq R_0$, the corresponding $v(x)=v_{x_0}(x)$ all satisfy

$$\Delta_n v\leqq e^{av^\delta}-C. \tag{1.5}$$

Comparing (1.1) and (1.5) and taking account of the fact that v is infinite for x on $|x-x_0|=|x_0|/2$ we may apply Lemma A and obtain for a.e. x_0 such that $0<|x_0|\leqq R_0$ the estimate

$$u(x_0) \leqq v(x_0) = a^{-\delta^{-1}} nV(x_0) = a^{-\delta^{-1}} n\left(\frac{n}{n-1}\log\frac{2}{|x_0|}\right)^{1/\delta}. \tag{1.6}$$

To complete the proof we show that (1.1) and (1.6) imply that $u^+\in L^\infty(|x|<R_0)$.

If we replace u by $U=\varkappa u$ $(\varkappa>1)$ then $\Delta_n U=\varkappa^{n-1}\Delta_n u\geqq \varkappa^{n-1}e^{a\varkappa^{-\delta}}U^\delta - C$. Thus, we may fix $\varkappa$ sufficiently large so that a.e. for $0<|x|<R_0$

$$\Delta_n U \geqq e^{\tilde{a}U^\delta} \quad (\tilde{a}>0) \tag{1.7}$$

and

$$U(x) \leqq \varkappa a^{-\delta^{-1}} n\left(\frac{n}{n-1}\log\frac{2}{|x|}\right)^{1/\delta}.$$

Thus, we may choose $M>0$ such that for any $\varepsilon>0$,

$$(U-M-\varepsilon\log(1/|x|)^+\in W_0^{1,n}(|x|<R_0).$$

With $\Delta_n(\varepsilon\log(1/|x|))=0$ along with (1.7) we may again apply Lemma A and conclude that $U(x)\leqq M+\varepsilon\log(1/|x|)$ a.e. in $|x|<R_0$. Since $\varepsilon>0$ was arbitrary and $u=\varkappa^{-1}U$ the proof is complete.

II. Proof of Theorem 1. Theorem 1 now follows in a standard way (cf. [11: p. 9]) from Lemma 1. Briefly, it suffices from [8; p. 269] to show that u is a weak locally L^∞ solution of (2), since f is continuous. Assume $q=0$. Now (3) and Lemma 1 imply that $u\in L^\infty_{\mathrm{loc}}(\Omega)$. Let $0\leqq\eta\in\mathscr{C}_0^\infty(\Omega)$ and $\zeta_m\in\mathscr{C}^\infty(\Omega)$ such that

$$\zeta_m(x)=\begin{cases}0 & \text{if } |x|<\dfrac{1}{2m}\\[2mm] 1 & \text{if } |x|>\dfrac{1}{m}\end{cases}\qquad 0\leqq\zeta_m\leqq 1,\ |\nabla\zeta|\leqq cm.$$

Then, if Λ is a relatively compact neighborhood of the origin in Ω, and containing supp η

$$\int_\Lambda \zeta_n|\nabla u|^{n-2}\nabla u\cdot\nabla\eta\,dx+\int_\Lambda \eta|\nabla u|^{n-2}\nabla u\cdot\nabla\zeta_n\,dx+\int_\Lambda f(x,u(x))\zeta_m\eta\,dx=0,$$

so it suffices to show that

$$\int_\Omega \eta|\nabla u|^{n-2}\nabla u\cdot\nabla\zeta_m\to 0 \quad\text{as}\quad m\to\infty. \tag{2.1}$$

We must show first that $|\nabla u| \in L^n_{\mathrm{loc}}(\Omega)$. Now, $u \in W^{1,n}_{\mathrm{loc}}(\Omega')$, so the rest follows from

$$\Big|\int_A f(x, u(x))\zeta_m^n u\,dx\Big| = \Big|\int_A |\nabla u|^{n-2}\nabla u\cdot(\zeta_m^n u)\Big|$$

$$= \Big|\int_A |\nabla u|^n \zeta_m^n dx + n\int_A \zeta_m^{n-1}|\nabla u|^{n-2}\nabla u\cdot\nabla\zeta_m\Big|$$

$$\geqq \|\zeta_m|\nabla u|\|^n_{L^n(A)} - n\|\zeta_m|\nabla u|\|^{n-1}_{L^n(A)}\|u\nabla\zeta_m\|_{L^n(A)}$$

since the left-hand side remains bounded, as well as the term $\|u\nabla\zeta_m\|_{L^n(A)}$, as $m\to 0$.

Thus, returning to (2.1) we have

$$\Big|\int_\Omega \eta|\nabla u|^{n-2}\nabla u\cdot\nabla\zeta_m dx\Big| \leqq \Big(\int_{1/2m<|x|<1/m}|\nabla u|^n\Big)^{(n-1)/n}\Big(\int_{1/2m<|x|<1/m}|\eta\nabla\zeta_m|^n\Big)^{1/n}$$

which tends to zero as $m\to\infty$.

III. Proof of Theorem 2(A). The proof is based on the following

Lemma 2. *If f satisfies* (5) *then the ordinary differential equation*

$$|\dot v|^{(p-2)}\Big((p-1)\ddot v(r) + \frac{(n-1)}{r}\dot v(r)\Big) = f(v(r)) \tag{3.1}$$

has solutions for $r>0$ with the following properties

$$\begin{cases}\dot v(0) = 0,\ v(0) = a,\ a\in \boldsymbol{R},\ \dot v > 0 \text{ if } r > 0\\ v(r)\to\infty \text{ as } r\to r_0 \text{ with } r_0<\infty,\ v\in\mathscr{C}^2\ (0, r_0).\end{cases} \tag{3.2}$$

Proof. We first construct a solution of (3.1) with $a=0$. For, we consider the formula

$$v(r) = \int_0^r \Big(\frac{1}{s^{n-1}}\int_0^s t^{n-1}f(v(t)dt)^{1/(p-1)}\Big)ds. \tag{3.3}$$

Applying to (3.3) the Picard iteration process with $v_0\equiv 0$ we obtain a local solution of

$$\dot v^{(p-2)}\Big((p-1)\ddot v(r) + \frac{(n-1)}{r}\dot v(r)\Big) = f(r) \tag{3.4}$$

with the properties

(3.5)

$$v(0) = 0,\ \dot v(0) = 0,\ \dot v(r) > 0 \text{ and } v\in\mathscr{C}^2 \text{ whenever defined for } r > 0.$$

If there is r_0 as in (3.2) we are done. Otherwise, with the usual existence and uniqueness theorems, v may be continued to a solution of (3.4) with the properties in (3.5) in a larger interval. Since $\dot v(r)>0$, this local process may be repeated indefinitely unless there is $r_0<\infty$ such that $v(r_0)=\infty$. We now prove that (5) forces this situation. We observe that (3.4) can be written as

$$\int_0^r (\dot{v}^{(p-1)} r^{(n-1)})_r \, dr = \int_0^r f(v) r^{(n-1)} \, dr. \tag{3.6}$$

Since f and v are nondecreasing, (3.6) implies

$$(p-1)\ddot{v} \geqq \dot{v}/r. \tag{3.7}$$

Substituting (3.7) into (3.8) we have

$$(p-1) n \ddot{v} \dot{v}^{(p-2)} \geqq f(v).$$

Multiplying by $\dot{v}$ and integrating we get that there is a positive constant C such that

$$\dot{v}\Big(\int_0^{v(r)} f(s)\, ds\Big)^{-1/p} > C.$$

From here we obtain

$$\int_0^{v(r)} \Big(\int_0^t f(s)\, ds\Big)^{-1/P} dt > Cr. \tag{3.8}$$

From (5) and (3.8) it follows that there exists $r_0 < \infty$ such that $v(r) \to \infty$ as $r \to r_0$.

To obtain solutions for arbitrary a we consider

$$|\dot{v}|^{(p-2)}\big((p-1)\ddot{v}(r) + ((n-1)/(r)\dot{v}(r)\big) = g(v(r))$$

with $g(t) = f(t+a)$. Since $g(t)$ also satisfies (5) there is v satisfying (3.1), (3.2) with $a=0$. Now it is enough to take $\bar{v} = v + a$. This completes the proof of the lemma.

To prove Theorem 2(A) we observe that $v(r)$ is a radial solution of $\Delta_p u = f(u)$ if and only if v satisfies (3.1) for $r > 0$. On the other hand it is easy to see that if v satisfies (3.1) and (3.2) then v, in fact, satisfies

$$\Delta_p v = f(v) \quad \text{in} \quad \mathscr{D}'(B(0, r_0))$$

with $B(0, r_0) = \{x \colon \|x\| < r_0\}$. Therefore, from Lemma A it now follows that

$$u(x) \leqq v(x) \quad \text{in} \quad B(0, r_0). \tag{3.9}$$

Taking in (3.2), $a = \operatorname{ess\,inf}_{x \in B(0, r_0)} u(x)$ we get a contradiction. Therefore, Theorem 2(A) is now complete.

IV. Proof of Theorem 2 (B). Given $-\infty < \alpha < \infty$, $\beta \neq 0$, the ordinary differential equation

$$|\dot{v}(r)|^{p-2}\Big((p-1)\ddot{v}(r) + \frac{(n-1)}{r}\dot{v}(r)\Big) = f(v) \tag{4.1}$$

can be solved uniquely with initial data

$$v(1) = \alpha, \quad \dot{v}(1) = \beta \tag{4.2}$$

and continued in each direction.

Now, for $\dot{v}>0$ (4.1) is the same as $(\dot{v}(r)^{p-1}r^{n-1})_r=r^{n-1}f(v(r))$ and for $\dot{v}<0$ (4.1) becomes $((-\dot{v}(r))^{p-1}r^{n-1})_r=-r^{n-1}f(v(r))$. Thus, we may continue v to the left and right from $r=1$ until either $\dot{v}=0$ or $v=\infty$. If neither of these occurs on a side of $r=1$, the continuation proceeds indefinitely in that direction.

We wish to show first that, given numbers α, M there exists $\beta=\beta(\alpha, M)$ such that the solution u of (4.1) with initial condition (4.2) satisfies

$$v\left(\frac{1}{2}\right) \geqq M. \tag{4.3}$$

In fact, with $\beta<0$ and $r<1$,

$$r^{n-1}(-\dot{v}(r))^{p-1} = (-\beta)^{p-1}+\int_r^1 t^{n-1}f(v(t))\,dt$$

which shows $\dot{v}$ stays negative. Hence from (4.1) we have $(-\dot{v})^{p-2}(p-1)\ddot{v}>f(v)$ so $(-\dot{v})^{p-1}(p-1)\ddot{v}>-\dot{v}f(v)$ and integrating we obtain for $0<r<1$

$$\frac{(p-1)}{p}(-\dot{v}(r))^p > \frac{(p-1)(-\beta)^p}{p} - \int_r^1 \dot{v}(t)f(v(t))\,dt = \frac{(p-1)(-\beta)^p}{p} + \int_\alpha^{v(r)} f(s)\,ds.$$

Thus,

$$-\dot{v}(r) > \left((-\beta)^p+\frac{p}{p-1}\int_\alpha^{v(r)} f(s)\,ds\right)^{1/p},$$

$$-\int_{1/2}^1 \dot{v}(r)\left(\int_\alpha^{v(r)} f(s)\,ds\right)^{-1/p} dr > \left(\int_{1/2}^1 (-\beta)^p\left(\int_a^{v(r)} f(s)ds\right)^{-1}+\frac{p}{p-1}\right)^{1/p} dr,$$

and

$$\int_\alpha^{v(1/2)} \left(\int_\alpha^t f(s)\,ds\right)^{-1/p} dt > \left(\int_{1/2}^1 (-\beta)^p \left(\int_\alpha^{v(r)} f(s)\,ds\right)^{-1}+\frac{p}{p-1}\right)^{1/p} dr. \tag{4.4}$$

For fixed α, it follows from (5) that the left hand side of (4.4) is bounded, independent of β. On the other hand, if $v(r)$ were to remain bounded with $-\beta$ large on the right-hand side, we would have a contradiction.

Having established $\beta<0$ so that (4.3) holds, we now apply (6) to show that there exists a value $r_0>1$ such that $\dot{v}(r)\to 0$ as $r\to r_0$.

Integrating the relation $((-\dot{v}(r))^{p-1}r^{n-1})_r=-r^{n-1}f(v)$ we obtain for $r>1$

$$(-\dot{v}(r))^{p-1}r^{n-1} = (-\beta)^{p-1}-\int_1^r t^{n-1}f(v(t))\,dt. \tag{4.5}$$

It follows that $r^{(n-1)/(p-1)}\dot{v}$ is increasing so for some $K>0$

$$v(r) = \alpha+\int_1^r \dot{v}(t)\,dt = \alpha+\int_1^r \dot{v}(t)t^{(n-1)/(p-1)}t^{-(n-1)/(p-1)}\,dt$$

$$\geqq \alpha+\beta\int_1^r t^{(1-n)/(p-1)}\,dt \geqq -K.$$

We then have

$$(-\dot{v}(r))^{p-1}r^{n-1} = (-\beta)^{p-1} - \int_1^r t^{n-1}f(v(t))\,dt \leqq (-\beta)^{p-1} - f(-K)\int_1^r t^{n-1}\,dt,$$

and the right-hand side will eventually be negative. Thus, $\dot{v}(r_0)=0$ for some $r_0>1$.

To summarize we have now shown that, given numbers α, M, there exists $\beta=\beta(\alpha, M)$ such that if v satisfies (4.1) and (4.2), then $v(1/2)\geqq M$ and there exists $r_0>1$, such that v may be continued from $r=1$ to $r=r_0$, at which point $\dot{v}$ becomes 0.

To complete the construction of v past r_0 we continue by the equation

$$v(r) = v(r_0) + \int_{r_0}^{r} \left(\frac{1}{t^{n-1}} \int_{r_0}^{t} s^{n-1} f(v(s))\right)^{1/(p-1)} dt$$

as in § 3. Also as in § 3 there exists $r_1>r_0$ such that $v(r)\to\infty$ as $r\to r_1$.

With v now completely described, the proof is now easily completed by comparison.

We may assume $S\subseteqq |x|<1/4$. Suppose u satisfies (4) outside S,

$$M > \operatorname*{ess\,sup}_{1/4\leqq|x|\leqq1} u(x), \quad \alpha < \operatorname*{ess\,inf}_{1/2<|x|<3/2} u(x)$$

and v is the radial function previously constructed with $\beta=\beta(\alpha, M)$. Then, v is $\mathscr{C}^1$ and satisfies (4.1) a.e.; hence v is a radial solution to (4). But $v(r)\to\infty$ as $r\to r_1$, so by choice of M and α, Lemma A gives a contradiction. Hence u cannot satisfy (4) in $\boldsymbol{R}^n-S$.

V. Some examples. Let $V(r)=\log\left((1+r)^\beta/r^\gamma\right)$, $(\beta, \gamma>0)$. Then,

$$\Delta_n V(r) = (n-1)\left|\frac{\gamma}{r^2} - \frac{\beta}{(1+r)^2}\right|^{n-2}\left(\frac{\beta}{r(1+r)^2}\right)$$

so, for $\beta-\gamma<-2n+1$ V is a radial subsolution $\Delta_n V\geqq e^V$ for r sufficiently large. This shows that $p<n$ is needed in Theorem 2 (B). If $\gamma<2n-3$, V is a subsolution for sufficiently small ϱ in $\{0<|x|<\varrho\}$.

Regarding Theorem 1, to show that $\delta>1$ is essential, we verify that the equation $\Delta_n u=e^u$ has a solution in some set $\{0<|x|<\varrho\}$, which is singular at $x=0$. In fact the radial form of $-\Delta_n u+e^u=0$ is the Euler equation for the functional $I[u]=\int(|u_r|^n+ne^u)r^{n-1}dr$. For $\gamma<2n-3$, let V be a subsolution as above and $M_r=V(r)$ $(0<r<\varrho)$. The functional $I[u]$ has a minimizing function $u_n(r)$ [4; p. 24] on $[\varrho/n, \varrho]$ for each $n=2, 3, \ldots$, with $u_n(\varrho)=M_\varrho$, $u_n(\varrho/n)=M_{\varrho/n}$. This $u_n(r)$ is a solution of the equation and by Lemma A $u_n(r)\geqq V(r)$ on $[\varrho/n, \varrho]$. To show that as $n\to\infty$ we obtain a solution with the desired properties, we need only bound the u_n's from above. To this end, let $\{|x-x_0|\leqq R\}$ be any closed ball in $\{0<|x|<\varrho\}$ and n be sufficiently large so that it is contained in $\{\varrho/n<|x|<\varrho\}$. Let v be the comparison function of § 1, with $\delta=1$, $a=1$ in (1.4). Since $v(R)=\infty$, it follows from

Lemma A that $u_n \leqq v$, for all sufficiently large n, in $\{|x-x_0|<R\}$. Thus, the u_n's are uniformly bounded in compact subsets of $\{0<|x|<\varrho\}$.

Finally, to see that condition (5) is sharp for Theorem 2(A), suppose f is a positive nondecreasing locally Lipschitz function with

$$\int_0^\infty dr\Big(\int_0^r f(s)\,ds\Big)^{-1/p} = \infty. \tag{5.1}$$

Then, there is a solution of (3.1) such that

$$\dot v(0) = 0,\ \dot v > 0 \quad \text{if} \quad r > 0,\ v \in \mathscr{C}^1[0, \infty] \cap \mathscr{C}^2(0, \infty). \tag{5.2}$$

Indeed, as in the proof of Lemma 2 we can construct a solution v of (3.1) that will have the properties in (5.2) unless there is r_0 such that $\lim_{r\to r_0} v(r)=\infty$. We note that if this happens then (5.1) does not hold. For, since $\dot v(r)>0$ from (3.1) we obtain $\dot v^{(p-2)}((p-1)\ddot v)<f(v)$. Hence, since without loss of generality we may assume $v(0)=0$, we have

$$\int_0^{v(r)} \Big(\int_0^t f(s)\,ds\Big)^{-1/p} dt < \Big(\frac{p}{(p-1)}\Big)^{1/p} r.$$

Making $r \to r_0$ we obtain a contradiction with (5.1).

On the other hand, it is not difficult to see that in fact we have

$$\Delta_p v = f(v) \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^n).$$

VI. Concluding remark. In the case $\delta=1$, comparison of a solution of $\Delta_n u=e^u$ in $\{0<|x|<\varrho\}$ with the function v of § 1 yields $e^{u(x)} \leqq C/|x|$ for some $C>0$. Hence, $e^{u(x)} \in L_{n/n-\varepsilon}$ $(\varepsilon>0)$ and by [9; Theorem 1], if u is a positive solution then $C_1 \log(1/|x|) \leqq u(x) \leqq C_2 \log(1/|x|)$. This estimate generalizes a theorem of Nitsche [5] for $n=2$.

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