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ON L^p-INTEGRABILITY IN PDE'S AND QUASIREGULAR MAPPINGS FOR LARGE EXPONENTS

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Introduction

We begin with presenting a local version of the celebrated lemma of F. W. Gehring invented with references to quasiconformal mappings [6].

Lemma 1 (Gehring). *Let Ω be a domain in $\boldsymbol{R}^n$, $q>1, f\in L^q(\Omega)$. Suppose that for each cube $Q\subset\Omega$ the inequality*

$$\left(\fint_Q |f|^q\right)^{1/q} \leqq M \fint_Q |f| \tag{0.1}$$

holds with a constant $M\geqq 1$ independent of Q. Hereafter $\fint_Q f$ stands for the average value of f, i.e., $\fint_Q f=f_Q=|Q|^{-1}\int_Q f$. Then there exist $p=p(n,q,M)>q$ and $C=C(n,q,M)\geqq 1$ such that

$$\left(\fint_Q |f|^p\right)^{1/p} \leqq C \fint_Q |f| \tag{0.2}$$

for each cube $Q\subset\Omega$.

We call (0.1) the inverse Hölder inequality. Following Gehring we notice that the derivatives of quasiconformal mappings satisfy (0.1) with $q=n$. In major cases of PDE's we only succeeded in proving weak forms of (0.1); this means that on each cube $Q\subset\Omega$ and for some $0<\sigma<1$, independent of Q, we have

$$\left(\fint_{\sigma Q} |f|^q\right)^{1/q} \leqq M \fint_Q |f|, \tag{0.3}$$

where σQ stands for a cube of the same centre as Q but contracted by the factor of σ. In general, M may depend on σ. In this case Gehring's lemma reads as follows:

This paper amplifies a fragment of the lecture by prof. B. Bojarski and T. Iwaniec, "Some new concepts in the analytical theory of QC-maps in $\boldsymbol{R}^n$, $n\geqq 3$, and differential geometry" [2], delivered at the Conference on Global Analysis, Garwitz, DDR, October, 1981. In a not far remote future we intend to give a more systematic and complete discussion of the topics.

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there exist $p=p(n, q, M, \sigma)>q$ and $C=C(n, q, M, \sigma)\geqq 1$ such that

$$\left(\fint_{\sigma Q} |f|^p\right)^{1/p} \leqq C \fint_Q |f|, \quad \text{for each cube} \quad Q\subset\Omega. \tag{0.4}$$

We call (0.3) the weak inverse Hölder inequality.

These two and a few other variants of Gehring's lemma were successfully adapted to PDE (see [4]). Here we shall discuss two typical examples which illustrate the practical use of the lemma. They will provoke us to further investigations.

Consider the divergence elliptic equation with measurable coefficients

$$\sum_{i,j=1}^{n} \frac{d}{dx_j}\left(a_{ij}(x)\frac{du}{dx_i}\right) = 0, \tag{0.5}$$

where $a_{ij}(x)=a_{ji}(x)$ satisfy the uniform ellipticity condition

$$\alpha\xi^2 \leqq \sum_{i,j} a_{ij}(x)\xi^i\xi^j \leqq \beta\xi^2, \quad 0<\alpha\leqq\beta<\infty \tag{0.6}$$

for almost all $x\in\Omega$ and all vectors $\xi\in \boldsymbol{R}^n$. We are looking for solutions u from $W_2^1(\Omega)$ for which (0.5) is understood in the sense of distributions, i.e.,

$$\int_\Omega \sum a_{ij}(x)\frac{du}{dx_i}\frac{d\varphi}{dx_j}\,dx = 0 \tag{0.7}$$

for any test function $\varphi\in \overset{0}{W}{}_2^1(\Omega)$ — the completion of $C_0^\infty(\Omega)$ in the space $W_2^1(\Omega)$. Inserting a suitable φ one can easily derive from (0.7) the Caccioppoli type estimate

$$\left(\fint_{\sigma\Omega} |\nabla u|^2\right)^{1/2} \leqq \frac{C(n,\alpha,\beta)}{(1-\sigma)|Q|^{1/n}}\left(\fint_Q |u-u_Q|^2\right)^{1/2}. \tag{0.8}$$

By Poincare's inequality, see Lemma 6 of § 4, we are led to

$$\left(\fint_{\sigma Q} |\nabla u|^2\right)^{1/2} \leqq \frac{C(n,\alpha,\beta)}{1-\sigma}\left(\fint_Q |\nabla u|^{(2n)/(n+2)}\right)^{(n+2)/(2n)}, \quad Q\subset\Omega, \tag{0.9}$$

so the hypothesis (0.3) of Gehring's lemma holds with $q=(n+2)/n$ and $f=|\nabla u|^{2n/(n+2)}$. Hence we conclude that $u\in W^1_{p',\mathrm{loc}}(\Omega)$ for some $p'>2$ and

$$\left(\fint_{\sigma Q} |\nabla u|^{p'}\right)^{1/p'} \leqq C(n,\alpha,\beta,\sigma,p')\left(\fint_Q |\nabla u|^2\right)^{1/2} \tag{0.10}$$

holds for each cube $Q\subset\Omega$.

The second example deals with quasiregular mappings.

Definition 1. *Assume that* $f\colon \Omega\to\boldsymbol{R}^n$ *is a mapping whose components* $f^1, f^2, \ldots, f^3$ *belong to* $W^1_{n,\mathrm{loc}}(\Omega)$. *Let* $Df=(df^i/dx_j)$ *denote the Jacobi matrix of* f *and let* $J(x,f)$ *be its determinant* (Jacobian). *Then* f *is said to be K-quasiregular*

($1 \leqq K < \infty$) *if and only if*

$$|Df(x)|^n \leqq n^{n/2} K J(x, f) \tag{0.11}$$

for almost all $x \in \Omega$.

We denoted by $|A|$ the norm of a matrix A, $|A|^2 = \sum |A_{ij}|^2 = \operatorname{Tr} A^* A$. The smallest K for which (0.11) holds will be called the analytical distortion of f. If $K=1$, the map f becomes a generalized conformal transformation (1-quasiregular). On account of Corollary 2 of § 5 we have

$$\Big(\int_{(1/2)Q} |Df(x)|^n\,dx\Big)^{1/n} \leqq C(n) K \Big(\int_{Q} |Df(x)|^{n/2}\,dx\Big)^{2/n} \tag{0.12}$$

for each cube $Q \subset \Omega$. Hence by Gehring's lemma we obtain the mentioned result of Gehring, that $f \in W^1_{p,\,\mathrm{loc}}(\Omega)$ for some $p > n$ and

$$\Big(\int_{(1/2)Q} |Df(x)|^p\,dx\Big)^{1/p} \leqq C(n, K, p) \Big(\int_{Q} |Df(x)|^n\,dx\Big)^{1/n}, \quad Q \subset \Omega. \tag{0.13}$$

In connection with the above examples there arises a natural question: How large can the exponent p be? The proper answer requires scrupulous analysis of (0.5) and (0.11). Let us remark that if the coefficients $a_{ij}(x)$ are smooth, the solution u is also smooth. In particular, $u \in W^1_{p,\,\mathrm{loc}}(\Omega)$ with p as large as one likes. In general we shall see that the closer to continuous functions the coefficients $a_{ij}(x)$ are, the larger p can be taken in (0.10). The idea of the proof will be exemplified by

Proposition 1. *Suppose that* $a_{ij}(x) = \delta_i^j + \varepsilon_{ij}(x)$, *where* δ_i^j *denotes Kronecker's symbol and* $\varepsilon_{ij}(x) = \varepsilon_{ji}(x)$ *are measurable functions small enough to satisfy*

$$\Big|\sum_{i,j} \varepsilon_{ij}(x) \xi^i \xi^j\Big| \leqq \varepsilon \xi^2, \quad \varepsilon < 1 \tag{0.14}$$

for almost all $x \in \Omega$ *and all* $\xi \in \boldsymbol{R}^n$. *Then there exists* $k = k(n) > 0$ *such that each solution* $u \in W^1_{2,\,\mathrm{loc}}(\Omega)$ *of* (0.7) *belongs to* $W^1_{p,\,\mathrm{loc}}(\Omega)$ *for every* $p \in (1, 2 + k \log(1/\varepsilon))$ *and*

$$\Big(\int_{(1/2)Q} |\nabla u|^p\Big)^{1/p} \leqq C(n, p) \Big(\int_{Q} |\nabla u|^2\Big)^{1/2} \tag{0.15}$$

holds for each cube $Q \subset \Omega$.

Commenting the role of the constant K in (0.11) we observe that for every $n \times n$-matrix A

$$|A|^n \geqq n^{n/2} \det A, \tag{0.16}$$

the equality holding for matrices proportional to orthogonal ones. Thus the Jacobi matrix $Df(x)$ of a 1-quasiregular mapping is (at almost every point $x \in \Omega$) a similarity transformation of $\boldsymbol{R}^n$. Therefore the concept of generalized conformal mapping is (up to smoothness conditions) much like the one familiar in the differential geometry, the more so because the Liouville theorem states that every 1-quasiregular mapping, if $n \geqq 3$, is either constant or a restriction to Ω of a Möbius transformation, i.e., the finite product of reflections with respect to spheres. For the simplest and very elementary proof see [3]. Generally the distortion K should be thought of as a distance measure between f and a Möbius transformation. That is why f is expected to possess good regularity properties when K is sufficiently close to 1. We shall prove

Proposition 2. *There exist* $p=p(n,K)>n$ *and* $C=C(n,K)\geqq 1$ *such that*

a) *every K-quasiregular mapping* $f\colon \Omega\to\boldsymbol{R}^n$ *belongs to* $W^1_{p,\,\mathrm{loc}}(\Omega)$,

b) *for any cube* $Q\subset\Omega$ *we have*

$$\left(\fint_{(1/2)Q}|Df|^p\right)^{1/n}\leqq C\left(\fint_{Q}|Df|^n\right)^{1/n}, \tag{0.17}$$

c) *the exponent* $p=p(n,K)$ *increases to infinity as* K *tends to* 1.

The principle of the proof will be the same as that of Proposition 1. However, the details will be quite different and perhaps interesting in their own right.

The problem of L^p-integrability $(p>n)$ of derivatives of quasiregular mappings was first solved by B. Bojarski [1] in two dimensional domains. Later Gehring and then N. Meyers and A. Elcrat [4] have extended it for arbitrary dimension. In 1976 Ju. G. Rešetnjak [9] examined the asymptotic behaviour of $p=p(n,K)$ in K close to 1. The exact value of $p=p(n,K)$ is unknown. Rešetnjak's method relies on deep arguments relating to quasiconformal theory, especially on a strong stability theorem, and it seems to be ineffective for PDE's.

In this paper we present a tool for dealing with problems of L^p-integrability in both quasiregular mappings as well as solutions of PDE's. The following lemma is the most important one in this paper.

Lemma 2. *Let* Ω *be an open subset of* $\boldsymbol{R}^n$, $f\in L^1_{\mathrm{loc}}(\Omega)$, $0<\sigma<1$, $1\leqq p<\infty$. *Suppose that for each cube* $Q\subset\Omega$ *the inequality*

$$\fint_{\sigma Q}|f(y)-f_{\sigma Q}|\,dy\leqq 10^{-6np}\fint_{Q}|f(y)|\,dy \tag{0.18}$$

holds. Then $f\in L^p_{\mathrm{loc}}(\Omega)$, *and for each cube* $Q\subsetneq\Omega$ *we have*

$$\left(\fint_{(1/2)Q}|f|^p\right)^{1/p}\leqq\frac{10^{8n^2p}}{\sigma^n}\fint_{Q}|f|. \tag{0.19}$$

For the case of $\sigma=1$ *see Corollary* 1 *of* § 2.

In other words, a function which can be approximated locally by constants with errors, small relative to its mean values, belongs to L^p with large p.

This extends somewhat the lemma of John-Nirenberg concerning the theory of BMO spaces. For Lemma 2 we were inspired by the inequality of Fefferman and Stein [5]. What we really need is a local version of their inequality; for the completeness of our arguments we will give a proof imitating the original one.

The proofs of Propositions 1 and 2 are divided into three stages. First we study the extremal cases, when $\varepsilon_{ij}(x)=0$ and $K=1$. We derive local estimates which in fact are consequences of the regularity of harmonic functions (Lemma 7) and of generalized conformal mappings (formula 4.5). Next we find that solutions of elliptic equations and quasiregular mappings behave stably with respect to coefficients (Lemma 8) and to distortion K (Lemma 10), respectively. And, finally, we complete the proofs on the basis of Lemma 2. This procedure works in many other cases not discussed here (see for instance [7]).

We are taking the opportunity here to show how this method works if Gehring's lemma is used. To this end we look at (0.1) again. For simplicity assume that $q=2$. The extremal case corresponds to $M=1$. Functions which satisfy the inverse Hölder inequality $(\fint_Q |f|^2)^{1/2} \leqq \fint_Q |f|$ must be constant in their absolute values. Since in general

$$\fint_Q \big||f|-|f|_Q\big| \leqq \Big(\fint_Q \big||f|-|f|_Q\big|^2\Big)^{1/2} = \Big(\fint_Q |f|^2-\Big(\fint_Q |f|\Big)^2\Big)^{1/2} \leqq \sqrt{M^2-1}\fint_Q |f|,$$

whence the stability property of (0.1). Now, Lemma 2 with $\sigma=1$ (see Corollary 1) implies that for each cube $Q\subset\Omega$

$$\Big(\fint_Q |f|^p\Big)^{1/p} \leqq 2\cdot 10^{n+1}\fint_Q |f|$$

whenever $1\leqq p< -(1/(10n))\log 4(M^2-1)$. In this way we are led to the following addendum to Gehring's lemma:

Proposition 3. *Under hypothesis of Lemma* 1, (0.2) *holds with the exponent p of order* $O(\log 1/(M-1))$.

Let us remark that one can obtain Proposition 1 by applying singular integral operators. However, having other applications in mind we use our method, which is fairly universal, particularly when non-linear equations are concerned.

1. Decomposition lemma

Let Q_0 be a cube in $\boldsymbol{R}^n$. We define by induction the families M_k, $k=0, 1, 2, \dots$ of open subcubes of the cube Q_0: $M_0=\{Q_0\}$. Suppose that the family M_k is given. Then we divide dyadically every cube of M_k into 2^n equal cubes. They form together the family M_{k+1}. The cubes of M_k are disjoint. In general every two cubes from the union $M=\bigcup_k M_k$ are either disjoint or one includes the other. Let us observe that any cube $Q\in M$ initiates uniquely the increasing sequence of cubes from M such that

$$Q = Q_l\subset Q_{l-1}\subset\dots\subset Q_1\subset Q_0,\quad Q_s\in M_s,\quad s=0, 1, \dots, l. \tag{1.1}$$

The volume of the cube $Q_s\in M_s$ is equal to $2^{-sn}|Q_0|$. Let f be an integrable function defined on Q_0. We will work with the following local maximal functions of Hardy—

Littlewood and Fefferman—Stein:

$$f^b(x) = \sup\left\{ \int_Q |f(y)|\,dy;\ x\in Q\subset Q_0\right\}$$

$$f^\#(x) = \sup\left\{ \int_Q |f(y)-f_Q|\,dy;\ x\in Q\subset Q_0\right\}.$$

The following version of Calderon—Zygmund lemma will be used in the next section.

Lemma 3 (Decomposition lemma). *For any* $\alpha\geqq a=|f|_{Q_0}$ *there exist disjoint cubes* $Q_j^\alpha\in M$, $j=1, 2, \ldots$ *such that*

a) $\alpha<\int_{Q_j^\alpha}|f(y)|\,dy\leqq 2^n\alpha$,

b) *If* $\alpha\geqq\beta\geqq a$, *then each cube* Q_j^α *is a subcube of one from the family* $\{Q_j^\beta;\ j=1, 2, \ldots\}$,

c) $|f(x)|\leqq\alpha$ *for almost all* $x\in Q_0-\bigcup_j Q_j^\alpha$,

d) $\sum_j |Q_j^\alpha|\leqq\lambda(\alpha)=\text{mes}\,\{x\in Q_0;\ f^b(x)>\alpha\}$ *for* $\alpha\geqq a$,

e) $\lambda(5^n\alpha)\leqq 5^n\sum_j|Q_j^\alpha|$ *for* $\alpha\geqq a$.

Proof. Let Q be an arbitrary cube from M. We examine the sequence

$$Q = Q_1\subset Q_{l-1}\subset\ldots\subset Q_1\subset Q_0,\quad Q_s\in M_s,\quad s = 0, l, \ldots, l. \tag{1.2}$$

The cube Q will be included in the family $\{Q_j^\alpha;\ j=1, 2, \ldots\}$ if and only if

$$\alpha < \int_Q |f| \quad\text{and}\quad \int_{Q_s}|f| \leqq \alpha \quad\text{for}\quad s = 0, 1, \ldots, 1-1. \tag{1.3}$$

Clearly any two cubes Q' and Q'' from $\{Q_j^\alpha\}$ are disjoint. In fact, if not, then it would be $Q'\subset Q''$ or $Q''\subset Q'$. Hence both would appear in the same sequence initiated by a smaller cube. This contradicts the obvious fact that no two cubes from (1.2) can be in $\{Q_j^\alpha;\ j=1, \ldots\}$.

Since $|f|_{Q_0}=a\leqq\alpha$, Q_0 cannot be a member of $\{Q_j^\alpha\}$, in particular

$$\int_Q |f| \leqq \frac{1}{|Q|}\int_{Q_{l-1}}|f|=2^n \int_{Q_{l-1}}|f| \leqq 2^n\alpha,$$

whence a) verified.

Now, let $\alpha\geqq\beta\geqq a$ and $Q\in\{Q_j^\alpha;\ j=1, 2, \ldots\}$. Then the $Q_{s'}$ from (1.2) with the smallest index $s'\in\{1, 2, \ldots, l\}$ for which $\beta<\int_{Q_{s'}}|f|$, must belong to $\{Q_j^\beta;\ j=1, 2, \ldots\}$, proving assertion b).

Let $x\notin\bigcup_j Q_j^\alpha$ and let $Q\in M$ be an arbitrary cube containing x. Then none of the cubes $Q\subset Q_{l-1}\subset\ldots\subset Q_1\subset Q_0$ belong to $\{Q_j^\alpha\}$, which means that $\int_{Q_s}|f|\leqq\alpha$ for each $s=0, 1, \ldots, l$, and in particular $\int_Q|f|\leqq\alpha$. By the Lebesgue theorem we conclude that

$$|f(x)| \leqq \sup\left\{\int_Q |f|;\ x\in Q\in M\right\} \leqq \alpha \quad\text{for almost every}\quad x\in Q_0-\bigcup_j Q_j^\alpha.$$

Actually we have shown that $\bigcup_j Q_j^\alpha = \{x\in Q_0;\ f^*(x)>\alpha\}$, where $f^*(x)=\sup\{f_Q|f|,\ x\in Q\in M\}$. Obviously we have $f^*(x)\leqq f^b(x)$. Hence $\sum_j |Q_j^\alpha|\leqq \operatorname{mes}\{x\in Q_0;\ f^b(x)>\alpha\}=\lambda(\alpha)$.

In order to prove condition e) we will show the inclusion

$$\{x\in Q_0;\ f^b(x)>5^n\alpha\}\subset \bigcup_j 5Q_j^\alpha. \tag{1.4}$$

Let $x\notin \bigcup_j 5Q_j^\alpha$ and let Q be an arbitrary cube such that $x\in Q\subset Q_0$. From a) and c) it follows that

$$\int_Q |f| = \int_{Q-\cup_j Q_j^\alpha} |f| + \sum_j \int_{Q\cap Q_j^\alpha} |f|$$
$$\leqq \alpha|Q| + \sum_{Q_j\cap Q\neq\emptyset} \int_{Q_j^\alpha} |f| \leqq \alpha|Q| + 2^n\alpha \sum_{Q_j^\alpha\cap Q\neq\emptyset} |Q_j^\alpha|.$$

Now we use the elementary fact: if $Q\not\subset 5Q_j^\alpha$ and $Q\cap Q_j^\alpha\neq\emptyset$, then $Q_j^\alpha\subset 2Q$. Therefore we can write

$$\int_Q |f| \leqq \alpha|Q| + 2^n\alpha \sum_{Q_j^\alpha\subset 2Q} |Q_j^\alpha| \leqq \alpha|Q| + 2^n\alpha|2Q| \leqq 5^n\alpha|Q|,$$

so $|f|_Q\leqq 5^n\alpha$ whenever $x\in Q-\bigcup_j 5Q_j^\alpha$. In other words, $f^b(x)\leqq 5^n\alpha$ for $x\notin\bigcup_j 5Q_j^\alpha$, which proves (1.4).

2. A local version of the Fefferman and Stein inequality

Lemma 4. *Suppose that $f^\#\in L^p(Q_0)$, $1\leqq p<\infty$. Then $f^b\in L^p(Q_0)$ and*

$$\Big(\int_{Q_0} |f^b|^p\Big)^{1/p} \leqq 10^{5np}\Big(\int_{Q_0} |f^\#|^p\Big)^{1/p} + 10^{n+1}\int_{Q_0} |f|. \tag{2.1}$$

Proof. First we shall show the following estimation:

$$\lambda(5^n\alpha) \leqq 5^n \operatorname{mes}\Big\{x\in Q_0;\ f^\#(x) > \frac{\alpha}{A}\Big\} + \frac{2\cdot 5^n}{A}\lambda(\alpha 2^{-n-1}) \tag{2.2}$$

for $\alpha\geqq 2^{n+1}a = 2^{n+1}|f|_{Q_0}$ and any positive A.

By d) and e) of Lemma 3 we see that it suffices to prove that

$$\sum_j |Q_j^\alpha| \leqq \operatorname{mes}\Big\{x\in Q_0;\ f^\#(x) > \frac{\alpha}{A}\Big\} + \frac{2}{A}\sum_j |Q_j^{\alpha 2^{-n-1}}|. \tag{2.3}$$

Fix a cube $Q\in\{Q_j^{\alpha 2^{-n-1}};\ j=1,2,\dots\}$. Then by a) we have $|f_Q|\leqq 2^n(\alpha 2^{-n-1})=\alpha/2$. Moreover, $\int_{Q_j^\alpha}|f|\geqq\alpha|Q_j^\alpha|$ for every cube Q_j^α. Thus

$$\int_{Q_j^\alpha} |f-f_Q| \geqq \Big(\alpha-\frac{\alpha}{2}\Big)|Q_j^\alpha| = \frac{\alpha}{2}|Q_j^\alpha|.$$

Hence by summing over those cubes Q_j^α which are contained in Q we get

$$\sum_{Q_j^\alpha\subset Q}|Q_j^\alpha| \leqq \frac{2}{\alpha}\sum_{Q_j^\alpha\subset Q}\int_{Q_j^\alpha}|f-f_Q| \leqq \frac{2}{\alpha}\int_Q |f-f_Q|.$$

Now we consider two cases:

Case 1. $f_Q|f-f_Q|\leqq\alpha/A$. Then $\sum_{Q_j^\alpha\subset Q}|Q_j^\alpha|\leqq(2/A)|Q|$.

Case 2. $f_Q|f-f_Q|>\alpha/A$. This means $f^\#(x)>\alpha/A$ for every point $x\in Q$, i.e., $Q\subset\{x\in Q_0;\ f^\#(x)>\alpha/A\}$. In particular

$$\sum_{Q_j^\alpha\subset Q}|Q_j^\alpha| \leqq \operatorname{mes} Q\cap\left\{x\in Q_0;\ f^\#(x)>\frac{\alpha}{A}\right\}.$$

In either case one can write

$$\sum_{Q_j^\alpha\subset Q}|Q_j^\alpha| \leqq \operatorname{mes} Q\cap\left\{x\in Q_0;\ f^\#(x)>\frac{\alpha}{A}\right\}+\frac{2}{A}|Q| \tag{2.4}$$

for each cube $Q\in\{Q_j^{\alpha 2^{-n-1}};\ j=1,2,\ldots\}$. Appealing to property b) of the decompositions $\{Q_j^\alpha;\ j=1,2,\ldots\}$, $\alpha\geqq 2^{n+1}|f|_{Q_0}$ we see that after summing (2.4) over $Q\in\{Q_j^{\alpha 2^{-n-1}};\ j=1,2,\ldots\}$ all cubes from $\{Q_j^\alpha;\ j=1,2,\ldots\}$ will be counted once, i.e., we obtain (2.3).

Now we proceed to the proof of (2.1). We begin with the well-known formula

$$\int_{Q_0}|F(y)|^p dy = p\int_0^\infty t^{p-1}\operatorname{mes}\{x\in Q_0;\ |F(x)|>t\}dt \quad \text{for} \quad F\in L^p(Q_0).$$

Take an arbitrary number $N>2^{n+1}a$. It follows from (2.2) that

$$p\int_{2^{n+1}a}^{N}\alpha^{p-1}\lambda(5^n\alpha)\,d\alpha$$

$$\leqq 5^n p\int_0^\infty \alpha^{p-1}\operatorname{mes}\left\{x;\ f^\#(x)>\frac{\alpha}{A}\right\}d\alpha+\frac{2\cdot 5^n p}{A}\int_0^N \alpha^{p-1}\lambda(\alpha 2^{-n-1})\,d\alpha.$$

Changing the variable α accurately in each integral we get

$$p\int_{2\cdot 10^n a}^{5^n N} t^{p-1}\lambda(t)\,dt$$

$$\leqq 5^{pn+p}A^p p\int_0^\infty t^{p-1}\operatorname{mes}\{x;\ f^\#(x)>t\}dt+\frac{2\cdot 10^{pn+p}p}{A}\int_0^{5^n N} t^{p-1}\lambda(t)\,dt.$$

This and the evident inequality

$$p\int_0^{2\cdot 10^n a} t^{p-1}\lambda(t)\,dt \leqq |Q_0|\,p\int_0^{2\cdot 10^n a} t^{p-1}dt = |Q_0|(2\cdot 10^n a)^p$$

yield

$$p\int_0^{5^n N} t^{p-1}\lambda(t)\,dt$$

$$\leqq 5^{pn+p}A^p \int_{Q_0} |f^{\#}|^p + \frac{2\cdot 10^{p+np}}{A} p \int_0^{5^n N} t^{p-1}\lambda(t)\,dt + 2^p 10^{np} a^p |Q_0|.$$

Now we put here $A=4\cdot 10^{np+p}$, giving

$$p\int_0^{5^n N} t^{p-1}\lambda(t)\,dt < 2\cdot 4^p \cdot 10^{np^2+p^2}\cdot 5^{np+p} \int_{Q_0} |f^{\#}|^p + 2^{p+1}\cdot 10^{np}\cdot a^p\,|Q_0|.$$

This shows that the integral $\int_0^\infty t^{p-1}\lambda(t)dt$ is finite, i.e., $f^b\in L^p(Q_0)$ and

$$\int_{Q_0} |f^b|^p \leqq 2\cdot 4^p 10^{np^2+p^2}\cdot 5^{np+p} \int_{Q_0} |f^{\#}|^p + 2^{p+1} 10^{np} |Q_0| \Big(\fint_{Q_0} |f| \Big)^p,$$

which is as sharp as (2.1). This completes the proof of the lemma.

Corollary 1. *Let* $f\in L^1(\Omega)$ *and suppose that for each cube* $Q\subset\Omega$

$$\fint_Q |f-f_Q| \leqq 2^{-1} 10^{-5np} \fint_Q |f|. \tag{2.5}$$

Then $f\in L^p_{\mathrm{loc}}(\Omega)$ *and for any* $Q_0\subset\Omega$

$$\Big(\fint_Q |f|^p \Big)^{1/p} \leqq 2\cdot 10^{n+1} \fint_{Q_0} |f| \tag{2.6}$$

holds.

Proof. If $f\in L^p_{\mathrm{loc}}(\Omega)$, then inequality (2.6) follows immediately from (2.1). In fact, by (2.5) we have $f^{\#}(x)\leqq 2^{-1}\cdot 10^{-5np} f^b(x)$ for every $x\in Q_0$. Then by (2.1)

$$\Big(\fint_{Q_0} |f^b|^p \Big)^{1/p} \leqq \frac{1}{2} \Big(\fint_{Q_0} |f^b|^p \Big)^{1/p} + 10^{n+1} \fint_{Q_0} |f|,$$

which implies (2.6) when one takes into account that $|f(x)|\leqq f^b(x)$. The general case follows from this particular one by an appropriate approximation. The details can be recovered from the proof of Lemma 2.

3. Proof of Lemma 2

Hereafter on frequent occasions we will appeal to

LEMMA 5. *Let* $U\in L^p(Q)$, $1\leqq p\leqq\infty$, $U\colon Q\to\mathbf{R}^k$. *Then*

$$\Big(\fint_Q |U(y)-U_Q|^p dy\Big)^{1/p} \leqq 2\Big(\fint_Q |U(y)-C|^p dy\Big)^{1/p} \tag{3.1}$$

for any constant vector $C\in\mathbf{R}^k$.

Proof. The proof is a simple consequence of Hölder's inequality;

$$\Big(\fint_Q |U-U_Q|^p\Big)^{1/p} = \Big(\fint_Q |U-C)-(U-C)_Q|^p\Big)^{1/p}$$
$$\leqq \Big(\fint_Q |U-C|^p\Big)^{1/p} + |(U-C)_Q| \leqq 2\Big(\fint_Q |U-C|^p\Big)^{1/p}.$$

Now we shall show how to reduce our problem to the case of $f\in L^p_{\mathrm{loc}}(\Omega)$. For this purpose we take an arbitrary subdomain $\Omega_0\subset\subset\Omega$. We also consider mollifier functions $\varphi_r\in C_0^\infty(\mathbf{R}^n)$ which approximate the Dirac measure, i.e., $\varphi_r(x)\geqq 0$, $\int\varphi_r(y)dy=1$, $\operatorname{supp}\varphi_r\subset B(0,r)$ for $r<\operatorname{dist}(\Omega_0,\partial\Omega)$. Then the convolution

$$F_r(x) = \int_{R^n} \varphi_r(y)|f(x-y)|\,dy$$

is well defined and it belongs to $L^p(\Omega_0)$ with $p\geqq 1$. Moreover, $|f(x)|=\lim_{r\to 0}F_r(x)$ for almost all $x\in\Omega_0$. We verify that hypotheses of Lemma 2 are valid for F_r and for the cubes $Q\subset Q_0$. By Fubini's theorem and by (3.1) we get

$$\fint_{\sigma Q} |F(x)-F_{\sigma Q}|\,dx \leqq \int_{R^n}\varphi(y)\fint_{\sigma Q}\Big||f(x-y)|-\fint_{\sigma Q}|f(z-y)|\,dz\Big|\,dx\,dy$$
$$\leqq 2\int_{R^n}\varphi(y)\fint_{\sigma Q}\Big||f(x-y)|-\Big|\fint_{\sigma Q} f(z-y)\,dz\Big|\Big|\,dx\,dy$$
$$\leqq 2\int_{R^n}\varphi(y)\fint_{\sigma Q}\Big|f(x-y)-\fint_{\sigma Q} f(z-y)\,dz\Big|\,dx\,dy,$$

where we omitted the index r in F and φ for notational simplicity. Since $f(x)$ satisfies (0.18), the shifted function $f(x-y)$ does it as well. Hence referring again to Fubini's theorem, we conclude that

$$\fint_Q |F(x)-F_{\sigma Q}|\,dx \leqq 2\cdot 10^{-6np}\fint_Q F(x)\,dx \quad \text{for} \quad Q\subset\Omega_0. \tag{3.2}$$

Our aim is to derive from (3.2) the estimate

$$\Big(\fint_{(1/2)Q} F(y)^p\,dy\Big)^{1/p} \leqq 10^{8n^2p}\cdot\sigma^{-n}\fint_{Q_0} F(y)\,dy \tag{3.3}$$

for all $F(y)=F_r(y)$ with $r<\operatorname{dist}(Q_0, \partial\Omega)$. As we see, the factor on the right hand side is independent of r, wherefore the lemma will follow if we let r tend to 0.

For the proof of (3.3) we need an auxiliary function H defined on $\boldsymbol{R}^n$ by

$$H(x)=\begin{cases} d^n(x)F(x) & \text{for } x\in Q_0 \\ 0 & \text{for } x\notin Q_0, \end{cases}$$

where $d(x)=\operatorname{dist}(x, \boldsymbol{R}^n-Q_0)$. Notice that $|d(x)-d(y)|\leqq|x-y|$. We intend to prove that

$$\text{(3.4)}\quad H^{\#}(x)\leqq\left[\frac{2^{n+2}}{N-\sigma}+2\cdot 10^{-6np}\left(\frac{N}{N-1}\right)^{n}\right]H^{b}(x)+2\cdot n^{n/2}(1+\sigma^{-1}N)^n \int_{Q_0} F$$

for every $x\in Q_0$ and any number $N>2$.

Let x be a point of Q_0 and let Q be a cube such that $x\in Q\subset Q_0$. Two cases are possible:

Case 1. $d(x)\leqq N\sigma^{-1}\operatorname{diam} Q$. Then

$$\fint_Q |H(y)-H_Q|\,dy\leqq 2H_Q\leqq 2|Q|^{-1}\sup_Q d^n(y)\int_Q F(y)\,dy.$$

But if $y\in Q$, we have $d(y)\leqq d(x)+|x-y|\leqq d(x)+\operatorname{diam} Q\leqq(1+\sigma^{-1}N)\operatorname{diam} Q$, and so $d^n(y)\leqq(1+\sigma^{-1}N)^n n^{n/2}|Q|$. Hence

$$\fint_Q |H(y)-H_Q|\,dy\leqq 2\cdot n^{n/2}(1+\sigma^{-1}N)^n\int_{Q_0} F(y)\,dy.$$

Case 2. $d(x)>N\sigma^{-1}\operatorname{diam} Q$. Then $\sigma^{-1}Q$ is a subcube of Q_0 and we are justified in using inequality (3.2) for $\sigma^{-1}Q$. From (3.1) it follows that

$$\fint_Q |H(y)-H_Q|\,dy\leqq 2\fint_Q |d^n(y)F(y)-d^n(x)F_Q|\,dy$$

$$\leqq 2\fint_Q |d^n(y)-d^n(x)|F(y)\,dy+2d^n(x)\fint_Q |F(y)-F_Q|\,dy$$

$$\leqq 2\frac{\sup_{y\in Q}|d^n(y)-d^n(x)|}{\inf_{y\in Q} d^n(y)}\fint_Q H(y)\,dy+\frac{2\cdot 10^{-6np}d^n(x)}{\inf_{y\in\sigma^{-1}Q} d^n(y)}\fint_{\sigma^{-1}Q} H(y)\,dy.$$

To estimate the factor in front of the first integral we observe that for any $y\in Q$, $d(y)\leqq d(x)+|x-y|\leqq d(x)+\operatorname{diam} Q<(1+\sigma N^{-1})d(x)$ and $d(y)\geqq d(x)-|x-y|\geqq d(x)-\operatorname{diam} Q>(1-\sigma N^{-1})d(x)$. Hence

$$\frac{\sup_Q|d^n(y)-d^n(x)|}{\inf_Q d^n(y)}\leqq\frac{\sup_Q d^n(y)}{\inf_Q d^n(y)}-1\leqq\frac{(1+\sigma N^{-1})^n}{(1-\sigma N^{-1})^n}-1\leqq\frac{2^{n+1}}{N-\sigma}.$$

Furthermore, if $y \in \sigma^{-1}Q$, then $d(y) \geqq d(x) - |x-y| \geqq d(x) - \operatorname{diam} \sigma^{-1}Q > (1-N^{-1})d(x)$, so

$$d^n(x) \leqq \left(\frac{N}{N-1}\right)^n \inf_{\sigma^{-1}Q} d^n(y).$$

Summarizing the second case we write

$$\int_Q |H(y)-H_Q|\,dy \leqq \frac{2^{n+2}}{N-\sigma} \int_Q H(y)\,dy + 2\cdot 10^{-6np}\left(\frac{N}{N-1}\right)^n \int_{\sigma^{-1}Q} H(y)\,dy \leqq$$

$$\leqq \left[\frac{2^{n+2}}{N-\sigma} + 2\cdot 10^{-6np}\left(\frac{N}{N-1}\right)^n\right] H^b(x).$$

The estimations for the two cases imply (3.4).

Now we insert (3.4) into (2.1), getting

$$\left(\int_{Q_0} H^b(y)^p\,dy\right)^{1/p} \leqq 10^{5np}\left[\frac{2^{n+2}}{N-\sigma} + 2\cdot 10^{-6np}\left(\frac{N}{N-1}\right)^n\right]\left(\int_{Q_0} H^b(y)^p\,dy\right)^{1/p}$$

$$+2\cdot n^{n/2} 10^{5np}(1+\sigma^{-1}N)^n \int_{Q_0} F(y)\,dy + 10^{n+1}\int_{Q_0} H(y)\,dy.$$

Take N large enough to make the factor in front of the first integral on the right-hand side $\leqq 1/2$, as for instance $N = 2^{n+1}10^{6np}$ giving

$$\left(\int_{Q_0} H^b(y)^p\,dy\right)^{1/p} \leqq 10^{7n^2p}\cdot\sigma^{-n}\int_{Q_0} F(y)\,dy + 2\cdot 10^{n+1}\int_{Q_0} H(y)\,dy.$$

Clearly $H(y) \leqq H^b(y)$ for almost all $y \in Q_0$. Thus

$$\left(\int_{Q_0} H(y)^p\,dy\right)^{1/p} \leqq 2\cdot 10^{n+1}\int_{Q_0} H(y)\,dy + 10^{7n^2p}\cdot\sigma^{-n}\int_{Q_0} F(y)\,dy. \tag{3.5}$$

Let us recall that $H(x) = d^n(x)F(x)$. By the simple observation $d^n(x) \leqq 2^{-n}|Q_0|$ for $x \in Q_0$ and $d^n(x) \geqq 4^{-n}Q_0$ for $x \in (1/2)Q_0$ we eliminate from (3.5) the auxiliary function H as follows:

$$\left(\int_{(1/2)Q_0} F(y)^p\,dy\right)^{1/p} \leqq \frac{4^n}{|Q_0|}\left(\int_{(1/2)Q_0} H(y)^p\,dy\right)^{1/p} \leqq \frac{4^n\cdot 2^{n/p}}{|Q_0|}\left(\int_{Q_0} H(y)^p\,dy\right)^{1/p}$$

$$\leqq \frac{2\cdot 4^n\cdot 2^{n/p}\cdot 10^{n+1}}{|Q_0|}\int_{Q_0} H(y)\,dy + \frac{4^n\cdot 2^{n/p}\cdot 10^{7n^2p}}{\sigma^n|Q_0|}\int_{Q_0} F(y)\,dy$$

$$\leqq \frac{2\cdot 4^n\cdot 2^{n/p}\cdot 10^{n+1}}{|Q_0|}\frac{|Q_0|}{2^n}\int_{Q_0} F(y)\,dy + \frac{4^n\cdot 2^{n/p}\cdot 10^{7n^2p}}{\sigma^n|Q_0|}\int_{Q_0} F(y)\,dy$$

$$\leqq \frac{10^{8n^2p}}{\sigma^n}\int_{Q_0} F(y)\,dy.$$

This is exactly the inequality which we claimed to prove.

4. On the regularity of harmonic functions and generalized conformal maps

We state without proof the following well-known Sobolev—Poincaré's lemma:

Lemma 6. *Let $B=B(x, R)$ be a ball in $\mathbf{R}^n$ with centre x and radius R. Suppose that $U\in W_q^1(B)$, $1\leqq q<n$. Then $U\in L^{nq/n-q}(B)$ and*

$$(4.1)\qquad \Big(\fint_B |U(y)-U_B|^{nq/(n-q)}\,dy\Big)^{(n-q)/(nq)} \leqq C(n, q)R\Big(\fint_B |\nabla U(y)|^q dy\Big)^{1/q}.$$

The same holds when B is replaced by a cube of diameter $2R$.

This will be essential for proving the following regularity result.

Lemma 7. *Let H be a harmonic function defined on a ball $B(x, R)\subset\mathbf{R}^n$ and having values in $\mathbf{R}^k$. Suppose that $H\in L^p(B(x, R))$, $2\leqq p\leqq 2n/(n-2)$. Then for every $B(x, r)\subset B(x, R)$ the inequality*

$$(4.2)\qquad \fint_{B(x,r)} \Big||H|^p-\fint_{B(x,r)}|H|^p\Big| \leqq C(n)\frac{r}{R}\fint_{B(x,R)}|H|^p$$

holds with a constant $C(n)$ depending only on n.

Proof. Let $B=B(x, r)$. By Lemma 5 and some standard inequalities we get

$$\begin{aligned}\fint_B \Big||H|^p-\fint_B|H|^p\Big| &\leqq 2\fint_B \big||H|^p-|H_B|^p\big| \leqq 2p\fint_B |H-H_B|(|H|+H_B|)^{p-1}\\ &\leqq 2p\Big(\fint_B|H-H_B|^p\Big)^{1/p}\Big(\fint_B(|H|+|H_B|)^p\Big)^{(p-1)/p}\\ &\leqq 2p\Big(\fint_B|H-H_B|^p\Big)^{1/p}\Big[\Big(\fint_B|H|^p\Big)^{1/p}+|H_B|\Big]^{p-1}\\ &\leqq 2^p p\,\Big(\fint_B|H-H_B|^p\Big)^{1/p}\Big(\fint_B|H|^p\Big)^{(p-1)/p}.\end{aligned}$$

Since $2\leqq p\leqq 2n/(n-2)$, we may use Sobolev—Poincaré's lemma giving

$$\Big(\fint_B|H-H_B|^p\Big)^{1/p} \leqq C(n)r\Big(\fint_B|\nabla H|^2\Big)^{1/2},$$

inserting it in the last inequalities we obtain

$$\fint_B\Big||H|^p-\fint_B|H|^p\Big| \leqq C(n, p)r\Big(\fint_B|\nabla H|^2\Big)^{1/2}\Big(\fint_B|H|^p\Big)^{(p-1)/p}.$$

Now we observe that functions $|H(x)|^p$ and $|\nabla H(x)|^2$ are subharmonic, i.e., $\Delta|H|^p= p|H|^{p-2}(|\nabla H|^2+(p-2)|\nabla|H||^2)\geqq 0$, $\Delta|\nabla H|^2=2|\nabla\nabla H|^2\geqq 0$. It implies that the integrals $\fint_B|H|^p$ and $\fint_B|\nabla H|^2$, $B=B(x, r)$, are increasing in r. In particular we can write

(4.3)
$$\fint_B \left||H|^p - \fint_B |H|^p\right| \leqq C(n, p)\, r \Big(\fint_{B(x,R/2)} |\nabla H|^2\Big)^{1/2} \Big(\fint_{B(x,R)} |H|^p\Big)^{(p-1)/p} \quad \text{for} \quad r \leqq R/2.$$

Since H satisfies the Laplace equation, we may use a Caccioppoli type inequality (compare it with (0.8)). If, in addition, Hölder's inequality is used, then

$$\Big(\fint_{B(x,R/2)} |\nabla H|^2\Big)^{1/2} \leqq \frac{C(n)}{R}\Big(\fint_{B(x,R)} |H|^2\Big)^{1/2} \leqq \frac{C(n)}{R}\Big(\fint_{B(x,R)} |H|^p\Big)^{1/p}.$$

Inserting it in (4.3) we get (4.2) for $r \leqq R/2$. The case $R/2 < r \leqq R$ is obvious.

For convenience let us reformulate the above result by replacing balls by cubes. Given a harmonic function $H \in L^p(Q)$, where Q is a cube in $\boldsymbol{R}^n$, we have

$$\fint_{\sigma Q} \left||H|^p - \fint_{\sigma Q} |H|^p\right| \leqq C(n, p)\, \sigma \fint_Q |H|^p \quad \text{for} \quad 0 < \sigma \leqq 1. \tag{4.4}$$

This follows from (4.2) after it has been applied to the balls which satisfy $\sigma Q \subset B(x, r) \subset B(x, R) \subset Q$, provided $0 < \sigma < 1/\sqrt{n}$, the case $1/\sqrt{n} \leqq \sigma \leqq 1$ being obvious.

It is now quite easy to infer the corresponding regularity result for generalized conformal mappings. We notice that for every such mapping, say $g\colon \Omega \to \boldsymbol{R}^n$, the function $H(x) = J(x, g)^{(n-2)/(2n)}$ is harmonic; for an elementary proof see [2]. Therefore referring once again to (4.4) with $p = 2n/(n-2)$ we immediately obtain

$$\fint_{\sigma Q} \left|J(x, g) - \fint_{\sigma Q} J(y, g)\, dy\right| dx \leqq C(n)\, \sigma \fint_Q J(x, g)\, dx \tag{4.5}$$

for any $0 < \sigma \leqq 1$ and each cube $Q \subset \Omega$.

5. The weak stability

Lemma 8. *Let* $u \in W_2^1(\Omega)$ *be a solution of*

$$\sum_{i,j} \frac{d}{dx_i}\left(a_{ij}(x) \frac{du}{dx_j}\right) = 0, \tag{5.1}$$

where $a_{ij}(x)$ *satisfy the hypotheses of Proposition* 1. *Then there exists a harmonic function* $h \in W_2^1(\Omega)$ *such that*

$$\int_\Omega |\nabla u(y) - \nabla h(y)|^2 dy \leqq \varepsilon^2 \int_\Omega |\nabla u(y)|^2 dy. \tag{5.2}$$

Proof. We define h to be the solution to Dirichlet problem; $\Delta h = 0$ on Ω and $u - h \in \overset{0}{W}{}_2^1(\Omega)$. It is a well-known fact in the calculus of variations that such a solution exists and is unique. We write the Laplace equation as follows:

$$\int_\Omega \langle \nabla h(y), \nabla\varphi(y)\rangle dy = 0 \quad \text{for any test function} \quad \varphi \in \overset{0}{W}{}_2^1(\Omega).$$

We subtract side by side the equation

$$\int_\Omega \langle \nabla u(y), \nabla\varphi(y)\rangle dy = -\int_\Omega \langle \mathscr{E}(y)\nabla u(y), \nabla\varphi(y)\rangle dy,$$

which is a weak form of (5.1), $\mathscr{E}(y)$ being the matrix of entries $\varepsilon_{ij}(y)$. This yields

$$\int_\Omega \langle \nabla u - \nabla h, \nabla\varphi\rangle = -\int_\Omega \langle \mathscr{E}\nabla u, \nabla\varphi\rangle, \quad \text{for each} \quad \varphi \in \overset{0}{W}{}_2^1(\Omega).$$

In view of the boundary condition on h we are allowed to set $\varphi = u - h$. By (0.14) and by Hölder's inequality we then get

$$\int_\Omega |\nabla u - \nabla h|^2 = -\int_\Omega \langle \mathscr{E}\nabla u, \nabla u - \nabla h\rangle \leqq \varepsilon\Big(\int_\Omega |\nabla u|^2\Big)^{1/2}\Big(\int_\Omega |\nabla u - \nabla h|^2\Big)^{1/2}.$$

This immediately leads to (5.2).

For further purposes we anticipate the following result:

$$\int_\Omega \big||\nabla u|^2 - |\nabla h|^2\big| \leqq 3\varepsilon \int_\Omega |\nabla u|^2; \tag{5.3}$$

this we infer from the lemma by applying the elementary inequality $\big||v|^2 - |w|^2\big| \leqq ((1+\varepsilon)/\varepsilon)|v-w|^2 + \varepsilon|v|^2$, for $v, w \in \boldsymbol{R}^n$. In fact

$$\int_\Omega \big||\nabla u|^2 - |\nabla h|^2\big| \leqq \frac{1+\varepsilon}{\varepsilon}\int_\Omega |\nabla u - \nabla h|^2 + \varepsilon\int_\Omega |\nabla u|^2$$

$$\leqq \frac{1+\varepsilon}{\varepsilon}\varepsilon^2 \int_\Omega |\nabla u|^2 + \varepsilon\int_\Omega |\nabla u|^2 \leqq 3\varepsilon\int_\Omega |\nabla u|^2.$$

Our nearest aim is to establish the stability of quasiregular mappings. For this we need a few non-standard facts on Sobolev's spaces.

Lemma 9. *Let Ω be an open subset in $\boldsymbol{R}^n$ and let f, g be mappings from $W^1_{n,\,\mathrm{loc}}(\Omega)$, $f=(f^1, \dots, f^n)$, $g=(g^1, \dots, g^n)$. Then for every $\varphi \in C_0^\infty(\Omega)$ the inequalities*

$$\Big|\int \varphi(x)[J(x,f) - J(x,g)]\,dx\Big| \leqq \int |f-g||\nabla\varphi|(|Df|^2 + |Dg|^2)^{n-1/2} \tag{5.4}$$

$$\leqq \|(f-g)\nabla\varphi\|_n(\|Df\|_n^2 + \|Dg\|_n^2)^{n-1/2}$$

hold, where $\|U\|_n = (\int_\Omega |U|^n)^{1/n}$.

Proof. By an approximation argument one can reduce the problem to the case of smooth f and g. Consequently, we may write

$$\int \varphi(x)[J(x,f)-J(x,g)]\,dx = \int \varphi(df^1 \wedge \ldots \wedge df^n - dg^1 \wedge \ldots \wedge dg^n)$$

$$= \int \varphi \sum_{k=1}^{n} df^1 \wedge \ldots \wedge df^{k-1} \wedge d(f^k-g^k) \wedge dg^{k+1} \wedge \ldots \wedge dg^n$$

$$= -\int \sum_{k=1}^{n} (f^k-g^k) df^1 \wedge \ldots \wedge df^{k-1} \wedge d\varphi \wedge dg^{k+1} \wedge \ldots \wedge dg^n$$

$$+ \int \sum_{k=1}^{n} df^1 \wedge \ldots \wedge df^{k-1} \wedge d\varphi(f^k-g^k) \wedge dg^{k+1} \wedge \ldots \wedge dg^n.$$

We easily see that the differential forms under the last summation sign are equal to $(-1)^k d[\varphi(f^k-g^k)df^1 \wedge \ldots \wedge df^{k-1} \wedge dg^{k+1} \wedge \ldots \wedge dg^n]$. Therefore, by Stokes' theorem the last integral vanishes. Now we shall use the inequality of Hadamard and others which can be easily recognized from calculations below:

$$\left|\sum_{k=1}^{n} (f^k-g^k) df^1 \wedge \ldots \wedge df^{k-1} \wedge d\varphi \wedge dg^{k+1} \wedge \ldots \wedge dg^n\right|$$

$$\leqq \sum_{k=1}^{n} |f^k-g^k|\,|\nabla f^1|\ldots|\nabla f^{k-1}|\,|\nabla\varphi|\,|\nabla g^{k+1}|\ldots|\nabla g^n|\,dx$$

$$\leqq |\nabla\varphi|\,|f-g| \left(\sum_{k=1}^{n} |\nabla f^1|^2\ldots|\nabla f^{k-1}|^2|\nabla g^{k+1}|^2\ldots|\nabla g^n|^2\right)^{1/2} dx$$

$$\leqq |\nabla\varphi|\,|f-g| \left[\sum_{k=1}^{n} \left(\frac{1}{n-1}\right)^{n-1} (|\nabla f^1|^2+\ldots+|\nabla f^{k-1}|^2+|\nabla g^{k+1}|^2+\ldots+|\nabla g^n|^2)^{n-1}\right]^{1/2} dx$$

$$\leqq (n-1)^{(1-n)/2}|\nabla\varphi|\,|f-g| \left[\sum_{k=1}^{n} (|\nabla f^1|^2+\ldots+\nabla f^{k-1}|^2+|\nabla g^{k+1}|^2+\ldots+|\nabla g^n|^2)\right]^{(n-1)/2} dx$$

$$= (n-1)^{(1-n)/2}|\nabla\varphi|\,|f-g| \left[\sum_{k=1}^{n} (n-k)|\nabla f^k|^2+(k-1)|\nabla g^k|^2\right]^{(n-1)/2} dx.$$

Thus

$$\left|\int \varphi(x)[J(x,f)-J(x,g)]\,dx\right|$$

$$\leqq (n-1)^{(1-n)/2}|f-g|\,|\nabla\varphi| \left[\sum_{k=1}^{n} (n-k)|\nabla f^k|^2+(k-1)|\nabla g^k|^2\right]^{(n-1)/2} dx.$$

Exchanging the roles of f and g and then adding by sides we easily get

$$2\left|\int \varphi(x)[J(x,f)-J(x,g)]\,dx\right.$$
$$\leqq 2(n-1)^{(1-n)/2}\int |f-g|\,|\nabla\varphi|\left[\sum_{k=1}^{n}(n-1)|\nabla g^k|^2+(n-1)|\nabla f^k|^2\right]^{(n-1)/2}dx$$
$$=2\int |f-g|\,|\nabla\varphi|(|Dg|^2+|Df|^2)^{(n-1)/2}dx$$
$$\leqq 2\Big(\int |(f-g)\nabla\varphi|^n\Big)^{1/n}\Big(\int (|Dg|^2+|Df|^2)^{n/2}\Big)^{(n-1)/n},$$

which immediately implies (5.4).

Corollary 2. *Let $f\colon \Omega\to\boldsymbol{R}^n$ be a K-quasiregular mapping. Then for each cube $Q\subset\Omega$ the inequality*

$$\Big(\fint_{(1/2)Q} |Df(y)|^n\,dy\Big)^{1/n}\leqq C(n)\,K\Big(\fint_{Q}|Df(y)|^{n/2}\,dy\Big)^{2/n} \tag{5.5}$$

holds with a constant $C(n)$ depending only on n.

Proof. Let $\xi(x)$ be a function of the class $C_0^\infty(Q)$ such that $0\leqq\xi(x)\leqq 1$, $\xi(x)=1$ for $x\in(1/2)Q$, $|\nabla\xi(x)|\leqq C(n)|Q|^{-1/n}$. Letting $\varphi(x)=\xi^n(x)$, $g(x)=f_Q$ in (5.4) we get by Hölder's inequality

$$\int_Q \xi^n(x)J(x,f)\,dx\leqq n\int_Q \xi^{n-1}(x)|\nabla\xi(x)|\,|f-f_Q|\,|Df(x)|^{n-1}dx$$
$$\leqq n\Big(\int_Q |\nabla\xi|^n|f-f_Q|^n\Big)^{1/n}\Big(\int_Q \xi^n|Df|^n\Big)^{(n-1)/n},$$

and applying (0.11)

$$\int_Q \xi^n|Df|^n\leqq n^{(n+2)/2}K\Big(\int_Q |\nabla\xi|^n|f-f_Q|^n\Big)^{1/n}\Big(\int_Q \xi^n|Df|^n\Big)^{(n-1)/n}.$$

Hence

$$\Big(\int_Q \xi^n|Df|^n\Big)^{1/n}\leqq\frac{n^{(n+2)/2}C(n)K}{|Q|^{1/n}}\Big(\int_Q |f-f_Q|^n\Big)^{1/n}.$$

By Poincaré's inequality (see Lemma 6 for $q=n/2$) we get

$$\Big(\fint_{(1/2)Q}|Df|^n\Big)^{1/n}\leqq C(n)K\Big(\fint_Q |Df|^{2/n}\Big)^{n/2},$$

which was stated in the Corollary.

As a consequence of (5.4) by Rellich—Sobolev's compactness theorem we deduce

Corollary 3. *Let f_j be mappings from Ω into $\boldsymbol{R}^n$, $f_j\in W_n^1(\Omega)$. Suppose that f_j converge weakly in $W_n^1(\Omega)$ to a map $f\in W_n^1(\Omega)$. Then for every $\varphi\in C_0^\infty(\Omega)$*

$$\lim_{j\to\infty}\int\varphi(x)J(x,f_j)\,dx=\int\varphi(x)J(x,f)\,dx. \tag{5.6}$$

In fact, the norms $\|Df_j\|_n$ are uniformly bounded and by the compactness theorem the sequence $(f_j-f)\nabla\varphi$ converges to zero strongly in $L^n(\Omega)$. Thus (5.6) immediately follows from (5.4) when we set $g=f_j$.

We derive from Corollary 3 the following weak stability result for quasiregular mappings.

Lemma 10. *There exists a function* $\beta=\beta(K)$, $\beta\colon [1,\infty)\to[0,\infty)$ (it may also depend on n) *such that*

a) $\lim_{K\to 1}\beta(K)=0$.

b) *For each cube* $Q\subset \boldsymbol{R}^n$ *and any* K*-quasiregular mapping* $f\in W_n^1(Q)$ *there exists a generalized conformal mapping* $g\in W_n^1(Q)$ *such that*

$$(5.7)\qquad \Big(\fint_{(1/2)Q}|Df-Dg|^n\Big)^{1/n}\leqq \beta(K)\Big(\fint_Q|Df|^n\Big)^{1/n}.$$

Proof. The proof of existence of g will be ineffective to the extent that we are unable to give a formula for $\beta(K)$ (see remarks after the proof). Assume that $Q=Q_0$ — the unit cube of $\boldsymbol{R}^n$ — and that $\fint_{Q_0}|Df|^n=1$, $\fint_{Q_0}=0$. These are illusive restrictions of generality because (5.7) is invariant with respect to translation and homothetic transformation of co-ordinates. Contradicting the lemma assume that we are given $\beta_0>0$ and a sequence $\{f_j\}$, $j=1,2,\dots$ of K_j-quasiregular mappings such that: $\lim K_j=0$, $\fint_{Q_0}|Df_j|^n=1$, $\fint_{Q_0}f_j=0$ and

$$(5.8)\qquad \Big(\fint_{(1/2)Q_0}|Df_j-Dg|^n\Big)^{1/n}\geqq\beta_0$$

for any 1-quasiregular mapping $g\in W_n^1(Q_0)$.

We can choose a subsequence f_{j_α}, $\alpha=1,2,\dots$ which weakly converges to a mapping $g\in W_n^1(Q_0)$. According to Corollary 3 we have

$$\lim_\alpha\int\varphi^n(x)J(x,f_{j_\alpha})\,dx=\int\varphi^n(x)J(x,g)\,dx$$

for every $\varphi\in C_0^\infty(Q_0)$; we shall assume that $\varphi(x)\geqq 0$. On the other hand the inequalities (0.11) and (0.16) yield

$$\int\varphi^n(x)|Dg(x)|^n\,dx\leqq\liminf_\alpha\int\varphi^n(x)|Df_{j_\alpha}(x)|^n\,dx$$

$$\leqq n^{n/2}\liminf_\alpha K_{j_\alpha}\int\varphi^n(x)J(x,f_{j_\alpha})\,dx=n^{n/2}\int\varphi^n(x)J(x,g)\,dx\leqq\int\varphi^n(x)|Dg(x)|^n\,dx.$$

It proves that each relation must be equality, in particular

$$\int\varphi^n(x)|Dg(x)|^n\,dx=n^{n/2}\int\varphi^n(x)J(x,g)\,dx.$$

As φ was an arbitrary non-negative function (of the class $C_0^\infty(Q_0)$), we conclude that $|Dg(x)|^n=n^{n/2}J(x,g)$ for almost every $x\in Q_0$, which shows that g is an 1-quasiregular mapping. Furthermore, since $\|\varphi Df_{j_\alpha}\|_n\to\|\varphi Dg\|_n$, with a view to uniform convexity of $L^n(Q_0)$ we infer that the sequence φDf_{j_α} actually converges to φDg in the sense of strong topology of $L^n(Q_0)$. Letting φ to be equal to 1 on $(1/2)Q_0$ we are led to

$$\lim_{\alpha}\Big(\int_{(1/2)Q_0}|Df_{j_\alpha}-Dg|^n\Big)^{1/n}=0,$$

which contradicts (5.8). The proof of Lemma 10 is complete.

By a slight modification of the above proof or indirectly from (5.7) one can derive the estimation

$$\int_{(1/2)Q}|J(x,f)-J(x,g)|\,dx \leqq C(n)K\beta(K)\int_Q J(x,f)\,dx. \tag{5.8}$$

The indirect proof is based on the inequality

$$|J(x,f)-J(x,g)| \leqq C(n)|Df(x)-Dg(x)|(|Df(x)|^{n-1}+|Dg(x)|^{n-1}).$$

Integrating it over $(1/2)Q$ and applying Hölder's inequality we get

$$\int_{(1/2)Q}|J(x,f)-J(x,g)|\,dx$$

$$\leqq C\Big(\int_{(1/2)Q}|Df-Dg|^n\Big)^{1/n}\Big[\Big(\int_{(1/2)Q}|Df|^n\Big)^{(n-1)/n}+\Big(\int_{(1/2)Q}|Dg|^n\Big)^{(n-1)/n}\Big],$$

and for the sake of (5.7) we are led to

$$\int_{(1/2)Q}|J(x,f)-J(x,g)|\,dx \leqq C(n)\beta(K)\int_Q|Df|^n \leqq n^{n/2}C(n)K\beta(K)\int_Q J(x,f)\,dx,$$

whence (5.8) verified.

Remarks. A result of the kind of Corollary 3 is known in the theory of quasiconformal mappings. Some other functionals which behave like Jacobian (so-called null Lagrangians) were investigated in non-linear elasticity.

Our Lemma 10 is the simplest one among several stability theorems for quasiregular mappings. The stronger ones require a much deeper study of the subject. We do not need them in this paper. However, it is interesting to see the chief points of these generalizations. Firstly, one can prove that the function $\beta=\beta(K)$ is of order $O(K-1)$. Secondly, the inequality (5.7) remains valid when the cube $(1/2)Q$ is replaced by Q; the same may be done in (5.5), i.e., the reverse Hölder inequalities are valid for derivatives of quasiregular mappings (see [8]). The last statement is not true for solutions of elliptic equations. For these remarks we recommend [9], [10].

6. Proofs of Propositions 1 and 2

Let $u\in W^1_{2,\mathrm{loc}}(\Omega)$ be a solution of (0.5). Then for each cube $Q\subset\subset\Omega$ u is a solution of the class $W_2^1(Q)$. Therefore by (5.3) we can find a harmonic function $h\in W_2^1(Q)$ (which of course depends on Q) such that $\fint_Q\big||\nabla u|^2-|\nabla h|^2\big|\leqq 3\varepsilon\fint_Q|\nabla u|^2$. This implies

$$\fint_Q|\nabla h|^2\leqq 4\fint_Q|\nabla u|^2 \tag{6.1}$$

and

$$\fint_{\sigma Q}\big||\nabla u|^2-|\nabla h|^2\big|\leqq\frac{3\varepsilon}{\sigma^n}\fint_Q|\nabla u|^2$$

for any $0<\sigma\leqq 1$. We apply (4.4) for $H=\nabla u$ and for $p=2$.

$$\fint_{\sigma Q}\Big||\nabla h|^2-\fint_{\sigma Q}|\nabla h|^2\Big|\leqq C(n)\sigma\fint_Q|\nabla h|^2.$$

Combining it with (6.1) we get

$$\fint_{\sigma Q}\Big||\nabla u|^2-\fint_{\sigma Q}|\nabla h|^2\Big|\leqq(3\varepsilon\sigma^{-n}+4C(n)\sigma)\fint_Q|\nabla u|^2.$$

Put $\sigma=\sqrt[n+1]{\varepsilon}$. Then by Lemma 5

$$\fint_{\sigma Q}\Big||\nabla u|^2-\fint_{\sigma Q}|\nabla u|^2\Big|\leqq\sqrt[n+1]{\varepsilon}(6+8C(n))\fint_Q|\nabla u|^2.$$

Now we deduce from Lemma 2 that for each cube $Q\subseteqq\Omega$

$$\Big(\fint_{(1/2)Q}|\nabla u|^{2q}\Big)^{1/q}\leqq\frac{10^{8n^2q}}{\sigma^n}\fint_Q|\nabla u|^2,$$

provided $\sqrt[n+1]{\varepsilon}(6+8C(n))\leqq 10^{-6nq}$, i.e.,

$$\Big(\fint_{(1/2)Q}|\nabla u|^p\Big)^{1/p}\leqq\frac{10^{2n^2p}}{\sigma^{n/2}}\Big(\fint_Q|\nabla u|^2\Big)^{1/2},$$

provided $\sqrt[n+1]{\varepsilon}(6+8C(n))\leqq 10^{-3np}$. The last condition holds for each $p\in[1,2+(6n(n+1))^{-1}\log(1/\varepsilon))$ whenever $\varepsilon\leqq\varepsilon_0=(6+8C(n))^{-2n-2}$. On the other hand, if $\varepsilon_0<\varepsilon<1$, then by (0.10) we infer that (0.15) is valid for $p\in[1,2+k(n)\log(1/\varepsilon))$, where $k=k(n)>0$ is small enough to satisfy $2+k(n)\log(1/\varepsilon_0)<p$: This proves Proposition 1. We prove Proposition 2 similarly. Let $f\colon\Omega\to\boldsymbol{R}^n$ be a K-quasiregular mapping. On account of Gehring's result, see (0.13), we may only consider the case of K close to 1, i.e., for small $\beta(K)$ (see Lemma 10). What we really need to assume on $\beta(K)$ will be easily seen from the calculations we are going to carry out. Let Q be an arbitrary cube in Ω. We may indicate a generalized conformal mapping $g\in W_n^1(Q)$

such that

$$\int_{(1/2)Q} |J(x, f)-J(x, g)|\, dx \leqq C(n)K\beta(K) \int_Q J(x, f)\, dx,$$

see (5.8). As in the previous proof we infer that

$$\int_{(1/2)Q} J(x, g)\, dx \leqq \left(2^n+C(n)K\beta(K)\right) \int_Q J(x, f)\, dx \tag{6.2}$$

and for $0<\sigma\leqq 1/2$

$$\int_{\sigma Q} |J(x, f)-J(x, g)|\, dx \leqq \frac{C(n)K\beta(K)}{2^n\sigma^n} \int_Q J(x, f)\, dx. \tag{6.3}$$

On the other hand, replacing Q and σ in (4.5) by $(1/2)Q$ and 2σ we see that

$$\int_{\sigma Q} \left|J(x, g)- \int_{\sigma Q} J(y, g)\, dy\right| dx \leqq C(n) \int_{(1/2)Q} J(x, g)\, dx, \quad \sigma \leqq \frac{1}{2}.$$

Combining it with (6.2) and (6.3) we get

$$\int_{\sigma Q} \left|J(x, f)- \int_{\sigma Q} J(y, g)\, dy\right| dx \leqq C(n)\left(\beta(K)\sigma^{-n}+\sigma\right) \int_Q J(y, f)\, dy$$

with a constant $C(n)$ depending only on n. Put here $\sigma=\sqrt[n+1]{\beta(K)}$. Eliminating the term $\int_{\sigma Q} J(y, g)dy$ by using Lemma 5, we are led to the hypothesis of Lemma 2, i.e.,

$$\int_{\sigma Q} \left|J(x, f)- \int_{\sigma Q} J(y, f)\, dy\right| dx \leqq C(n) \sqrt[n+1]{\beta(K)} \int_Q J(y, f)\, dy.$$

Now we conclude, assuming K is sufficiently close to 1, that there exists $p=p(n, K)>n$ such that

$$\left(\int_{(1/2)Q} J(x, f)^p\, dx\right)^{1/p} \leqq C(n, p) \int_Q J(x, f)\, dx$$

for each cube $Q\subsetneqq\Omega$ and that $\lim_{K\to 1^-} p(n, K)=\infty$ The whole conclusion of Proposition 2 follows from the inequality $|Df(x)|^n\leqq n^{n/2}KJ(x, f)\leqq K|Df(x)|^n$ defining f to be a K-quasiregular mapping.

References

[1] Bojarskiĭ, B. V.: Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients. - Mat. Sb. N. S. 43 (85), 1957, 451—503. (Russian.)

[2] Bojarskiĭ, B., and T. Iwaniec: Some new concepts in the analytical theory of QC-maps in R^n, $n\geqq 3$, and differential geometry. - The Conference on Global Analysis, Garwitz, DDR, October, 1981.

[3] Bojarskiĭ, B., and T. Iwaniec: Another approach to Liouville's theorem. - Math. Nachr. (to appear).

[4] ELCRAT, A., and N. G. MEYERS: Some results on regularity for solutions of non-linear elliptic systems and quasiregular functions. - Duke Math. J. 42, 1975, 121—136.

[5] FEFFERMAN, C., and E. M. STEIN: H^p spaces of several variables. - Acta Math. 129, 1972, 137—193.

[6] GEHRING, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. - Acta Math. 130, 1973, 265—277.

[7] IWANIEC, T.: Projections onto gradient fields and L^p-estimates for degenerated elliptic operators. - Studia Math. (to appear).

[8] MARTIO, O.: On the integrability of the derivative of a quasiregular mapping. - Math. Scand. 35, 1974, 43—48.

[9] REŠETNJAK, JU. G.: Stability estimates in Liouville's theorem, and the L^p-integrability of the derivatives of quasiconformal mappings. - Sibirsk. Math. Ž. 17, 1976, 868—896 (Russian).

[10] REŠETNJAK, JU. G.: Stability estimates in the class W_p^1 in Liouville's conformal mapping theorem for a closed domain. - Sibirsk. Math. Ž. 17, 1976, 1382—1394, 1439 (Russian).

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