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QUASICONFORMAL SEMIFLOWS

JUKKA SARVAS

1. Introduction

Let R^n be the n-dimensional euclidean space $n\geqq 2$, and let F: $[0,1]\times R^n\to R^n$ be a function such that the differential equation

$$(1.1)\qquad \dot{\varphi}(t)=F(t,\varphi(t)),\quad \varphi(0)=x,\quad \left(\text{here}\quad \dot{\varphi}(t)=\frac{\partial\varphi}{\partial t}(t)\right)$$

has a unique solution in $[0,1]$ for every initial value $x\in R^n$. Let $\varphi(t,x)$, $0\leqq t\leqq 1$, denote this solution for $x\in R^n$. The mapping $(t,x)\mapsto\varphi(t,x)$, $(t,x)\in[0,1]\times R^n$, is called a *semiflow* generated by F.

Write $F_t(x)=F(t,x)$ and $\varphi_t(x)=\varphi(t,x)$. If F is continuous, bounded and $\|SF_t\|_\infty\leqq k<\infty$ for all $t\in[0,1]$, then F generates a semiflow φ such that the mappings φ_t: $R^n\to R^n$ are *quasiconformal,* abbreviated qc, with maximal dilatation

$$(1.2)\qquad K(\varphi_t)\leqq e^{nkt}\quad\text{for}\quad 0\leqq t\leqq 1.$$

Here S is a differential operator defined by Ahlfors as follows: If f: $R^n\to R^n$ has first order partial derivatives $D_i f(x)$, $i=1,2,\dots,n$, at x and $Df(x)$ is its Jacobian matrix at x (i.e. the $n\times n$-matrix with column vectors $D_i f(x)$), then

$$Sf(x)=\frac{1}{2}\left(Df(x)+\widetilde{Df(x)}\right)-\frac{1}{n}\,tr\left(Df(x)\right)I,$$

where $\widetilde{Df(x)}$ is the transpose and tr $(Df(x))$ the trace of $Df(x)$ and I is the identity matrix. Results of type (1.2) are proved by Ahlfors [A_1], Reimann [R_2] and Semenov [Se].

The result (1.2) raises the following question: If f: $R^n\to R^n$ is qc, is there a qc semiflow φ: $[0,1]\times R^n\to R^n$ of the above type such that $f=\varphi_1$? Actually, this is the case if $n=2$, see [G—R]. For $n\geqq 3$ the question is still open.

The purpose of the first part of this paper is to enlarge the class of qc semiflows to cover the case where F need not be continuous and $\|SF_t\|_\infty$ not uniformly bounded. One aim of this generalization is to provide a larger and more flexible class of semiflows in which to consider the above open question. We shall prove the following result. Suppose that

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(i) F: $[0, 1]\times R^n\to R^n$ is locally integrable and

(ii) for almost every $t\in[0, 1]$ the mapping F_t: $R^n\to R^n$ is continuous and in $W^{1,1}_{\mathrm{loc}}(R^n)$, and $\int_0^1 \|SF_t\|_\infty dt<\infty$.

(iii) If $n=2$, we also assume that $\lim_{|x|\to\infty}|x|^{-3}|F_t(x)|=0$ for almost every $t\in[0, 1]$.

Then F generates a semiflow $\varphi: [0, 1]\times \bar{R}^n\to \bar{R}^n$ such that φ_t: $\bar{R}^n\to \bar{R}^n$ is qc with $K(\varphi_t)\leqq\exp\left(n\int_0^t\|SF_s\|_\infty ds\right)$ for all $t\in[0, 1]$. Here $W^{1,1}_{\mathrm{loc}}(R^n)$ refers to the Sobolev space of mappings f: $R^n\to R^n$ which have locally integrable distributional first order partial derivatives $D_i f$, $i=1, 2, \ldots, n$. The space $\bar{R}^n$ is the one point compactification of R^n. In the special case $F(t, x)=f(x)$ with f: $R^n\to R^n$ this result also improves $[A_2$, p. 9], where a stronger growth condition than (iii) was required for all $n\geqq 2$.

The method of proving the above result applies to other operators than S, too. In the second part of this paper (Chapters 6 and 7) operators $S_\alpha f=(1/2)(Df+\overleftarrow{Df})+\alpha\,\mathrm{tr}\,(Df)I$, $\alpha\neq -1/n$, and $\mathrm{div} f=\mathrm{tr}\,(Df)$ will be studied; the S_α operators lead to 'Lipschitz semiflows' and the div operator to semiflows φ with a bounded Jacobian determinant J_{φ_t}. The case of div is interesting because the generating process of the corresponding semiflow can be reversed. In fact, making use of the possibility to express the fundamental solution of the differential equation $\mathrm{div} f=u$ in terms of Riesz potentials, we will prove that for any measurable ϱ: $R^n\to R^+$ with $\log\varrho\in L^1\cap L^\infty$ there is a semiflow φ such that the Lebesgue (volume) derivative of φ_t equals $(\varrho(x))^t$ for a.e. $x\in R^n$. Related results are proved by Riemann [R_1] for a continuous ϱ and by Moser [M] for a smooth ϱ; however, their methods differ from the method used here.

Notation. For an $n\times n$-matrix A we use the sup-norm $\|A\|=\sup_{|x|=1}|Ax|$, $x\in R^n$. For a measurable function f: $D\to R^m$, $D\subset R^n$, we write $f\in L^p(D)$, $1\leqq p<\infty$, if the L^p-norm $\|f\|_p=\left(\int_D|f|^p dm\right)^{1/p}$ is bounded; for $p=\infty$ we write $f\in L^\infty(D)$ if $\|f\|_\infty=\mathrm{ess\,sup}_{x\in D}|f(x)|<\infty$. Often we abbreviate $L^p(D)=L^p$. We write $f\in C^1$ or $f\in C^\infty$ if f has continuous partial derivatives of first order or of all orders, respectively. The notation $f\in C_0^\infty$ means that $f\in C^\infty$ and f has a compact support in its open domain of definition.

2. Smooth qc semiflows

Suppose that $F: [0, 1]\times R^n\to R^n$ is continuous and generates the semiflow φ: $[0, 1]\times R^n\to R^n$. If the derivative $DF_t(x)$ exists everywhere and it is continuous in (t, x), then φ is unique and (see, for instance, [C—L])

(2.1) $$\varphi_t: R^n\to R^n \quad \text{is diffeomorphism and}$$

$$(D\varphi_t(x))^{\cdot} = DF_t(\varphi_t(x))D\varphi_t(x) \quad \text{for} \quad 0\leqq t\leqq 1, \quad \text{and}$$

for all $x \in R^n$. The next theorem shows how $SF_t(x)$ affects $K(\varphi_t)$. Let $D, D' \subset R^n$ be domains. Recall that the maximal dilatation $K(f)$ is defined for a diffeomorphism $f\colon D \to D'$ by $K(f) = \max\{K_I(f), K_0(f)\}$ with $K_0(f) = \sup_{x\in D} K_0(x, f)$, $K_0(x,f) = \|Df(x)\|^n/|\det Df(x)|$, and $K_I(f) = K_0(f^{-1})$.

2.2. Theorem. *If φ is as above, then*

$$K_0(x, \varphi_t) \leqq \exp\Big(n \int_0^t \big\|SF_s(\varphi_s(x)\big\|ds\Big), \quad 0 \leqq t \leqq 1, \quad x \in R^n, \tag{2.3}$$

and $K(\varphi_t) \leqq \exp\big(n \int_0^t a(s)\,ds\big)$ *with* $a(s) = \sup_{x\in R^n} \|SF_s(x)\|$.

Proof. (Ahlfors's method). Write $\Phi(t) = D\varphi_t(x)$ and $X(t) = \big(\det \Phi(t)\big)^{-1/n}\Phi(t)$. Then $K_0(x, \varphi_t) = \|X(t)\|^n$. By (2.1)

$$\dot\Phi(t) = A(t)\Phi(t), \quad \Phi(0) = I, \quad \text{with} \quad A(t) = DF_t(\varphi_t(x)). \tag{2.4}$$

We apply the formula $(\det M(t))^\cdot = (\det M(t))\ \mathrm{tr}\,[M(t)^{-1}\dot M(t)]$ were $M(t)$ is a differentiable matrix-valued function with $\det M(t) \neq 0$. After differentation we get (note that $\det \Phi(t) \neq 0$)

$$X(t)^\cdot = \Big[A(t) - \frac{1}{n}\,\mathrm{tr}\big(A(t)\big)I\Big]\,X(t), \quad t \in [0,1].$$

By transposing this equation we get $\widetilde{X(t)}^\cdot = \widetilde{X(t)}\,[\widetilde{A(t)} - (1/n)\,\mathrm{tr}\,\big(A(t)\big)I]$, $0 \leqq t \leqq 1$. Therefore

$$[\widetilde{X(t)}X(t)]^\cdot = \widetilde{X(t)}^\cdot X(t) + \widetilde{X(t)}X(t)^\cdot \tag{2.5}$$

$$= \widetilde{X(t)}\Big[A(t) + \widetilde{A(t)} - \frac{2}{n}\,\mathrm{tr}\big(A(t)\big)I\Big]X(t) = 2\widetilde{X(t)}SF_t\big(\varphi_t(x)\big)X(t), \quad 0 \leqq t \leqq 1.$$

Note that $\|M\|^2 = \|\tilde M M\|$ and $\|\tilde M\| = \|M\|$ for any matrix M. We get

$$\big(\|X(t)\|^2\big)^\cdot = \big(\|\widetilde{X(t)}X(t)\|\big)^\cdot \leqq \|\big(\widetilde{X(t)}X(t)\big)^\cdot\|$$

$$\leqq 2\|\widetilde{X(t)}SF_t\big(\varphi_t(x)\big)X(t)\| \leqq 2\|SF_t\big(\varphi_t(x)\big)\|\,\|X(t)\|^2,$$

for a.e. $t \in [0, 1]$. This implies

$$\big(\log\|X(t)\|^2\big)^\cdot = \frac{\big(\|X(t)\|^2\big)^\cdot}{\|X(t)\|^2} \leqq 2\|SF_t\big(\varphi_t(x)\big)\|, \quad \text{for a.e.} \quad t \in [0, 1],$$

and integration yields (2.3). From that we get $K_0(\varphi_t) \leqq \exp\big(n\int_0^t a(s)\,ds\big)$ with $a(s)$ as in the theorem.

It is easy to see that for any $t \in [0, 1]$ the function $\psi(s, y) = \varphi(t-s, x)$, $y = \varphi_t(x)$, is the solution of $\dot\psi(s, y) = -F\big(t-s, \psi(s, y)\big)$, $\psi(0, y) = y$, for all $0 \leqq s \leqq t$ and $y \in R^n$. Because $\psi_t(y) = \varphi(0, x) = x$ for $y = \varphi_t(x)$, we have $\psi_t = \varphi_t^{-1}$. Apply the above result to ψ_t and get $K_0(\psi_t) \leqq \exp\big(n\int_0^t a(s)\,ds\big) = K$. Thus $K(\varphi_t) = \max\{K_0(\varphi_t), K_0(\varphi_t^{-1})\} \leqq K$.

3. Non-smooth semiflow

In this chapter we relax the assumptions on F and show that it still generates a unique semiflow φ such that $\varphi_t\colon R^n\to R^n$ is a homeomorphism with certain equicontinuity properties which are important for our later constructions. Suppose that $F\colon [0,1]\times R^n\to R^n$ is measurable and

(3.1) (i) for all $x\in R^n$ the function $t\mapsto F(t,x)$ is measurable,

(ii) for a.e. $t\in[0,1]$ we have: $|F(t,x)|\leqq a(t)$ for all $x\in R^n$, and $|F(t,x_1)-F(t,x_2)|\leqq a(t)|x_1-x_2|\left(1+\log^+\frac{1}{|x_1-x_2|}\right)$ for all $x_1,x_2\in R^n$ with $a\colon [0,1]\to R\in L^1$.

Here $\log^+ u=0$ if $0<u\leqq 1$ and $\log^+ u=\log u$ if $u\geqq 1$. Then (see [C—L, Chapter 2]) the initial value problem

$$\dot\varphi(t)=F(t,\varphi(t))\quad\text{for a.e.}\quad t\in[0,1],\quad\text{and}\quad \varphi(t_0)=x, \tag{3.2}$$

has an absolutely continuous solution $\varphi\colon [0,1]\to R^n$ for any $t_0\in[0,1]$ and $x\in R^n$. Write $\psi(t_0,t,x)$ $t\in[0,1]$, for this solution, i.e.

$$\Psi(t_0,t,x)=x+\int_{t_0}^{t}F(s,\Psi(t_0,s,x))\,ds,\quad t\in[0,1], \tag{3.3}$$

which is the equivalent integral form of (3.2) (for absolutely continuous solutions). If $t_0=0$, we again write $\psi(0,t,x)=\varphi(t,x)=\varphi_t(x)$. With these notations and the assumptions (3.1) we get two theorems.

3.4. Theorem. *For* $t_1,t_2\in[0,1]$ *and* $x_1,x_2\in R^n$

$$|\varphi(t_1,x_1)-x_1|\leqq\int_0^{t_1}a(t)\,dt\leqq\int_0^1 a(t)\,dt=I,\quad\textit{and} \tag{3.5}$$

$$|\varphi(t_1,x_1)-\varphi(t_2,x_2)|\leqq w_I(|x_1-x_2|)+\left|\int_{t_1}^{t_2}a(t)\,dt\right|, \tag{3.6}$$

where $w_I\colon [0,\infty)\to[0,\infty)$ *is an increasing function depending only on* I *such that if* $0\leqq r\leqq\exp(1-e^I)=\alpha$ *then* $w_I(r)=\alpha r^{\exp(-I)}$.

Proof. The formula (3.5) follows directly from (3.1) and (3.3). To prove (3.6) suppose first that $t_1=t_2=t\in[0,1]$. We may suppose $x_1\neq x_2$. Let $\sigma(t)=|\varphi(t,x_1)-\varphi(t,x_2)|$, $t\in[0,1]$. Then by (3.1)

$$\begin{aligned}\sigma(t)&=\left|x_1-x_2+\int_0^t[F(s,\varphi(s,x_1))-F(s,\varphi(s,x_2))]\,ds\right|\\&\leqq|x_1-x_2|+\int_0^t a(s)h(\sigma(s))\,ds\end{aligned} \tag{3.7}$$

with $h(u)=u(1+\log^+ 1/u)$, $u>0$. Next we employ a standard differential inequality technique. Define

$$\tau(t) = |x_1-x_2|+\int_0^t a(s)h(\sigma(s))\,ds$$

for $t\in[0, 1]$. Then τ is absolutely continuous and $\dot\tau(t)=a(t)h(\sigma(t))\leqq a(t)h(\tau(t))$ for a.e. $t\in[0, 1]$ by (3.7) and because h is increasing. This implies $\dot\tau(t)/h(\tau(t))\leqq a(t)$, and integrating yields (the change of variable in the integral is possible because τ is increasing and absolutely continuous)

$$I \geqq \int_0^t a(s)\,ds \geqq \int_0^t \frac{\dot\tau(s)}{h(\tau(s))}\,ds = \int_{\tau(0)}^{\tau(t)} \frac{ds}{h(s)} = G(\tau(t))-G(|x_1-x_2|),$$

where $G\colon (0, \infty)\to(0, \infty)$ is the increasing homeomorphism

$$G(u) = \int_1^u \frac{ds}{h(s)} = \begin{cases} -\log\left(\log \dfrac{e}{u}\right) & \text{if } 0 < u \leqq 1, \\ \log u \quad \text{if} \quad 1 \leqq u. \end{cases}$$

Therefore, $\sigma(t)\leqq\tau(t)|\leqq G^{-1}(G(|x_1-x_2|)+I)=w_I(|x_1-x_2|)$ for all $t\in[0, 1]$. This gives (3.6) for $t_1=t_2=t$. If $t_1\leqq t_2$, we get by (3.1)

$$|\varphi(t_1, x_1)-\varphi(t_2, x_2)| \leqq |\varphi(t_1, x_1)-\varphi(t_1, x_2)|+\int_{t_1}^{t_2} |F(s, \varphi(s, x_2))|\,ds$$

$$\leqq w_I(|x_1-x_2|)+\left|\int_{t_1}^{t_2} a(t)\,dt\right|,$$

which proves the theorem.

3.8. Theorem. *The above φ is the unique semiflow generated by F, and the mappings $\varphi_t\colon R^n\to R^n$, $0\leqq t\leqq 1$, are homeomorphisms such that*

$$|\varphi_t^{-1}(x)-x| \leqq \int_0^1 a(t)\,dt, \tag{3.9}$$

$$|\varphi_{t_1}^{-1}(x_1)-\varphi_{t_2}^{-1}(x_2)| \leqq w_I\left(|x_1-x_2|+\left|\int_{t_1}^{t_2} a(t)\,dt\right|\right) \tag{3.10}$$

for all $t_1, t_2\in[0, 1]$ and $x_1, x_2\in R^n$. Here w_I is as in Theorem 3.4.

Proof. Theorem 3.4 implies that the solutions of (3.3) are unique for $t_0=0$ for any $x\in R^n$, and also that the function $x\mapsto\varphi_t(x)=\psi(0, t, x)$, $x\in R^n$, is continuous.

To apply Theorem 3.4 to $\psi(t_0, t, x)$ of (3.3), note that for any $t_0\in[0, 1]$ the solution of the initial value problem

$$\theta(s, x) = x-\int_0^s F(t_0-u, \theta(u, x))\,du, \quad 0 \leqq s \leqq t_0, \tag{3.11}$$

is given by $\theta(s, x)=\psi(t_0, t_0-s, x)$. Theorem 3.4 applies to $\theta(s, x)$ and we get

$$\text{(3.12)} \quad |\psi(t_0, t, x_1)-\psi(t_0, t, x_2)| = |\theta(t_0-t, x_1)-\theta(t_0-t, x_2)| = w_I(|x_1-x_2|)$$

for all $0\leqq t\leqq t_0\leqq 1$. We see that the solution of (3.3) in unique for $t\in[0, t_0]$, and the assignment $x\mapsto\psi(t_0, 0, x)$, $x\in R^n$, defines a continuous mapping. Due to the uniqueness, it is easy to see that $\varphi_{t_0}\colon R^n\to R^n$ has an inverse mapping which is given by $\varphi_{t_0}^{-1}(y)=\psi(t_0, 0, y)$, $y\in R^n$, and thus $\varphi_{t_0}\colon R^n\to R^n$ is a homeomorphism for all $t_0\in[0, 1]$. The inequality (3.9) follows immediately from (3.3) and (3.1). To prove (3.10) note that by uniqueness $\psi(t_2, 0, x)=\psi(t_1, 0, \psi(t_2, t_1, x))$ for all $0\leqq t_1\leqq t_2\leqq 1$, $x\in R^n$, and using (3.12) and (3.1) we get

$$|\varphi_{t_1}^{-1}(x_1)-\varphi_{t_2}^{-1}(x_2)| = |\psi(t_1, 0, x_1)-\psi(t_2, 0, x_2)|$$

$$= |\psi(t_1, 0, x_1)-\psi(t_1, 0, \psi(t_2, t_1, x_2))| \leqq w_I(|x_1-\psi(t_2, t_1, x_2)|)$$

$$= w_I\Big(\Big|x_1-x_2-\int_{t_2}^{t_1} F(s, \psi(t_2, s, x_2))\,ds\Big|\Big) \leqq w_I\Big(|x_1-x_2|+\Big|\int_{t_1}^{t_2} a(t)\,dt\Big|\Big).$$

This proves the theorem.

4. Potentials

A large class of functions which satisfy conditions (3.1) are obtained by formation of potentials with $(1-n)$-homogeneous kernels.

4.1. Definition. Let G be the set of all (kernel) functions g with the properties

(i) $$g\colon R^n\setminus\{0\}\to R\in C^1,$$

(ii) $$g(tx) = t^{1-n}g(x) \quad \text{for all} \quad t>0, \quad x\neq 0.$$

4.2. Lemma. *If* $g\in G$, $x, z, \in R^n$ *and* $0<|z|\leqq|x|/2$, *then*

$$|g(x+z)-g(x)| \leqq d|z||x|^{-n} \quad \textit{with} \quad d=\max\{\|Dg(y)\| : 1/2\leqq|y|\leqq 3/2\}<\infty.$$

Proof. We have $|g(x+z)-g(x)|=|x|^{1-n}|g(|x|^{-1}x+|x|^{-1}z)-g(|x|^{-1}x)$. Here $1/2\leqq|(|x|^{-1}x+|x|^{-1}z)|\leqq 3/2$ because $||x|^{-1}z|\leqq 1/2$. Use the mean value theorem and the lemma follows.

4.3. Theorem. *Suppose that* $g\in G$,

(i) $$f\colon R^n\to R\in L^1\cap L^\infty,$$

(ii) $$F(x) = (g*f)(x) = \int_{R^n} g(x-y)f(y)\,dy, \quad x\in R^n, \quad \text{i.e. } F \textit{ is}$$

the g-potential of f.

Then the convolution integral defining F converges absolutely and

$$|F(x)| \leqq c_1(\|f\|_1+\|f\|_\infty), \quad x\in R^n, \quad \text{and} \tag{4.4}$$

$$|F(x_1)-F(x_2)| \leqq c_2(\|f\|_1+\|f\|_\infty)|x_1-x_2|\left(1+\log^+\frac{1}{|x_1-x_2|}\right) \tag{4.5}$$

for all $x_1, x_2\in R^n$, *with* $c_1, c_2<\infty$ *depending only on* g *and* n.

Proof. Note that $|g(z)|=|z|^{1-n}g(|z|^{-1}z)\leqq c|z|^{1-n}$ for $z\neq 0$ with $c=\max_{|y|=1}|g(y)|<\infty$. We get for any $x\in R^n$

$$|F(x)| = \Big|\int_{R^n} g(x-y)f(y)dy\Big| \leqq c\int_{R^n}\frac{|f(y)|}{|x-y|^{n-1}}dy$$

$$\leqq c\int_{|x-y|\leqq 1}\frac{\|f\|_\infty}{|x-y|^{n-1}}dy+c\int_{|x-y|>1}|f(y)|dy \leqq c(\omega_{n-1}\|f\|_\infty+\|f\|_1),$$

where $\omega_{n-1}=m_{n-1}S^{n-1}$, which proves (4.4). To prove (4.5) let $z=x_1-x_2\neq 0$. Then (use change of variable $x=x_2-y$ below)

$$|F(x_1)-F(x_2)| = \Big|\int_{R^n}(g(x_1-y)-g(x_2-y))f(y)dy\Big|$$

$$\leqq \int_{R^n}|g(x_1-y)-g(x_2-y)|\,|f(y)|dy = \int_{R^n}|g(x+z)-g(x)|\,|f(x_2-x)|dx$$

$$= \int_{|x|\leqq r_0}|g(x+z)-g(x)|\,|f(x_2-x)|dx+\int_{|x|>r_0}|g(x+z)-g(x)|\,|f(x_2-x)|dx,$$

where $r_0=\max\{1, 2|z|\}$ (then $r_0/|z|\geqq 2$). Call the integrals on the last line above I_1 and I_2. We first estimate I_1 (below use change of variable $u=x/|z|$):

$$I_1 \leqq \int_{|x|\leqq r_0}|g(x+z)-g(x)|\,\|f\|_\infty dx = \frac{\|f\|_\infty}{|z|^{n-1}}\int_{|x|\leqq r_0}\left|g\left(\frac{x}{|z|}+\frac{z}{|z|}\right)-g\left(\frac{x}{|z|}\right)\right|dx$$

$$= \frac{\|f\|_\infty}{|z|^{n-1}}|z|^n\int_{|u|\leqq r_0/|z|}\left|g\left(u+\frac{z}{|z|}\right)-g(u)\right|du$$

$$= \|f\|_\infty|z|\left[\int_{|u|\leqq 2}\left|g\left(u+\frac{z}{|z|}\right)-g(u)\right|du+\int_{2\leqq|u|\leqq r_0/|z|}\left|g\left(u+\frac{z}{|z|}\right)-g(u)\right|du\right]$$

$$\leqq \|f\|_\infty|z|\left[2\int_{|u|\leqq 3}|g(u)|\,du+\int_{2\leqq|u|\leqq r_0/|z|}d|u|^{-n}du\right],$$

where we have used Lemma 4.2 in the last inequality. Because g is $(1-n)$-homogeneous, $\int_{|u|\leqq 3}|g(u)|\,du=c'<\infty$. We also get

$$\int_{2\leqq|u|\leqq r_0/|z|}|u|^{-n}du = \omega_{n-1}\log\left(\frac{r_0}{2|z|}\right) \leqq \omega_{n-1}\log^+\frac{1}{|z|}.$$

Therefore, $I_1 \leqq \|f\|_\infty |z|(2c' + d\omega_{n-1} \log^+(1/|z|))$. Next we estimate I_2. Note that $|g(x+z)-g(x)| \leqq d|z|\,|x|^{-n}$ in I_2 by Lemma 4.2, and note that $|x|^{-n} \leqq 1$ in I_2 because $|x| \geqq r_0 \geqq 1$. We get

$$I_2 = \int_{|x|>r_0} |g(x+z)-g(x)|\,|f(x_2-x)|\,dx \leqq d|z| \int_{R^n} |f(x_2-x)|\,dx = d|z|\,\|f\|_1.$$

Combine the above estimates and get $|F(x_1)-F(x_2)| \leqq I_1 + I_2 \leqq \max\{2c', d\omega_{n-1}, d\}(\|f\|_1 + \|f\|_\infty)(1+\log^+(1/|z|))\,|z|$. This proves the theorem.

5. Non-smooth quasiconformal semiflow

In this chapter Ahlfors's formula for recovering f from Sf plays a crucial role. We need it in the following form:

5.1. Lemma. *If $f\colon R^n \to R^n \in W^{1,1}_{\text{loc}}(R^n)$, $Sf \in L^1 \cap L^\infty$, f is continuous and $f(x) \to 0$ as $|x| \to \infty$, then for all $x \in R^n$*

$$f(x) = c_n \sum_{k=1}^{n} \left(\int_{R^n} \sum_{i,j=1}^{n} \gamma_{ij}^k(x-y)(Sf)_{ij}(y)\,dy \right) e_k. \tag{5.2}$$

Here $e_1, \ldots, e_n$ is the standard basis of R^n, $c_n = n/(2(n-1)\omega_{n-1})$ and $\gamma_{ij}^k(x) = |x|^{-n}(\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k) - (n+2)|x|^{-n-2}x_i x_j x_k$, where $\delta_{ij}=0$ if $i \neq j$ and $\delta_{ij}=1$ if $i=j$.

Proof. Let $f=(f_1, \ldots, f_n)$. By Ahlfors [A_1, p. 80]

(5.3)

$$f_k(x) = c_n \int_{B^n(x,r)} \sum_{i,j} \gamma_{ij}^k(x-y)(Sf)_{ij}(y)\,dy - c_n \int_{S^{n-1}(x,r)} [\gamma^k(x-y)f(y)] \cdot \left(\frac{y-x}{|y-x|}\right) dy$$

for any $r>0$. Here $\gamma^k(x-y)$ is the linear map $R^n \to R^n$ with matrix $(\gamma_{ij}^k(x-y))$, and the dot $(\cdot)$ refers to the scalar product in R^n. Let $I_1(r)$ and $I_2(r)$ be the integrals above. Because $\gamma_{ij}^k \in G$ and $(Sf)_{ij} \in L^1 \cap L^\infty$ for all i, j, we get as in Theorem 4.3

$$I_1(r) \to c_n \int_{R^n} \sum_{i,j} \gamma_{ij}^k(x-y)(Sf)_{ij}(y)\,dy \quad \text{as} \quad r \to \infty,$$

and the integral over R^n converges absolutely. If $r \geqq 1$, we get

$$|I_2(r)| \leqq c' c_n \int_{S^{n-1}(x,r)} \frac{1}{r^{n-1}} |f(y)|\,dy = c' c_n \int_{S^{n-1}(x,1)} |f(ry)|\,dy,$$

where $c' = \max_{|x|=1} \|\gamma^k(x)\|$. Because $|f(x)| \to 0$ as $|x| \to \infty$, then $I_2(r) \to 0$ as $r \to 0$. Therefore (5.3) yields (5.2) as $r \to \infty$, and the lemma is proved.

In the next lemma we need mollifiers. Let $\theta\colon R\times R^n\to R^n$ be a C^∞-function such that

(i) $$\theta(u,v)\geqq 0 \quad\text{and}\quad \theta(u,v)=0 \quad\text{for}\quad |(u,v)|\geqq 1$$

(ii) $$\int_{R\times R^n}\theta(u,v)\,d(u,v)=1.$$

Define

(5.4) $$\theta_\varepsilon(u,v)=\varepsilon^{-n-1}\theta(\varepsilon^{-1}u,\varepsilon^{-1}v),\quad\text{for all}\quad \varepsilon>0.$$

Then $f*\theta_\varepsilon\in C^\infty$ with $(f*\theta_\varepsilon)(t,x)=\int_{R\times R^n}f(t-u,x-v)\theta_\varepsilon(u,v)\,d(u,v)$ for any function f which is locally integrable in $R\times R^n$.

5.5. Lemma. *Suppose that*

(i) $F\colon[0,1]\times R^n\to R^n$ *is locally in* L^1 *and* (*recall* $F_t(x)=F(t,x)$)

(ii) *for almost every* $t\in[0,1]$ *the function* $F_t\colon R^n\to R^n$ *is continuous,* $\lim_{|x|\to\infty}F_t(x)=0$, $F_t\in W^{1,1}_{\text{loc}}(R^n)$ *and* $\int_0^1 a(t)dt<\infty$ *with* $a(t)=\|SF_t\|_1+\|SF_t\|_\infty$. *Then F generates a unique continuous semiflow* $\varphi\colon[0,1]\times R^n\to R^n$ *such that* $\varphi_t\colon R^n\to R^n$ *is qc with*

(5.6) $$K(\varphi_t)\leqq\exp n\Big(\int_0^t\|SF_s\|_\infty ds\Big)\quad\text{for all}\quad 0\leqq t\leqq 1.$$

Proof. Let $F^\varepsilon=F*\theta_\varepsilon$ for $\varepsilon>0$ (set $F(t,x)=0$ if $t\notin[0,1]$). Write $SF(t,x)$ for $SF_t(x)$. By Lemma 5.1 and Fubini's theorem we get (write $F=(F_1,\dots,F_n)$)

(5.7) $$\begin{aligned}F_k^\varepsilon(t,x)&=\int_{R\times R^n}F_k(t-u,x-v)\theta_\varepsilon(u,v)d(u,v)\\&=\int_{R\times R^n}\Big(c_n\int_{R^n}\sum_{i,j}\gamma_{ij}^k(y)\big(SF(t-u,x-v-y)\big)_{ij}dy\Big)\theta_\varepsilon(u,v)d(u,v)\\&=c_n\int_{R^n}\sum_{i,j}\gamma_{ij}^k(y)\Big(\int_{R\times R^n}\big(SF(t-u,x-v-y)\big)_{ij}\theta_\varepsilon(u,v)\,d(u,v)\Big)dy\\&=c_n\sum_{i,j}\gamma_{ij}^k*\big((SF*\theta_\varepsilon)_{ij}\big)_t(x).\end{aligned}$$

Let $a_{ij}^\varepsilon(t)=\big\|\big((SF*\theta_\varepsilon)_{ij}\big)_t\big\|_1+\big\|\big((SF*\Theta_\varepsilon)_{ij}\big)_t\big\|_\infty$, $t\in[0,1]$. It is easy to see that for all $1\leqq i,j\leqq n$

(5.8) $$\int_0^1 a_{ij}^\varepsilon(t)\,dt\leqq\int_0^1 a(t)\,dt\quad\text{(see the assumption (ii))},$$

(5.9) $$\Big|\int_{t_1}^{t_2}a_{ij}^\varepsilon(t)\,dt\Big|\leqq\delta(|t_1-t_2|)\quad\text{for all}\quad t_1,\ t_2\in[0,1],$$

where $\delta\colon[0,1]\to R$ is an increasing function with $\lim_{s\to 0}\delta(s)=0$ so that

$$\Big|\int_{t_1}^{t_2}a(t)\,dt\Big|\leqq\delta(|t_1-t_2|)$$

for t_1, $t_2\in[0,1]$ (such δ exists because $a\in L^1[0,1]$).

Now, Theorem 4.3 and (5.7) imply

$$(5.10)\qquad \begin{cases} |F_t^\varepsilon(x)| \leqq c' \sum_{i,j} a_{ij}^\varepsilon(t) \quad (\text{here } c' < \infty \text{ depends only on } n), \\ |F_t^\varepsilon(x_1) - F_t(x_2)| \leqq c'\big(\sum_{i,j} a_{ij}^\varepsilon(t)\big)|x_1 - x_2| \left(1 + \log^+ \dfrac{1}{|x_1 - x_2|}\right) \end{cases}$$

for $t \in [0, 1]$ and $x, x_1, x_2 \in R^n$. Then Theorem 3.4 with (5.8) and (5.9) yields that F^ε generates a semiflow φ^ε with

$$(5.11)\qquad \begin{cases} |\varphi^\varepsilon(t, x) - x| \leqq c_1 \int_0^1 a(t)\,dt = I \text{ and} \\ |\varphi^\varepsilon(t_1, x_1) - \varphi^\varepsilon(t_2, x_2)| \leqq c_2\big(w_I(|x_1 - x_2|) + \delta(|t_1 - t_2|)\big) \end{cases}$$

for all $t, t_1, t_2 \in [0, 1]$ and $x, x_1, x_2 \in R^n$, where constants $c_1, c_2 < \infty$ only depend on n. By Ascoli's theorem $(\varphi^\varepsilon)_{\varepsilon>0}$ form a normal family and, therefore, there is a continuous function $\varphi\colon [0, 1] \times R^n \to R^n$ and a subsequence $(\varphi^{\varepsilon_i})$ with $\varepsilon_i \to 0$ as $i \to \infty$ such that $\varphi^{\varepsilon_i} \to \varphi$ locally uniformly in $[0, 1] \times R^n$ as $i \to \infty$.

We want to show that φ is a semiflow generated by F, i.e.

$$(5.12)\qquad \varphi(t, x) = x + \int_0^t F(s, \varphi(s, x))\,ds \quad \text{for} \quad t \in [0, 1],\ x \in R^n.$$

Since $\varphi^{\varepsilon_i}(t, x) = x + \int_0^t F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x))ds$ and $\varphi^{\varepsilon_i} \to \varphi$ as $i \to \infty$, we only need to prove that

$$(5.13)\qquad I_i = \left|\int_0^t F(s, \varphi(s, x))\,ds - \int_0^t F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x))\,ds\right| \to 0$$

as $i \to \infty$. Applying Lemma 5.1 and Theorem 4.3 we get for a.e. $t \in [0, 1]$

$$(5.14)\qquad \begin{cases} |F_t(x)| \leqq c'' a(t), \\ |F_t(x_1) - F_t(x_2)| \leqq c'' a(t) h(|x_1 - x_2|) \end{cases}$$

for all $x_1, x_2 \in R^n$ with $h(u) = u(1 + \log^+(1/u))$, $u > 0$. We get

$$(5.15)\quad I_i = \left|\int_0^t \Big[\int_{R^{n+1}} \big(F(s, \varphi(s, x)) - F(s-u, \varphi^{\varepsilon_i}(s, x) - v)\big)\theta_{\varepsilon_i}(u, v)\,d(u, v)\Big]\,ds\right|$$

$$\leqq \int_{R^{n+1}} \Big(\int_0^t |F(s, \varphi(s, x)) - F(s-u, \varphi(s-u, x))|\,ds\Big)\,\theta_{\varepsilon_i}(u, v)\,d(u, v)$$

$$+ \int_{B^{n+1}(\varepsilon_i)} \Big(\int_0^t |F(s-u, \varphi(s-u, x)) - F(s-u, \varphi^{\varepsilon_i}(s, x) - v)|\,ds\Big)\,\theta_{\varepsilon_i}(u, v)\,d(u, v).$$

Let the integrals on the right-hand side of (5.15) be I_i' and I_i''. By (5.14) we get

$$I_i'' \leqq \int_{B^{n+1}(\varepsilon_i)} \Big(\int_0^t c'' a(s-u) h\big(|\varphi(s-u, x) - \varphi^{\varepsilon_i}(s, x) - v|\big)\,ds\Big)\,\theta_{\varepsilon_i}(u, v)\,d(u, v).$$

Here $h(|\varphi(s-u, x)-\varphi^{\varepsilon_i}(s, x)-v|)\to 0$ uniformly as $i\to\infty$ because then $\varepsilon_i\to 0$, $u, v\to 0$ and $\varphi^{\varepsilon_i}\to\varphi$ locally uniformly. Because $a\in L^1$, $I_i''\to 0$ as $i\to\infty$. To estimate I_i' write $G(r)=F(r, \varphi(r, x))$, $r\in R$ (recall $F(t, y)=0$ if $t\notin[0, 1]$). Then $G\in L^1(R)$ by (5.14) (the measurability is not difficult to prove). Because $G\in L^1$ there is an increasing function δ_G: $[0, \infty)\to R$ such that $\int_0^t |G(s-u)-G(s)|ds\leqq\delta_G(|u|)\to 0$ as $|u|\to 0$ for all $t\in[0, 1]$. We get

$$I_i' = \int_{B^{n+1}(\varepsilon_i)} \Big(\int_0^t |G(s)-G(s-u)|\,ds\Big)\,\theta_{\varepsilon_i}(u, v)\,d(u, v)$$

$$\leqq \int_{B^{n+1}(\varepsilon_i)} \delta_G(|u|)\theta_{\varepsilon_i}(u, v)\,d(u, v) \leqq \delta_G(\varepsilon_i) \int_{B^{n+1}(\varepsilon_i)} \theta_{\varepsilon_i}\,d(u, v)$$

$$= \delta_G(\varepsilon_i) \to 0 \quad \text{as} \quad i\to\infty.$$

We have shown that $I_i\to 0$ as $i\to\infty$, and therefore φ is generated by F (it is not difficult to see that (i) and (ii) of (5.5) imply (i) of (3.1)). From (5.14) and Theorem 3.8 it follows that φ is unique and φ_t: $R^n\to R^n$ are homeomorphims for all $t\in[0, 1]$. Finally, we prove (5.6). Fix $t\in[0, 1]$. We know that $\varphi_t^{\varepsilon_i}\to\varphi_t$ locally uniformly in R^n. By Theorem 2.2 we have

$$K(\varphi_t^{\varepsilon_i}) \leqq \exp\Big(n\int_0^t \|SF_s^{\varepsilon_i}\|_\infty\,ds\Big).$$

Because $D_{x_k}(F*\theta_\varepsilon)=(D_{x_k}F)*\theta_\varepsilon$, we have $SF_s^{\varepsilon_i}=((SF)*\theta_{\varepsilon_i})_s$, and therefore it is easy to see that $\limsup_{i\to\infty}\int_0^t\|SF_s^{\varepsilon_i}\|_\infty ds\leqq\int_0^t\|SF_s\|_\infty ds$. By a well-known limit theorem for qc-mappings φ_t is qc and $K(\varphi_t)\leqq\liminf_{i\to\infty}K(\varphi_t^{\varepsilon_i})\leqq\exp\big(n\int_0^t\|SF_s\|_\infty ds\big)$. The theorem is proved.

Next we prove our main result on qc semiflows. We consider $\bar R^n=R^n\cup\infty$ as a smooth manifold with local coordinates $x\mapsto x$ for $x\in R^n$, and $x\mapsto J(x)\in R^n$ for $x\in\bar R^n\setminus\{0\}$, where J is the reflection in S^{n-1}, i.e. $J(x)=x/|x|^2$, $x\neq 0$. Suppose that a vector field F on $\bar R^n$ is given in local coordinates by functions $f(x)$, $x\in R^n$, and $f^*(J(x))$, $x\in\bar R^n\setminus\{0\}$. Then by the usual transformation formula for contravariant vectors [A_2, p. 8]

$$(5.17)\qquad f^*(y) = [DJ^{-1}(y)]^{-1}f(J(y)) = |y|^2(I-2Q(y))f(J(y)),$$

$y\neq 0$, where $Q(y)$ is the matrix $(Q(y))_{ij}=|y|^{-2}y_iy_j$, $1\leqq i, j\leqq n$. Furthermore, a function $\varphi: [a, b]\to\bar R^n$ is the solution of the differential equation $\dot\varphi(t)=F(\varphi(t))$ if $\dot\varphi(t)=f(\varphi(t))$ for $\varphi(t)\in R^n$ and $(J\circ\varphi)^\cdot(t)=f^*(J\circ\varphi(t))$ for $\varphi(t)\in\bar R^n\setminus\{0\}$. Especially, if $\varphi(t)\neq 0, \infty$, then $\dot\varphi(t)=f(\varphi(t))$ if and only if $(J\circ\varphi)^\cdot(t)=f^*(J\circ\varphi(t))$.

Consider a continuous function f: $R^n\to R^n$ as a vector field on R^n. Suppose that f can be extended continuously to $\bar R^n$ as a vector field, i.e. there exists $\lim_{x\to 0}f^*(x)\in R^n$ for f^* defined by (5.17), and suppose that f generates a unique flow $\varphi(t, x)$, $0\leqq t\leqq 1$, $x\in\bar R^n$, on $\bar R^n$. Then f^* generates the flow $\psi(t, x)$, and we have

$\psi_t(x)=J\circ\varphi_t\circ J(x)$ for (t,x) with $\varphi_t\circ J(x)\neq\infty$. Similar remarks are valid for semiflows on $\bar{R}^n$.

We are now ready to prove our main result for qc semiflows.

5.18. Theorem. *Suppose that* $F\colon [0,1]\times R^n\to R^n$ *is locally integrable in* $[0,1]\times R^n$ *and for a.e.* $t\in[0,1]$ *the function* $F_t\colon R^n\to R^n$ *is continuous and in* $W^{1,1}_{\text{loc}}(R^n)$ *and* $\int_0^1\|SF_t\|_\infty dt<\infty$. *If* $n=2$, *we also assume that* $\lim_{|x|\to\infty}|x|^{-3}|F_t(x)|=0$ *for a.e.* $t\in[0,1]$. *Then F generates a unique semiflow* $\varphi(t,x)$, $0\leqq t\leqq 1$, $x\in\bar{R}^n$, *such that* $\varphi_t\colon \bar{R}^n\to\bar{R}^n$ *is quasiconformal with*

$$K(\varphi_t)\leqq\exp\Big(n\int_0^t\|SF_s\|_\infty ds\Big)\quad \textit{for}\quad 0\leqq t\leqq 1.$$

Proof. We first show that for every $x_0\in R^n$, $r_0\geqq 1$,

$$\int_0^1\sup_{|x-x_0|\leqq r_0}|F_t(x)|\,dt<\infty. \tag{5.19}$$

Clearly it suffices to prove (5.19) for $r_0=1$. Let $t\in[0,1]$ such that F_t is continuous and $\|SF_t\|_\infty<\infty$. Then by (5.3) we get for every $x\in B^n(x_0,1)$

$$|F_t(x)|\leqq c_n'\|SF_t\|_\infty m_n B^n(x,r)+c_n' r^{-n+1}\int_{S^{n-1}(x,r)}|F_t(y)|\,dm_{n-1}(y)$$

for $r>0$. By Fubini's theorem choose $r\in[1,2]$ such that

$$\int_{S^{n-1}(x,r)}|F_t(y)|\,dm_{n-1}(y)=\int_{1\leqq|y-x|\leqq 2}|F_t(y)|\,dm_n(y)\leqq\int_{B^n(x_0,3)}|F_t(y)|\,dm_n(y).$$

Thus we get

$$\int_0^1\sup_{|x-x_0|\leqq 1}|F_t(x)|\,dt\leqq\int_0^1 c'\Big(\|SF_t\|_\infty+\int_{B^n(x_0,3)}|F_t(y)|\,dy\Big)<\infty$$

because $\int_0^1\|SF_t\|_\infty dt<\infty$ and F is locally integrable in $[0,1]\times R^n$. Here $c'<\infty$ only depends on n. This proves (5.19).

The inequality (5.19) enables us to apply truncating to F. Let $x_0\in R^n$ and $\psi\in C_0^\infty(B^n(x_0))$ with $\psi(x)=1$ for $|x-x_0|\leqq 1/2$. Then for a.e. $t\in[0,1]$ we get $\|S(\psi F_t)(x)\|\leqq 2n|D\psi(x)|\,|F_t(x)|+|\psi(x)|\,\|SF_t(x)\|$ for a.e. $x\in B^n(x_0,1)$, which implies by (5.19) that $\int_0^1\|S(\psi F_t)\|_\infty dt<\infty$. Therefore, we may apply Lemma 5.5 to ψF and conclude that F locally generates a unique semiflow.

Next we consider the semiflow generated by F in a neighborhood of ∞. We define the reflection F^* of F in S^{n-1} by (5.17): $F^*(t,y)=|y|^2(I-2Q(y))F(t,y/|y|^2)$, $y\neq 0$. Let $t\in[0,1]$ such that F_t is continuous and $\|SF_t\|_\infty<\infty$. Then $\operatorname{ess\,sup}_{y\neq 0}\|SF_t^*(y)\|=\|SF_t\|_\infty$ by [A_1, (1.8)], and $\lim_{|x|\to 0}|x|\,|F_t^*(x)|=0$ by assumption if $n=2$ and by [Sa, Theorem 3.15] if $n\geqq 3$. By [Sa, Lemma 3.2] then F_t^* has a continuous extension to R^n such that $F_t^*\in W^{1,1}_{\text{loc}}(R^n)$, and so $\|SF_t^*\|_\infty=\|SF_t\|_\infty$. Applying (5.19) and the representation (5.3) we see that F^* is locally integrable in

$[0, 1]\times R^n$. Now, the above truncating method applies to F^*, and we see that F^* generates a unique semiflow locally. This means that we can extend F to $[0, 1]\times \bar{R}^n$ such that F_t is a continuous vector field on $\bar{R}^n$ for a.e. $t\in[0, 1]$ and F generates a unique semiflow on $\bar{R}^n$ locally. Because $\bar{R}^n$ is a compact manifold without boundary, F generates a unique global semiflow $\varphi(t, x)$, $0\leqq t\leqq 1$, $x\in \bar{R}^n$. The local uniqueness also implies, by the usual reasoning, that mappings $\varphi_t\colon \bar{R}^n\to\bar{R}^n$ are home omorphisms for $0\leqq t\leqq 1$. We still have to prove that they are quasiconformal.

Suppose first that $\varphi(t, 0)\in R^n$ for $0\leqq t\leqq 1$. Define $H(t, y)=F(t, y+\varphi(t, 0))-F(t, \varphi(t, 0))$ for $0\leqq t\leqq 1$, $y\in R^n$. Then $H(t, 0)=0$ for every $t\in[0, 1]$. Applying (5.19) to H and the inequality [A_1, Theorem 1] or [Sa, (2.3)] we get

$$|H(t, x)| \leqq h(t)|x|\left(1+\log\frac{1}{|x|}\right) \tag{5.20}$$

for $|x|\leqq 1, 0\leqq t\leqq 1$, with $\int_0^1 h(t)dt<\infty$. Define $H^*(t, x)=|x|^2(I-2Q(x))H(t, x/|x|^2)$. Then we see, as above, that $H_t^*\colon R^n\to R^n$ is continuous, $H_t^*\in W^{1,1}_{\text{loc}}(R^n)$, $\|SH_t^*\|_\infty=\|SH_t\|_\infty=\|SF_t\|_\infty$ for a.e. $t\in[0, 1]$ and $H^*\in L^1_{\text{loc}}$. With (5.20) we also conclude that

$$|H^*(t, x)| \leqq k(t)|x|(1+\log^+|x|+\log^+(1/|x|)) \tag{5.21}$$

for $x\in R^n$, $0\leqq t\leqq 1$, with $\int_0^1 k(t)dt<\infty$. By the above reasoning H^* generates a unique continuous semiflow $\psi(t, x)$, $0\leqq t\leqq 1$, $x\in\bar{R}^n$. We claim that $\psi(t, x)\in R^n$ for all $t\in[0, 1]$, $x\in R^n$. Let $x\in R^n$, and let $s\in(0, 1]$ such that $\psi(t, x)\in R^n$ for $0\leqq t\leqq s$. Then by (5.21)

$$\begin{aligned}|\psi(t, x)| &= \left|x+\int_0^t H^*(u, \psi(u, x))\,du\right| \\ &\leqq 1+|x|+\int_0^t k(u)|\psi(u, x)|(1+\log^+|\psi(u, x)|+\log^+(|\psi(u, x)|^{-1}))\,du\end{aligned} \tag{5.22}$$

for $0\leqq t\leqq s$. Define $\tau(t)$ to be the right-hand side of (5.22), $0\leqq t\leqq s$. Then $\dot\tau(t)\leqq k(t)\tau(t)(1+\log\tau(t))$ for a.e. $t\in[0, s]$, and we get, as in the proof of Theorem 3.4, $\int_0^1 k(u)\,du\geqq\log\left[(1+\log\tau(s))/(1+\log\tau(0))\right]$, which implies $\log|\psi(s, x)|\leqq \log\tau(s)\leqq(1+\log(1+|x|))\exp\left(\int_0^1 k(u)du\right)$. This proves $\psi(s, x)\in R^n$ for all $s\in[0, 1]$, $x\in R^n$.

We know now that $\psi_t\colon R^n\to R^n$ is a homeomorphism for every $t\in[0, 1]$. Furthermore, ψ_t is qc with

$$K(\psi_t)\leqq\exp\left(n\int_0^t \|SF_s\|_\infty ds\right). \tag{5.23}$$

To see this, apply truncating to H^* and then apply Lemma 5.5. Note that the dilatation of ψ_t in a neighbourhood of $x\in R^n$ depends only on the values of SH_t in a neighbourhood of $\{\psi(s, x)\colon 0\leqq s\leqq t\}$ in $[0, 1]\times R$; see (2.3) and the proof of Lemma 5.5. This reasoning proves (5.23).

Because H^* generates $\psi(t, x)$, the flow generated by H on $\bar{R}^n$ is $\theta(t, x)=(J\circ\psi_t\circ J)(x)$. Define $\tilde{\varphi}(t, x)=\theta(t, x)+\varphi(t, 0)$ for $x\neq\infty$ and $\theta(t, x)\neq\infty$. Then $\tilde{\varphi}(0, x)=x$ and

$$\dot{\tilde{\varphi}}(t, x) = \dot{\theta}(t, x)+\dot{\varphi}(t, 0) = H(t, \theta(t, x))+F(t, \varphi(t, 0))$$
$$= F(t, \theta(t, x)+\varphi(t, 0)) = F(t, \tilde{\varphi}(t, x))$$

for a.e. $t\in[0, 1]$. Therefore, $\tilde{\varphi}$ is a semiflow generated by F, and thus $\tilde{\varphi}=\varphi$ by uniqueness. Especially, $\varphi_t(x)=\theta_t(x)+\varphi_t(0)=(J\circ\psi_t\circ J)(x)+\varphi_t(0)$ for $x\in R^n$ such that $(J\circ\psi_t\circ J)(x)\in R^n$. Then φ_t: $\bar{R}^n\to\bar{R}^n$ is qc and $K(\varphi_t)\leqq\exp\left(n\int_0^t\|SF_s\|_\infty ds\right)$ by (5.23).

If $\varphi(t, 0)\notin R^n$ for some $t\in[0, 1]$, we use factorization. For $0<t\leqq 1$ and an integer $m\geqq 1$ we get, by uniqueness, $\varphi_t=\varphi_1^m\circ\varphi_1^{m-1}\circ\ldots\circ\varphi_1^1$, where $\varphi_1^k(x)=\varphi^k(1, x)$ and $\varphi^k(t, x)$ is the semiflow generated by

$$F^k(s, x) = \begin{cases} F(s+(k-1)t/m, x), & 0\leqq s\leqq t/m, \\ 0 & \text{otherwise} \end{cases}$$

for all $x\in R^n$. By the existence of local solutions in R^n (see [C—L, p. 43]), we can choose so large m that $\varphi^k(s, 0)\in R^n$ for all $0\leqq s\leqq 1$ and $k=1, 2, \ldots, m$. Then by the above result every φ_1^k is qc, and so is φ_t, and because

$$K(\varphi_1^k) \leqq \exp\left(n\int_0^1 \|SF_s^k\|_\infty ds\right) = \exp\left(n\int_{(k-1)t/m}^{kt/m} \|SF_s\|_\infty ds\right),$$

we get $K(\varphi_t)\leqq K(\varphi_1^m)\ldots K(\varphi_1^1)\leqq\exp\left(n\int_0^t\|SF_s\|_\infty ds\right)$. The theorem is proved.

6. Operators S_α and div

The method of proving Theorem 2.2 and Lemma 5.5 applies to linear combinations of operators $Tf=(1/2)(\widetilde{Df}+Df)$ and $\operatorname{div} f=\operatorname{tr}(Df)$ as well. Therefore, it is of some interest to see how operators S_α,

$$S_\alpha f = \frac{1}{2}(Df+\widetilde{Df})+\alpha\operatorname{tr}(Df)I, \quad \alpha\neq-\frac{1}{n},$$

and div affect the generated mappings φ_t.

We deal first with S_α. It turns out that if we replace S with S_α, $\alpha\neq-1/n$, in Lemma 5.5, then again F generates a unique semiflow φ and φ_t: $R^n\to R^n$ is a bi-Lipschitz-mapping with the Lipschitz constant

$$L(\varphi_t) \leqq \exp\left(c_{n,\alpha}\int_0^t \|S_\alpha F_s\|\,ds\right), \quad c_{n,\alpha} = \left|\frac{\alpha n}{\alpha n+1}\right|+1, \tag{6.1}$$

for all $0\leqq t\leqq 1$.

We outline the proof of (6.1). Note first that if we consider S, S_α and T as linear mappings from the space $R^{n\times n}$ of real $n\times n$-matrices into $R^{n\times n}$, then

(i) $$S\circ S_\alpha = S,$$

(ii) $$\|SM\| \leqq 2\|M\|, \quad M\in R^{n\times n},$$

(iii) $$\|TM\| \leqq \left(\frac{|\alpha n|}{|\alpha n+1|}+1\right)\|S_\alpha M\|, \quad \alpha \neq -1/n, \quad M\in R^{n\times n}.$$

Now, (i) and (ii) imply $\|SF_t\|\leqq 2\|S_\alpha F_t\|$ and, therefore, F and F^ε generate φ and φ^ε and $\varphi^{\varepsilon_i}\to\varphi$ locally uniformly, as in the proof of Lemma 5.5. As in the proof of Theorem 2.2, we get in the smooth case $(\|D\varphi_t\|^2)^\cdot=(\|\widetilde{D\varphi_t D\varphi}\|)^\cdot\leqq 2\|\widetilde{D\varphi_t}\|\,\|TF_t(\varphi_t(x))\|\,\|D\varphi_t\|$, $0\leqq t\leqq 1$, which implies $\|D\varphi_t\|\leqq\exp\left(\int_0^t\|TF_s\|_\infty ds\right)=l_t$; thus $L(\varphi_t)\leqq l_t$, and (6.1) follows from (iii) in the smooth case. Because $\varphi_t^{\varepsilon_i}\to\varphi_t$ locally uniformly, we get (6.1) also in the non-smooth case.

For the operator div we get in the smooth case (apply the differentation formula after (2.4) to (2.1))

(6.2) $$\det\left(D\varphi_t(x)\right)^\cdot = \operatorname{div} F_t(\varphi_t(x))\det D\varphi_t(x), \quad \det D\varphi_0(x)=1,$$

which implies

(6.3) $$J\varphi_t(x)=\det D\varphi_t(x)\leqq\exp\left(\int_0^t\|\operatorname{div} F_s\|_\infty ds\right), \quad 0\leqq t\leqq 1.$$

The non-smooth case does not follow immediately from (6.3) because, in general, F cannot be recovered from div F (i.e. the differential equation div $F=\varrho$ is underdetermined) and there is no representation of type (5.2). Therefore, we have to restrict ourselves to the functions F which have such a representation.

Let $\gamma=(\gamma_1, \dots, \gamma_n)$ with

(6.4) $$\gamma_i(x)=\frac{1}{\omega_{n-1}}|x|^{-n}x_i, \quad x\neq 0, \quad i=1,2,\dots,n.$$

Then every $\gamma_i\in G$. For $f\colon R^n\to R\in L^1\cap L^\infty$ we form a vector valued Riesz potential

$$\gamma*f(x)=\sum_{i=1}^n(\gamma_i*f)(x)e_i=\sum_{i=1}^n\left(\int_{R^n}\gamma_i(x-y)f(y)dy\right)e_i$$

for all $x\in R^n$. Then $\gamma*f\colon R^n\to R^n$ is continuous and γ is the fundamental solution for the operator div, i.e. (see e.g. [St, p. 125])

(6.5) $$\operatorname{div}(\gamma*f)=f \quad \text{for} \quad f\in C_0^\infty.$$

We want to extend (6.5) to a non-smooth case. We have

$$D_i\gamma_j(x)=\frac{1}{\omega_{n-1}}(\delta_{ij}|x|^{-n}-n|x|^{-n-2}x_ix_j), \quad x\neq 0,$$

and it is easy to see that

(6.6) (1) $D_i\gamma_j(tx) = t^{-n}D_i\gamma_j(x)$, $t>0$, $x\neq 0$, and

(2) $\int_{S^{n-1}} D_i\gamma_j(x)\,dx = 0$

for all $1\leq i, j\leq n$. This means that $D_i\gamma_j$ is a Calderon—Zygmund kernel and defines a singular integral operator Γ_{ij} by

$$(6.7)\qquad (\Gamma_{ij}h)(x) = \lim_{r\to 0}\int_{|x-y|\geq r} D_i\gamma_j(x-y)h(y)\,dy \quad \text{(limes in } L^p)$$

for $h\colon R^n\to R\in L^p$. Then $\Gamma_{ij}\colon L^p\to L^p$ is a continuous linear operator for all $1<p<\infty$ [St, p. 39, Theorem 3]. Next we conclude, as in [A_1, p. 84], that $\gamma_i * f\in W^{1,p}$ for all $1<p<\infty$ (i.e. it has distributional first order partial derivatives in L^p) and

$$(6.8)\qquad D_i(\gamma_j * f) = \frac{1}{n}\delta_{ij}f - \Gamma_{ij}(f) \quad \text{for} \quad f\in L^1\cap L^\infty.$$

(Note that $L^p\supset L^1\cap L^\infty$ for all $1\leq p\leq\infty$.) Therefore, $\operatorname{div}(\gamma * f) = \sum_i D_i(\gamma_i * f) = f - \sum_i \Gamma_{ii}(f)$. But $\sum_i \Gamma_{ii} = 0$ because

$$\sum_{i=1}^{n}(D_i\gamma_i(x)) = (1/\omega_{n-1})\sum_{i=1}^{n}(\delta_{ii}|x|^{-n} - n|x|^{-n-2}x_i^2) = 0.$$

So we have proved:

6.9. Lemma. *If $f\colon R^n\to R\in L^1\cap L^\infty$, then $\gamma * f\colon R^n\to R^n$ is continuous, belongs to $W^{1,p}(R^n)$ for all $1<p<\infty$ and* $\operatorname{div}(\gamma * f) = f$.

We are now ready to prove the non-smooth form of (6.3). Recall that for a homeomorphism $f\colon R^n\to R^n$ the limit

$$\lim_{r\to 0}\frac{m(fB^n(x,r))}{m(B^n(x,r))} = \mu_f(x)$$

exists for a.e. $x\in R^n$ and is called the Lebesgue derivative of f. If f is differentiable at x, then $|J_f(x)| = \mu_f(x)$.

6.10. Theorem. *Let $\varrho\colon [0,1]\times R^n\to R$ be locally integrable and $\int_0^1(\|\varrho_s\|_1 + \|\varrho_s\|_\infty)\,ds<\infty$ (here $\varrho_s(x) = \varrho(s,x)$). Define $F(t,x) = (\gamma * \varrho_t)(x)$, $(t,x)\in[0,1]\times R^n$. Then F generates a continuous semiflow φ such that $\varphi_t\colon R^n\to R^n$ is a bi-absolutely continuous homeomorphism with*

$$\mu_{\varphi_t}(x)\leq e^I \quad \textit{and} \quad \mu_{\varphi_t^{-1}}(x)\leq e^I \quad \textit{with} \quad I = \int_0^t \|\varrho_s\|_\infty\,ds$$

for a.e. $x\in R^n$ and for all $0\leq t\leq 1$.

Proof. Because F is measurable and the function $\gamma * \varrho_t$ is continuous for a.e. $t \in [0, 1]$, it is not difficult to see that for every $x \in R^n$ the mapping $t \mapsto \gamma * \varrho_t(x)$, $t \in [0, 1]$, is measurable. Let θ_ε be a mollifier as in (5.4) for $\varepsilon > 0$, and $F^\varepsilon = F * \theta_\varepsilon$ (Theorem 4.3 shows that F is locally integrable). Here set $\varrho_t(x) = 0$ for $t \notin [0, 1]$. By Fubini's theorem $F^\varepsilon(t, x) = F * \theta_\varepsilon = [\gamma * (\varrho * \theta_\varepsilon)_t](x)$. Therefore, we can proceed in the same way as in the proof of Lemma 5.5. Let φ^ε and φ be the semiflows generated by F^ε and F, respectively, and choose $\varphi^{\varepsilon_i} \to \varphi$ locally uniformly as $\varepsilon_i \to 0$. Then $\varphi_t\colon R^n \to R^n$ is a homeomorphism and $\varphi_t^{\varepsilon_i}\colon R^n \to R^n$ is a diffeomorphism for which

$$(6.11) \qquad J_{\varphi_t^{\varepsilon_i}}(x) \leqq \exp\Big(\int_0^{t+\varepsilon_i} \|\varrho_s\|_\infty \, ds\Big) \leqq \exp\Big(\int_0^1 \|\varrho_s\|_\infty \, ds\Big)$$

by (6.3) because $\operatorname{div} F_s^\varepsilon = (\varrho * \theta_\varepsilon)_s$ and $\int_0^t \|(\varrho * \theta_\varepsilon)_s\|_\infty ds \leqq \int_0^{t+\varepsilon} \|\varrho_s\|_\infty ds$. Therefore, the next lemma proves the theorem for φ_t, and a similar reasoning applies to φ_t^{-1}.

6.12. Lemma. *Let $\varphi, \varphi_k\colon R^n \to R^n$, $k = 1, 2, \ldots$, be homeomorphisms such that every φ_k is absolutely continuous and $\varphi_k \to \varphi$ locally uniformly as $k \to \infty$. If $\|\mu_{\varphi_k}\|_\infty \leqq M < \infty$ for every k, then φ is absolutely continuous and $\mu_\varphi \leqq M$ a.e. If, in addition, $\mu_{\varphi_k} \to J$ a.e. as $k \to \infty$, then $\mu_\varphi = J$ almost everywhere.*

Proof. We first show that for any $x \in R^n$ and $r > 0$

$$(6.13) \qquad m(\varphi B^n(x, r)) = \lim_{k \to \infty} m(\varphi_k B^n(x, r)).$$

Here m refers to the Lebesgue measure in R^n. Fix x and r. Let $r > \varepsilon > 0$. Take so big k_0 that $\varphi_k B^n(x, r-\varepsilon) \subset \varphi B^n(x, r) \subset \varphi_k B^n(x, r+\varepsilon)$ for all $k \geqq k_0$. Then

$$\big|m(\varphi B^n(x, r)) - m(\varphi_k B^n(x, r))\big| \leqq m\big[\varphi_k\big(B^n(x, r+\varepsilon) \setminus B^n(x, r-\varepsilon)\big)\big]$$

$$\leqq \int_{r-\varepsilon \leqq |x-y| \leqq r+\varepsilon} \mu_{\varphi_k}(y)\, dm(y) \leqq M m[B^n(x, r+\varepsilon) \setminus B^n(x, r-\varepsilon)] \to 0$$

as $\varepsilon \to 0$. This proves (6.13). It follows that $m\big(\varphi B^n(x, r)\big) \leqq M m\big(B^n(x, r)\big)$ for all x and r, and it is easy to see that φ is absolutely continuous. It also follows that $\mu_\varphi \leqq M$. If $\mu_{\varphi_i} \to J$ almost everywhere as $i \to \infty$, then for an open ball B we get

$$m(\varphi B) = \lim_{i \to \infty} m(\varphi_i B) = \lim_{i \to \infty} \int_B \mu_{\varphi_i} dm = \int_B J dm$$

by Lebesgue's dominated convergence theorem. This implies that $\mu_\varphi = J$ a.e., and the lemma is proved.

7. Semiflow with prescribed Jacobian determinant

7.1. Theorem. *Suppose that* $J\colon R^n \to R$ *is measurable,* $J(x)>0$ *a.e. and* $\log J \in L^1 \cap L^\infty$. *Then there is* $g\colon [0,1]\times R^n \to R^n \in L^1 \cap L^\infty$ *such that* $F(t,x)=\gamma * g_t(x)$, $(t,x)\in[0,1]\times R^n$, *generates a continuous semiflow* φ *such that* $\varphi_t\colon R^n \to R^n$ *is a bi-absolutely continuous homeomorphism with*

$$\mu_{\varphi_t}(x) = (J(x))^t \quad \textit{for a.e.} \quad x \in R^n, \tag{7.2}$$

for all $0 \leqq t \leqq 1$. *If, in addition, J is continuous and*

$$\int_0^a \frac{1}{r} s(r)dr < \infty, \quad \textit{for some} \quad 0 < a < \infty,$$

with $s(r)=\sup\{|J(x)-J(y)|\colon |x-y|\leqq r\}$, *then* φ_t *is a diffeomorphism and* $J_{\varphi_t}(x) = (J(x))^t$ *for all* $x \in R^n$, $0 \leqq t \leqq 1$.

Proof. The theorem will be proved in three steps. For the first and second steps we suppose that $J \in C^\infty$, $J>0$ and that $\log J$ has a compact support.

Step 1. Let $0<\varepsilon<1$. We start by constructing $G\colon [0,1]\times R^n \to R^n$ and a unique semiflow ψ generated by G such that

$$G(s,x) = [\gamma * (\log J \circ \psi_{s-\varepsilon}^{-1})](x), \quad (s,x)\in[0,1]\times R^n. \tag{7.3}$$

Define $\psi_t=\mathrm{id}$ for $t\leqq 0$. We prove that (7.3) defines G and ψ uniquely. Let $h(s,x)=(\log J \circ \psi_{s-\varepsilon}^{-1})(x)$. We show that

$$\begin{cases} \|h_s\|_\infty \leqq \|\log J\|_\infty = a_1 \quad \text{and} \\ \|h_s\|_1 \leqq \|\log J\|_1 e^{a_1} \quad \text{for} \quad 0 \leqq s \leqq 1. \end{cases} \tag{7.4}$$

Clearly (7.4) is true for $0\leqq s\leqq\varepsilon$. Next assume that h_s, G_s and ψ_s are defined and (7.4) is true for $0\leqq s\leqq\varepsilon$, and that $h\colon [0,t]\times R^n\to R^n$ is measurable. By Theorem 6.10 the function $\psi_s\colon R^n\to R^n$ is a bi-absolutely continuous homeomorphism and $\mu_{\psi_s}\leqq\exp(a_1 s)$ for $0\leqq s\leqq t$. For $t\leqq s\leqq t+\varepsilon$ we get $\|h_s\|_\infty=\|\log J\|_\infty$ and

$$\|h_s\|_1 = \int_{R^n} |(\log J)\circ\psi_{s-\varepsilon}^{-1}|\,dx = \int_{R^n} |\log J|\,\mu_{\psi_{s-\varepsilon}}dx$$

$$\leqq \exp(a_1)\|\log J\|_1.$$

So (7.4) is true for $t\leqq s\leqq t+\varepsilon$. Define G_s by (7.3) for $t\leqq s\leqq t+\varepsilon$. Again Theorem 6.10 implies that G_s generates ψ_s for $0\leqq s\leqq t+\varepsilon$ in a unique way. This completes the contruction of G and ψ so that (7.3) holds for $0\leqq s\leqq 1$. Denote G, ψ by F^ε and φ^ε for $\varepsilon\in(0,1)$.

By (7.4) Theorems 4.3, 3.4 and 3.8 imply that $(\varphi^\varepsilon)_{0<\varepsilon<1}$ and $(\varkappa^\varepsilon)_{0<\varepsilon<1}$, $\varkappa^\varepsilon(t, x)=(\varphi_t^\varepsilon)^{-1}(x)$, are normal families. Choose subsequences such that $\varepsilon_i\to 0$ and

(7.5) $$\varphi^{\varepsilon_i} \to \varphi:[0, 1]\times R^n \to R^n \quad \text{and}$$
$$\varkappa^{\varepsilon_i} \to \varkappa:[0, 1]\times R^n \to R^n$$

locally uniformly as $i\to\infty$. We see that φ and $\varkappa$ are continuous, $\varphi_t^{-1}=\varkappa_t$, and therefore φ_t: $R^n\to R^n$ are homeomorphisms for $0\leqq t\leqq 1$. Define

(7.6) $$F(t, x) = [\gamma*(\log J\circ\varphi_t^{-1})](x), \quad (t, x)\in[0, 1]\times R^n.$$

Because $\log J\in C_0^\infty$, we see by (3.9) that there is a compact $C\subset R^n$ such that

(7.7) $$\operatorname{spt}[\log J\circ(\varphi_t^{\varepsilon_i})^{-1}]\subset C \quad \text{for all} \quad \varepsilon_i, \quad 0\leqq t\leqq 1.$$

Then also $\operatorname{spt}\log J\circ\varphi_t^{-1}\subset C$ and $\log J\circ\varphi_t^{-1}\in L^1\cap L^\infty$, $0\leqq t\leqq 1$. Due to (7.5) (7.7) and the representation

$$F_t^{\varepsilon_i}(x) = \left[\gamma*\left(\log J\circ(\varphi_{t-\varepsilon_i}^\varepsilon)^{-1}\right)\right](x)$$

we see that $F^{\varepsilon_i}(t, x)\to F(t, x)$ locally uniformly as $i\to\infty$. Therefore,

$$\varphi^{\varepsilon_i}(t, x) = x+\int_0^t F^{\varepsilon_i}\left(s, \varphi^{\varepsilon_i}(s, x)\right)ds \to x+\int_0^t F\left(s, \varphi(s, x)\right)ds \quad \text{as} \quad i\to\infty.$$

Also $\varphi^{\varepsilon_i}(t, x)\to\varphi(t, x)$ as $i\to\infty$, and thus we see that φ is generated by F.

Step 2. We want to show that the above F is continuous and has $DF_t(x)$ which is continuous in (t, x). If this is the case, then φ_t: $R^n\to R^n$ is a diffeomorphism, and by (6.2) and (7.6)

$$\left(\det D\varphi_t(x)\right)^{\cdot} = \operatorname{div} F_t\left(\varphi_t(x)\right)\det D\varphi_t = [\log J\circ\varphi_t^{-1}\left(\varphi_t(x)\right)]\det D\varphi_t = \log J(x)\det D\varphi_t$$

for all $0\leqq t\leqq 1$ with $\det D\varphi_0(x)=1$. Then integration yields

(7.8) $$\det D\varphi_t(x) = J_{\varphi_t}(x) = e^{t\log J(x)} = \left(J(x)\right)^t, \quad 0\leqq t\leqq 1.$$

The continuity of F is easy to see, and we prove only the existence and continuity of $DF_t(x)$. By Theorem 3.8 $\varphi_t^{-1}(x)$ is continuous in (t, x) and

(7.9) $$|\varphi_t^{-1}(x_1)-\varphi_t^{-1}(x_2)| \leqq c|x_1-x_2|^\alpha \quad \text{for} \quad |x_1-x_2|\leqq\beta,$$

where $c<\infty$, $\alpha, \beta>0$ depend only on $\|\log J\|_1$ and $\|\log J\|_\infty$. Write $F=(F_1, \dots, F_n)$, and the formulas (7.6) and (6.8) imply that $D_{x_i}F_j(t, x)=n^{-1}\delta_{ij}g_t(x)-(\Gamma_{ij}g_t)(x)$ with $g_t(x)=\left(\log J\circ\varphi_t^{-1}(x)\right)$. Thus we only need to show that $\Gamma_{ij}(g_t)(x)$ is continuous in (t, x). We have $\Gamma_{ij}g_t=\lim_{\varepsilon\to 0}T_\varepsilon(g_t)$ (limes in L^p, $1<p<\infty$) with

$$T_\varepsilon(g_t)(x) = \int_{|x-y|\geqq\varepsilon} D_i\gamma_j(x-y)g_t(y)dy, \quad x\in R^n.$$

We show that $T_\varepsilon(g_t)(x)$ is continuous in (t, x) and $T_\varepsilon(g_t)(x)$ converges uniformly in (t, x) as $\varepsilon\to 0$. Clearly this yields the continuity of $\Gamma_{ij}g_t(x)$ in (t, x).

Let $0<\varepsilon<\varrho<\infty$ and write

$$T_\varepsilon g_t(x) = \int_{\varepsilon\leq|x-y|\leq\varrho} D_i\gamma_j(x-y)g_t(y)\,dy + \int_{\varrho\leq|x-y|} D_i\gamma_j(x-y)g_t(y)\,dy.$$

It is obvious that the first integral above is continuous in (t, x). For the second one we get by (6.6) and Theorem 6.10

$$\int_{\varrho\leq|x-y|} |D_i\gamma_j(x-y)||g_t(y)|\,dy \leq \frac{c_{ij}}{\varrho^p}\int_{R^n} |\log J\circ\varphi_t^{-1}(y)|\,dy$$

$$= \frac{c_{ij}}{\varrho^n}\int_{R^n} |\log J(x)|\,\mu_{\varphi_t}(x)\,dx \leq c_{ij}e^{\|\log J\|_\infty}\|\log J\|_1\frac{1}{\varrho^n} \to 0$$

as $\varrho\to\infty$ and uniformly in (t, x). This proves the continuity of $(T_\varepsilon g_t)(x)$ in (t, x). We still have to show that $|T_{\varepsilon_1}g_t(x)-T_{\varepsilon_2}g_t(x)|=\left|\int_{\varepsilon_1\leq|x|\leq\varepsilon_2} D_i\gamma_j(x-y)g_t(y)\,dy\right|\to 0$ as $0<\varepsilon_1\leq\varepsilon_2\to 0$ uniformly in (t, x). Because $\log J\in C_0^\infty$,

$$\int_0^a \frac{1}{r}s(r)\,dr<\infty, \quad \text{for all} \quad 0<a<\infty, \tag{7.10}$$

with $s(r)=\sup\{|\log J(x)-\log J(y)|\colon |x-y|\leq r\}$. We get by (6.6)

$$\left|\int_{\varepsilon_1\leq|x-y|\leq\varepsilon_2} D_i\gamma_j(x-y)g_t(y)\,dy\right| = \left|\int_{\varepsilon_1}^{\varepsilon_2}\Big(\int_{S^{n-1}} D_i\gamma_j(ry)g_t(x-ry)\,dy\Big)r^{n-1}\,dr\right|$$

$$= \left|\int_{\varepsilon_1}^{\varepsilon_2}\Big(\int_{S^{n-1}} D_i\gamma_j(y)[g_t(x-ry)-g_t(x)]\,dy\Big)r^{-1}\,dr\right|$$

$$\leq \int_{\varepsilon_1}^{\varepsilon_2}\int_{S^{n-1}} \frac{c_{ij}}{r}\left|(\log J)(\varphi_t^{-1}(x-ry))-(\log J)(\varphi_t^{-1}(x))\right|\,dy\,dr$$

$$\leq c_{ij}\omega_{n-1}\int_{\varepsilon_1}^{\varepsilon_2}\frac{1}{r}s(cr^\alpha)\,dr = c_{ij}\omega_{n-1}\alpha^{-1}\int_{c\varepsilon_1^\alpha}^{c\varepsilon_2^\alpha}\frac{1}{u}s(u)\,du \to 0$$

as $0<\varepsilon_1\leq\varepsilon_2\to 0$ due to (7.10). This completes the proof of Step 2.

Step 3. Here we deal with the general J satisfying the assumptions of the theorem. Because $\log J\in L^1\cap L^\infty$ we have $J-1\in L^1\cap L^\infty$ (because $|\log J(x)|\geq e^{-\|\log J\|_\infty}|J(x)-1|$ a.e.). For any $\varepsilon>0$ define

$$J^\varepsilon = 1+[(J-1)\chi_\varepsilon]*\theta_\varepsilon,$$

where χ_ε is the characteristic function of $B^n(1/\varepsilon)$ and θ_ε is a mollifier in R^n as in (5.4). Then $J^\varepsilon>0$ and

(7.11) 1° $\log J^\varepsilon\in C_0^\infty(R^n)$,

2° $\|\log J^\varepsilon\|_\infty\leq\|\log J\|_\infty$, $\|\log J^\varepsilon\|_1\leq e^{2\|\log J\|_\infty}\|\log J\|_1$,

3° $J^\varepsilon\to J$ a.e. as $\varepsilon\to 0$.

Then by Steps 1 and 2 there is F^ε which generates φ^ε such that

(7.12) $$F^\varepsilon(t,x) = [\gamma * (\log J^\varepsilon \circ (\varphi_t^\varepsilon)^{-1})](x)$$

and $\varphi_t^\varepsilon\colon R^n \to R^n$ is a homeomorphism with

(7.13) $$J_{\varphi_t^\varepsilon}(x) = (J^\varepsilon(x))^t, \quad x \in R^n, \quad 0 \leqq t \leqq 1.$$

This implies by 2° of (7.11)

(7.14) $$\|\log J^\varepsilon \circ (\varphi_t^\varepsilon)^{-1}\|_1 \leqq \exp(3\|\log J\|_\infty)\|\log J\|_1 \quad \text{and}$$
$$\|\log J^\varepsilon \circ (\varphi_t^\varepsilon)^{-1}\|_\infty \leqq \|\log J\|_\infty .$$

By (7.12), (7.14) and Theorems 3.4, 3.8 and 4.3 we see that $(\varphi_t^\varepsilon)_{\varepsilon>0}$ and $((\varphi_t^\varepsilon)^{-1})_{\varepsilon>0}$ form normal families and we can choose subsequences such that $\varepsilon_i \to 0$ and

(7.15) $$\varphi^{\varepsilon_i}(t,x) \to \varphi(t,x) \quad \text{and} \quad (\varphi_t^{\varepsilon_i})^{-1}(x) \to \varkappa(t,x)$$

locally uniformly in $[0,1]\times R^n$ as $i\to\infty$. Therefore, φ and $\varkappa$ are continuous, $\varphi_t\colon R^n\to R^n$ is a homeomorphism and $\varphi_t^{-1}(x)=\varkappa(t,x)$. Furthermore, by parts 2° and 3° of (7.11), (7.13) and Lemma 6.12 we see that φ_t is absolutely continuous and $\mu_{\varphi_t}(x)=(J(x))^t$ for a.e. $x\in R^n$ and all $0\leqq t\leqq 1$. This proves (7.2). Define

$$F(t,x) = (\gamma * g_t)(x)$$

with $g(t,x)=(\log J\circ\varphi_t^{-1})(x)$, $(t,x)\in[0,1]\times R^n$. We want to show that φ is generated by F and g satisfies the claims of the theorem.

We first show that for any $(t,x)\in[0,1]\times R^n$

(7.16) $$F(t,x) = \lim_{i\to\infty} F^{\varepsilon_i}(t,x).$$

Let $0<r_1<r_2<\infty$ and $B_i=B^n(x,r_i)$, $i=1,2$. Write $F^{\varepsilon_i}=(F_1^{\varepsilon_i},\dots,F_n^{\varepsilon_i})$ and get

$$F_k^{\varepsilon_i}(t,x) = \int_{R^n}\gamma_k(x-y)[\log J^{\varepsilon_i}\circ(\varphi_t^{\varepsilon_i})^{-1}](y)\,dy = \Big(\int_{B_1}\Big)+\Big(\int_{B_2\setminus B_1}\Big)+\Big(\int_{R^n\setminus B_2}\Big).$$

It is not difficult to see that the integrals $\int_{B_1}$ and $\int_{R^n\setminus B_2}$ tend to 0 uniformly in ε_i as $r_1\to 0$ and $r_2\to\infty$. For the middle integral we get

(7.17) $$I_i = \int_{B_2\setminus B_1} = \int_{(\varphi_t^{\varepsilon_i})^{-1}(B_2\setminus B_1)} \gamma_k(x-\varphi_t^{\varepsilon_i}(y))[\log J^{\varepsilon_i}(y)](J^{\varepsilon_i}(y))^t\,dy.$$

Note that in the integral $|x-\varphi_t^{\varepsilon_i}(y)|\geqq r_1$. By (3.9) we see that $(\varphi_t^{\varepsilon_i})^{-1}(B_2\setminus B_1)$ is contained in a compact set C which is independent of i. On the other hand, for a fixed r_1, the integrand on the right-hand side of (7.17) is uniformly bounded and tends to $\gamma(x-\varphi_t(y))[\log J(y)]J(y)^t$ for a.e. $y\in R^n$ as $i\to\infty$. Then Lebesgue's dominated convergence theorem yields

$$\lim_{i\to\infty} I_i = \int_{\varphi_t^{-1}(B_2\setminus B_1)} \gamma_k(x-\varphi_t(y))(\log J(y))(J(y))^t dy = \int_{B_2\setminus B_1}\gamma_k(x-y)(\log J\circ\varphi_t^{-1})(y)\,dy.$$

Furthermore, the last integral above tends to $F_k(t, x)$ as $r_1 \to 0$ and $r_2 \to \infty$. This proves (7.16).

It follows that F is measurable, and then also $g(t, x) = \log J(\varphi_t^{-1}(x)) = \operatorname{div} F_t(x)$ is measurable in $[0, 1] \times R^n$. Because $\|\log J \circ \varphi_t^{-1}\|_1 \leqq \|J^t\|_\infty \|\log J\|_1 \leqq \exp(\|\log J\|_\infty)\|\log J\|_1 < \infty$, $g \in L^1([0, 1] \times R^n)$. Also $\|g_t\|_\infty \leqq \|\log J\|_\infty$ because φ_t is absolutely continuous. Next we show that φ is generated by F. We have

$$\varphi^{\varepsilon_i}(t, x) = x + \int_0^t F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x))\, ds,$$

where we write $F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x)) = F^{\varepsilon_i}(s, \varphi(s, x)) + F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x)) - F^{\varepsilon_i}(s, \varphi(s, x))$. Here $F^{\varepsilon_i}(s, \varphi(s, x) \to F(s, \varphi(s, x))$ boundedly as $i \to \infty$, and Theorem 4.3 with (7.14) implies that $|F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x)) - F^{\varepsilon_i}(s, \varphi(s, x))| \leqq \exp(3a_2)(a_1 + a_2)h[\varphi^{\varepsilon_i}(s, x) - \varphi(s, x)] \to 0$ uniformly in s as $i \to \infty$, where $a_1 = \|\log J\|_1$ and $a_2 = \|\log J\|_\infty$ and $h(u) = u(1 + \log^+ u^{-1})$, $u > 0$. Therefore, we obtain

$$\varphi(t, x) = \lim_{i \to \infty} \varphi^{\varepsilon_i}(t, x) = \lim_{i \to \infty} \Big[x + \int_0^t F^{\varepsilon_i}(s, \varphi^{\varepsilon_i}(s, x))\, ds\Big] = x + \int_0^t F(s, \varphi(s, x))\, ds$$

for all $0 \leqq t \leqq 1$, $x \in R^n$. Thus φ is generated by F.

Finally, we suppose that J satisfies the regularity condition $\int_0^a (1/r)s(r)dr < \infty$ for some $0 < a < \infty$. Because $|\log J(x+y) - \log J(x)| \leqq \exp(\|\log J\|_\infty |J(x+y) - J(x)|$, we can prove that $\varphi_t\colon R^n \to R^n$ is a diffeomorphism and $J_{\varphi_t}(x) = (J(x))^t$ precisely the same way as in Step 2. This completes the proof of the entire theorem.

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University of Helsinki
Department of Mathematics
SF-0010 Helsinki 10
Finland

University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48109
USA

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