Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 7, 1982, 291—300

# REGULAR $n$-GONS AND FUCHSIAN GROUPS

MARJATTA NÄÄTÄNEN

## 1. Introduction

In [2] A. F. Beardon found the greatest lower bound for the radius of a hyperbolic disc inscribed in a hyperbolic triangle of a given area. Here we find the corresponding upper bound for a convex $n$-gon $P$, $n \geqq 3$. We also consider the greatest lower bound for the radius of a closed disc containing a convex $n$-gon $P$ of a given area. Both are attained when $P$ is regular, i.e., the sides are of equal length and the angles are equal.

We apply the results for Fuchsian groups of signature (2,0), and calculate in Theorem 5.1 the minimal trace in the group with the regular octagon with diametrally opposite pairings of sides as a fundamental domain.

The formulas for hyperbolic geometry used in this paper can be found in Chapter 6 of [1]. The hyperbolic metric is denoted by $\varrho$, the hyperbolic area of an $n$-gon $P$ by $|P|$.

If $G$ is a Fuchsian group of signature (2,0), then $G$ has only hyperbolic elements, and we denote by $D(z)$ its Dirichlet region with center $z$. For $g \in G$, we denote the trace by $\tau(g)$, the transformation length by $T_g$, and the axis of $g$ by $a_g$.

The author is grateful to Dr. A. F. Beardon and Professor Troels Jørgensen for heplful discussions.

## 2. Octagonal Dirichlet regions

Let $D$ be a convex octagonal Dirichlet region with center 0 for a group $G$ with signature (2,0). Then $G$ has only hyperbolic elements and the genus is 2; hence $|D| = 4\pi$. By Euler's formula all vertices are equivalent and hence they are at equal distance $R$ from 0. We claim that

$$\cosh R \geqq (1+\sqrt{2})^2$$

with equality if and only if $D$ is regular. For this we use the following two lemmas.

Lemma 2.1. *Let the length $c$ of the hypotenuse of a right-angled triangle $T$ be given. Then the maximal area of $T$ is attained when $T$ is isosceles. Then the sides $a$, $b$*

doi:10.5186/aasfm.1982.0724

*and the angles* $\alpha, \beta$ *of* $T$ *fulfil*

$$\cosh a = \cosh b = (\cosh c)^{1/2} = \cot \alpha = \cot \beta$$

*and the maximal area* $A(c)$ *is*

$$A(c) = \frac{\pi}{2} - \overline{\operatorname{arc}} \cos ((\cosh c - 1)/\sinh c)^2.$$

$A(c)$ *is strictly increasing and for* $\cosh c = (1+\sqrt{2})^2$, *we get* $\alpha = \beta = \pi/8$ *and* $A(c) = \pi/4$.

*Proof.* Let the angles of $T$ be $\pi/2, \alpha, \beta$. By using the formulas $\tanh c \cos \beta = \tanh a$, $\sinh c \sin \alpha = \sinh a$, $\cos \alpha = \cosh a \sin \beta$ we obtain after a simple calculation

$$\cos^2 (\alpha+\beta) = (\cosh c - 1)^2 \tanh^2 a (\cosh^2 c - \cosh^2 a)/\sinh^4 c.$$

Differentiating the right-hand side with respect to $a$ we see that, for $c$ fixed, $\cos (\alpha+\beta)$ attains its maximum if and only if $\cosh a = (\cosh c)^{1/2}$. Since in a right-angled triangle $\cosh c = \cosh a \cosh b = \cot \alpha \cot \beta$, this means that $T$ is isosceles and $\cot \alpha = (\cosh c)^{1/2}$. Then

$$\cos (\alpha+\beta) = ((\cosh c - 1)/\sinh c)^2,$$

and we obtain the claimed formula for $A(c)$. Also, as $c \to 0$, $A(c) \to 0$ and $\alpha, \beta \to \pi/4$; as $c \to \infty$, $A(c) \to \pi/2$ and $\alpha, \beta \to 0$; and $A(c)$ is strictly increasing.

Lemma 2.2. *Let $D$ be a convex octagon with all vertices at equal distance $R$ from* 0. *Denote* $R_0 = \cosh^{-1} (1+\sqrt{2})^2$. *Then if* $R < R_0$, $|D| < 4\pi$, *and if* $R = R_0$, $|D| \leqq 4\pi$, *with equality if and only if $D$ is regular.*

*Proof.* We triangulate $D$ into 16 right-angled triangles with one vertex at 0 and hypotenuse $R$. By Lemma 2.1, $A(R_0) = \pi/4$, and if $R < R_0$, each triangle has area less than $\pi/4$; hence $|D| < 4\pi$.

For $R = R_0$ each triangle has area at most $\pi/4$. Hence $|D| \leqq 4\pi$, with equality if and only if each triangle has area $\pi/4$, i.e., has angles $\alpha = \beta = \pi/8$. Then the sum of the angles at 0 is $2\pi$ and $D$ is regular with each angle $\pi/4$, the circumscribed circle has radius $R_0$, and the inscribed circle has radius $r$, $\cosh r = 1+\sqrt{2}$.

Hence we have

Theorem 2.1. *Let $D$ be a convex octagon with* $|D| = 4\pi$ *and with all vertices of $D$ at equal distance from* 0. *Then the smallest disc containing $D$ has radius* $\cosh^{-1} (1+\sqrt{2})^2$, *attained when $D$ is regular.*

Remark. It follows that no Fuchsian group of signature (2,0) can have a convex octagonal Dirichlet region included in a closed disc with radius $R < \cosh^{-1} (1+\sqrt{2})^2$, and for $R = \cosh^{-1} (1+\sqrt{2})^2$, the only occurring octagon is the regular one.

Remark. From Lemma 2.1 we also obtain that if $P$ is an octagon with all vertices on the circle with center 0, radius $\cosh^{-1}(1+\sqrt{2})^2$, and if $P$ is triangulated as in Lemma 2.2, then $|P|$ attains its maximum if and only if each triangle attains its maximum area. This corresponds to the regular case.

## 3. Smallest disc containing a convex $n$-gon with prescribed area

The results of Chapter 2 can be done generally:

Lemma 3.1. *Let $P$ be a convex $n$-gon, $n \geqq 3$, with area $A$. If $P$ has a circumscribed circle $C$ and the center $O \in P \setminus \partial P$, then the radius $R$ of $C$ attains its minimum value $R(A)$ if and only if $P$ is regular,*

$$\cosh R(A) = \cot\frac{\pi}{n}\cot\left(((n-2)\pi - A)/2n\right). \tag{3.1}$$

*Hence $R(A)$ is a strictly increasing function of $A$ and vice versa.*

*Proof.* We do the proof for $n=3$, since the cases $n>3$ are treated similarly with only a larger number of parameters.

We triangulate $P$ into three pairs of right-angled triangles with angles $\alpha_i$ at $O$, $\theta_i$ at the vertices of $P$, and $\pi/2$ at the midpoints of the sides of $P$. Then

$$\cosh R = \cot\alpha_i \cot\theta_i, \quad i = 1, 2, 3$$

$$\sum_{i=1}^{3}\alpha_i = \pi, \quad A = \pi - 2\sum_{i=1}^{3}\theta_i.$$

Hence

$$\pi = \sum_{i=1}^{3}\cot^{-1}(\cosh R \tan\theta_i), \tag{3.2}$$

where the angles are subject to the constraints

$$\sum_{i=1}^{3}\theta_i = (\pi - A)/2 < \pi/2, \quad \theta_i \geqq 0, \quad i = 1, 2, 3. \tag{3.3}$$

The equation (3.2) determines $R$ uniquely as a function of $(\theta_1, \theta_2, \theta_3)$, subject to (3.3), and the problem is to minimize $R$ over the triangle $\Delta$ in $\boldsymbol{R}^3$ with vertices $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$, $a = (\pi - A)/2$.

We denote $\sigma = \cosh R$ and compute the minimum value of $\sigma$ on $\Delta$. We consider a horizontal section

$$\theta_3 = a - 2c,\ \theta_2 = c - t,\ \theta_1 = c + t,\ -c \leqq t \leqq c$$

of $\Delta$. On this segment $\sigma$ is a function of $t$, and differentiation of each side of (3.2) yields

$$\sigma'(t)\sum_{i=1}^{3}\tan\theta_i(1+\sigma^2\tan^2\theta_i)^{-1} = \sigma\left[(\sin^2\theta_2(\sigma^2-1)+1)^{-1}-(\sin^2\theta_1(\sigma^2-1)+1)^{-1}\right].$$

Hence $\sigma'(t)>0$ if $\theta_1>\theta_2$, or equivalently, $t>0$; and $\sigma'(t)<0$ if $t<0$. Thus $\sigma$ attains its minimum value on this segment when $\theta_1=\theta_2$. Hence we can consider the original problem in the intersection of $\Delta$ with the plane $\theta_1=\theta_2$, which means one parameter less. In the case $n=3$, $\sigma$ and $R$ are now functions of one variable, and by a similar differentiation we see that they obtain the minimum value for $\theta_1=\theta_2=\theta_3=(\pi-A)/6$.

In the case $n>3$ we divide $P$ with rays from $O$ into $2n$ right-angled triangles. Since in the minimal case $P$ is regular, the angle $\alpha$ at $O$ is $\pi/n$ and the other angle $\theta$ satisfies the equation

$$\theta = ((n-2)\pi-A)/2n$$

because of the area condition. The formula $\cosh R=\cot\alpha\cot\theta$ gives the result.

Theorem 3.1. *Let $P$ be a convex $n$-gon of a given area $A$. The smallest closed disc $C$ containing $P$ is obtained when $P$ is regular. The radius $R$ fulfils* (3.1).

*Proof.* If the center $O$ of $C$ is in $P\setminus\partial P$, and if all vertices of $P$ are on $\partial C$, the result follows from Lemma 3.1.

If $O\notin P\setminus\partial P$, there exists a side $s$ of $P$ such that one of the two half-planes with $s$ on its boundary contains $P$ but not $O$. By using the half-plane model with the continuation of $s$ as a vertical line we see that there exists a disc $C'$ with radius $R'\leqq R$ such that $s$ is on the vertical diagonal of $C'$. Let the center of $C'$ be $O'$ and let the vertices of $s$ be $A_1, A_2$, $\varrho(O', A_1)\geqq\varrho(O', A_2)$. Continue $s$ through $A_2$, and the adjacent side with vertex $A_1$ through $A_1$, until they hit $\partial C'$, say at the points $A_2', A_1'$. Draw rays from the midpoint of $s$ through the other vertices of $P$ and denote the points where they hit $\partial C'$ by $A_3', \ldots, A_n'$. Let $P'$ be the polygon with vertices $A_1', \ldots, A_n'$. Then $|P|<|P'|$, all vertices of $P'$ are on $\partial C'$, $O'\in P'\setminus\partial P'$. By (3.1),

$$R' \geqq R(|P'|) \geqq R(|P|).$$

It also follows:

Lemma 3.2. *If $G$ is a Fuchsian group with signature* (2,0), *then the only fundamental polygon which is a regular $n$-gon and can be divided into $2n$ isosceles right-angled triangles is the regular octagon.*

*Proof.* Since the area is $4\pi$, $n=8$ gives the only solution for equal angles in (3.1).

Lemma 3.3. *For given $R>0$ and $n\in N$, $n\geqq3$, the maximum area for a convex $n$-gon $P$ included in a closed disc with radius $R$ is attained when $P$ is regular.*

*Proof.* Let $R>0$ be given, and let $P$ be a convex regular $n$-gon with circumscribed circle with radius $R$. Then $|P|$ and $R$ are connected by (3.1). If $P'$ is a convex $n$-gon, which is not regular, and $|P'|\geqq|P|$, then by previous lemmas $P'$ is not included in any closed disc with radius $R$.

There is a simple application to "covering":

Theorem 3.2. *Let G be a Fuchsian group of signature* (2.0). *The translates under G of the closed disc* $\bar{B}(z, r)$ *cover the hyperbolic plane if and only if* $r\geqq R(G, z)$, *where* $R(G, z)=\max_p \varrho(z, P)$ *and P is a vertex of* $D(z)$.

*Proof.* It follows from the definition of $R(G, z)$ that the closed disc $\bar{B}(z, R(G, z))$ covers $D(z)$. The image of this disc under $g\in G$ is the closed disc $\bar{B}(g(z), R(G, z))$, which covers $D(g(z))$.

It remains to be shown that $R(G, z)$ is the smallest radius with the covering property. It follows from the definition of $D(z)$ that if $P$ is a vertex of $D(z)$ and $g(P)$ is in the cycle of $P$, then

$$\varrho(P, z) = \varrho(P, g^{-1}(z)) \leqq \varrho(P, f(z))$$

for all $f\in G\setminus I$. Hence the radius of a closed disc with center in the orbit of $z$ has to be at least $R(G, z)$ in order to cover the vertex with maximal distance from $z$.

Theorem 3.3. *Let G be a Fuchsian group of signature* (2,0). *The group which minimizes* $R(G, z)$ *of Theorem* 3.2 *is a group with* $D(z)$ *for some z the regular 18-gon with any of the eight possible identification patterns.*

*Proof.* Theorem 3.1 and the fact that $R(4\pi)$ of (3.1) is a decreasing function of $n$ when $8\leqq n\leqq 18$, give the result, and for the minimal radius $R$ the formula

$$\cosh R = \frac{1}{\sqrt{3}} \cot\frac{\pi}{18}.$$

There are eight possible identification patterns for $D(z)$ ([3]).

## 4. Convex $n$-gons containig a maximal disc

Lemma 4.1. *Let P be an octagon, which is a fundamental domain for a Fuchsian group with signature* (2,0). *Suppose that P has an inscribed disc C with radius r. Then* $\cosh r\leqq 1+\sqrt{2}$ *and equality corresponds to P being regular.*

*Proof.* We first consider the case when $C$ touches each side in its midpoint. Since the sides of $P$ are congruent in pairs, we can divide $P$ into 16 right-angled triangles, each congruent with 3 others. Hence we can choose as parameters the angles $\alpha_i$

at the center of $C$ and the remaining angles $\theta_i, i=1, 2, 3, 4$. Since $|P|=4\pi$,

$$\sum_{i=1}^{4} \theta_i = \frac{\pi}{2}.$$

Also

$$\sum_{i=1}^{4} \alpha_i = \frac{\pi}{2}.$$

By trigonometry, $\cosh r \sin \alpha_i = \cos \theta_i$. Hence

$$\sum_{i=1}^{4} \sin^{-1} \frac{\cos \theta_i}{\cosh r} = \frac{\pi}{2} \tag{4.1}$$

determines $r$ as a function of $(\theta_1, \theta_2, \theta_3, \theta_4)$ subject to the constraints

$$\theta_i \geqq 0, \ \sum_{r=1}^{4} \theta_i = \frac{\pi}{2}.$$

We want to maximize $r$ over this subset of $\boldsymbol{R}^4$. Let $c \in [0, \pi/2]$ and consider a section

$$\theta_4 = \frac{\pi}{2} - c, \ \theta_3 = c - 2k, \ \theta_2 = k + t, \ \theta_1 = k - t, \ 2k \in [0, c], \ |t| \leqq k.$$

We denote $\sigma = (\cosh r)^{-1}$. On this segment $\sigma$ is a function of $t$ and differentiation of each side of (4.1) yields

$$\sigma'(t) \sum_{i=1}^{4} \frac{\cos \theta_i}{(1-\sigma^2 \cos^2 \theta_i)^{1/2}} = \frac{\sigma \sin \theta_2}{(1-\sigma^2 \cos^2 \theta_2)^{1/2}} - \frac{\sigma \sin \theta_1}{(1-\sigma^2 \cos^2 \theta_1)^{1/2}}.$$

Hence $\sigma'(t) > 0$ if $\theta_2 > \theta_1$ or equivalently, $t > 0$; $\sigma'(t) < 0$ if $t < 0$, and $\sigma$ attains its minimum value on the segment when $\theta_1 = \theta_2$.

Next we assume that

$$\sum_{i=1}^{4} \theta_i = \frac{\pi}{2}, \ \theta_1 = \theta_2 = \left(\frac{\pi}{2} - 2k\right)/2, \ \theta_3 = k - t, \ \theta_4 = k + t, \ |t| \leqq k.$$

On this segment, as above, $\sigma$ attains its minimum value when $\theta_3 = \theta_4$. Hence we can assume $\theta_1 = \theta_2$, $\theta_3 = \theta_4$, and we can examine $\sigma$ as a function of $\theta_1$ and $\theta_3 = \pi/4 - \theta_1$. A derivation like the one above yields that $\sigma$ attains its minimum value when $\theta_i = \pi/8$, $i = 1, 2, 3, 4$ and hence $P$ is regular. The formula for $r$ becomes

$$\cosh r \sin \frac{\pi}{8} = \cos \frac{\pi}{8}$$

and hence $\cosh r = 1 + \sqrt{2}$.

The assumption of the inscribed disc touching $P$ at the midpoints is irrelevant — by increasing the number of parameters we can do a similar proof without it.

Remark. A similar proof shows that in the set of convex $n$-gons with given area $A$, and having an insribed disc, the largest disc is obtained for the regular $n$-gon. The radius $r$ of the largest disc fulfils

$$\cosh r = \frac{\cos(((n-2)\pi - A)/2n)}{\sin \pi/n}. \tag{4.2}$$

Hence, for $n$ fixed, $r$ is strictly increasing as a function of $A$. We denote the maximal radius by $r=r(A)$.

Theorem 4.1. *Let $P$ be a convex $n$-gon with given area $A$. Then if $P$ is regular, it contains the largest disc, and the radius $r$ fulfils* (4.2).

*Proof.* The result follows from the previous remark if the disc $C$ touches all sides of $P$. If all sides of $P$ do not touch $C$, we can find a convex $n$-gon, $P'$, $P' \subset P$, $|P'|<|P|$, with all sides of $P'$ touching $C$. Then, with the notation in the remark, if $r$ is the radius of $C, r \leqq r(|P'|) < r(|P|)$.

Remark. It follows that if $P$ is a convex octagon which is a fundamental domain for a Fuchsian group with signature (2,0), and if $P$ contains a disc with radius $r$, then

$$\cosh r \leqq 1+\sqrt{2} \tag{4.3}$$

with equality when $P$ is regular.

Remark. We can now examine the ratio $\cosh R/\cosh r$ when $P$ is a convex $n$-gon, $n \geqq 3$, of given area $A$, $0<A<(n-2)\pi$, $R$ and $r$ are, respectively, the radii of closed discs $C_1, C_2, C_2 \subset P \subset C_1$.

Due to Theorems 3.1 and 4.1 we obtain that

$$\min_P \cosh R/\cosh r$$

is attained when $P$ is regular. Then, with the notation used,

$$\cosh R/\cosh r = \cos \alpha/\sin \theta,$$

where $\alpha=\pi/n$, $\theta=((n-2)\pi-A)/2n$.

For $A$ given, we denote $\delta(n)=\cosh R/\cosh r$. Then $\delta(n)$ is strictly decreasing.

Lemma 4.2. *Let $P$ be a convex $n$-gon, which is a fundamental domain of a Fuchsian group of signature* (2,0). *Then with the notation above*

$$\max_n \min_P \cosh R/\cosh r = 1+\sqrt{2}.$$

*Proof.* Since the minimum number of sides for $P$ is eight, and $\delta(n)$ is decreasing, the maximum is attained when $P$ is the regular octagon.

There is an application to "packing":

Theorem 4.2. *Let $G$ be a Fuchsian group with signature* (2,0). *The translates under $G$ of the open disc $B(z, r)$ do not overlap if and only if $r \leqq r(G, z)$, where* $r(G, z) = (1/2) \min \{\varrho(z, g(z)) | g \in G \setminus I\}$.

*Proof.* It follows from the definitions of $r(G, z)$ and $D(z)$ that $B(z, r(G, z)) \subset D(z)$. Hence the images under $G$ of $B(z, r(G, z))$ are distinct, since

$$g(B(z, r(G, z))) = B(g(z), r(G, z)) \subset D(g(z)).$$

Since $r(G, z) = \varrho(z, g(z))/2$, where $g(z) \neq z$ is the point closest to $z$ in the orbit of $z$, the images of $B(z, r)$ are not disjoint if $r > r(G, z)$.

Theorem 4.3. *Let $G$ be a Fuchsian group with signature* (2,0). *The group which maximizes $r(G, z)$ of Theorem* 4.2 *is a group with $D(z)$ for some $z$ the regular* 18-*gon with any of the eight identification patterns.*

*Proof.* Theorem 4.1 and the fact that $r$ of (4.2) is increasing as a function of $n$ give the result and for the maximal radius the formula

$$\cosh r = \left(2 \sin \frac{\pi}{18}\right)^{-1}.$$

## 5. The group of the regular octagon

Theorem 5.1. *There exists a group $G$ with signature* (2,0) *and* $\min \{|\tau(g)| \,|\, g \in G \setminus I\} = 2(1+\sqrt{2})$.

*Proof.* We derive generators and relations between them for a group $G$ with $D(0)$ a regular octagon with diametrically opposite pairings. Then $\partial D(0)$ consists of arcs of eight circles of equal size, intersecting at angles $\pi/4$.

Let two of the circles have centers on the real axis and let $f_1$ pair the corresponding sides $s, s'$ of $\partial D(0)$. Then $f_1(\boldsymbol{R}) = \boldsymbol{R}$ and hence we can assume

$$f_1(z) = \frac{az+c}{cz+a} \quad a^2 - c^2 = 1, \; a, c \in \boldsymbol{R}.$$

Then the triangle with vertices 0, an edpoint of $s$ and $\boldsymbol{R} \cap s$ has angles $\pi/8$, $\pi/8$, $\pi/2$ and the distance from 0 to $\boldsymbol{R} \cap s$ is $T_{f_1}/2$.

By (4.3)

$$\tau(f_1) = 2 \cosh \frac{1}{2} T_{f_1} = 2(1+\sqrt{2}) > 4.8. \tag{5.1}$$

Hence $a = 1+\sqrt{2}$, $c = -\sqrt{2}(1+\sqrt{2})^{1/2}$.

Let $g(z) = (\exp(i5\pi/4))z$. Then $f_{k+1} = g^{-k} \circ f_1 \circ g^k$, $k = 1, 2, 3$, pair the remaining diagonally opposite sides of $\partial D(0)$ and $\tau(f_k) = \tau(f_1)$, $k = 2, 3, 4$. The cycle of the vertex gives the relation $f_1 f_2 f_3 f_4 f_1^{-1} f_2^{-1} f_3^{-1} f_4^{-1} = I$. By Euler's formula the genus is two.

Next we show that the traces of the generators are minimal in the set $\{|\tau(g)| \| g \in G \setminus I\}$. By (5.1) we can consider transformationlengths, and such $g \in G \setminus I$ that $T_g < T_{f_1}$. We denote $2r = T_{f_1}$.

Since an image of $a_g$ has to meet $\overline{D(z)}$, where $z$ is a vertex of $D(0)$, we can assume $a_g \cap \overline{D(z)} \neq \emptyset$ by conjugation. By symmetry we can then consider the case when $D(z)$ is the regular octagon with sidelengths $2r$ and vertices the images of 0 under $I, f_4, f_4 f_3, f_4 f_3 f_2, f_4 f_3 f_2 f_1, f_1 f_2 f_3, f_1 f_2$ and $f_1$. Hence for some $h$ $\varrho(h(0), a_g) < r$ and by conjugation we can assume $\varrho(0, a_g) < r$. By hyperbolic geometry

$$\sinh \frac{1}{2} \varrho(0, g(0)) = \cosh \varrho(0, a_g) \sinh \frac{1}{2} T_g .$$

Together with the assumption $T_g < 2r$ it gives

$$\sinh \frac{1}{2} \varrho(0, g(0)) < \cosh r \sinh r = \sqrt{2}\,(1+\sqrt{2})^{3/2} < 5.23.$$

To finish we have to calculate $\tau(g)$ for such $g \in G \setminus I$ that $\varrho(0, g(0)) < 4.7$. By elementary calculations and symmetry it suffices to calculate $|\tau(f_1 f_2)|$ and $|\tau(f_1 f_2 f_3)|$. These are not smaller than $|\tau(f_1)|$.

Lemma 5.1. *If $G$ is a Fuchsian group with signature* (2,0) *and such that* D(0) *is the regular octagon, then the diametrically opposite pairings give the group where maximum of* $\min \{|\tau(g)| \| g \in G \setminus I\}$ *is attained, and is* $2(1+\sqrt{2})$.

*Proof.* Each midpoint of a side of $D(0)$ is mapped to a midpoint of a side. Hence we can consider the case $z = re^{i\varphi}$ with $\cosh r = 1+\sqrt{2}$ and $w = \bar{z}$ in the orbit of $z$. By hyperbolic geometry

$$\cosh^2 \frac{1}{2} \varrho(z, w) = \frac{|1 - z\bar{w}|^2}{(1-|z|^2)(1-|w|^2)} = \frac{|1 - r^2 e^{2i\varphi}|^2}{(1-r^2)^2},$$

which has its maximum when $\varphi = \pi/2$ or $\varphi = 3\pi/2$, corresponding to the diametrically opposite pairings. By (5.1) the maximum is $\cosh(\varrho(z, w)/2) = 1+\sqrt{2}$.

If all pairings are not diametrally opposite, there exists a mapping $g \in G \setminus I$ with $\cosh(T_g/2) < 1+\sqrt{2}$. Since

$$\cosh \frac{1}{2} T_g = \tau(g)/2,$$

$\tau(g) < 2(1+\sqrt{2})$. Theorem 5.1 now gives the result.

Theorem 5.2. *Let G be a Fuchsian group of signature* (2,0) *and let G have a Dirichlet region D(z) which is a convex octagon. Then*

$$\max_{G} \min \{|\tau(g)| \,|\, g \in G \setminus I\} = 2(1+\sqrt{2}),$$

*attained when G is the regular octagon group of Theorem* 5.1.

*Proof.* There is a pair of equivalent points on the maximal circle with center $z$ contained by the octagonal Dirichlet region. Hence the result follows from the Remark of Theorem 4.1 and from Lemma 5.1.

**References**

[1] BEARDON, A. F.: The geometry of discrete groups. - To appear.

[2] BEARDON, A. F.: Hyperbolic polygons and Fuchsian groups. - J. London Math. Soc. (2) 20, 1979, 247—254.

[3] JORGENSEN, T., and M. NÄÄTÄNEN: Surfaces of genus 2: generic fundamental polygons. - Quart. J. Math. Oxford Ser. (2) (to appear).

University of Helsinki
Department of Mathematics
SF—00100 Helsinki 10
Finland

Received 30 March 1982