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# QUASICONFORMAL MAPPINGS WITH FREE BOUNDARY COMPONENTS

RICHARD FEHLMANN

## 1. Introduction

Let $\Gamma$ denote a closed subset of the boundary of $D=\{w||w|<1\}$ and $T$ a compact subset of $D$, such that $D-T$ is a domain. We consider a quasiconformal mapping $F$, $z=F(w)$, which maps $D-T$ into $D$, such that $\delta D$ is mapped onto itself. The class of all such mappings which agree with $F$ on $\Gamma$ is denoted by $Q_F$. We do not indicate the dependence on $T$ and $\Gamma$, since these sets are fixed throughout this paper (except in §6). If $F$ has minimal maximal dilatation in the class $Q_F$, we call $F$ absolutely extremal. We use the notation "absolutely extremal" to indicate that the image domains of competing mappings $G\in Q_F$ are not fixed. So $F$ is absolutely extremal, if

$$K[F] = K_0 := \inf_{G\in Q_F} K[G],$$

where $K[G]$ denotes the maximal dilatation of $G$.

By normality we conclude that $Q_F$ contains at least one absolutely extremal mapping. If there is only one such mapping, it is called uniquely absolutely extremal.

To derive necessary and sufficient conditions for a mapping to be absolutely extremal, we use the method of E. Reich [5] in connection with a similar problem. We consider the inverse mapping $f=F^{-1}$, $w=f(z)$, which is defined in $F(D-T)$, and its complex dilatation

$$\varkappa(z) = \frac{f_{\bar z}(z)}{f_z(z)} = -\frac{F_{\bar w}(w)}{F_w(w)}.$$

The following Banach space of holomorphic functions plays a basic role in this problem too:

$\mathscr{B} := \mathscr{B}_{F(\Gamma)} := \{\varphi|\varphi$ holomorphic in $D$, $\varphi dz^2$ real on $\delta D - F(\Gamma)$,

$$\|\varphi\| = \iint_D |\varphi(z)|\,dxdy < \infty\}.$$

First we derive a general necessary condition which leads to the possibility of the existence of a so-called "substantial" boundary point [4]. After this we derive a second necessary condition in the case when there is no such boundary point: Then

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there is a quadratic differential $\varphi\in\mathscr{B}$ such that $f$ is a Teichmüller mapping with complex dilatation $\varkappa=k_0\bar{\varphi}/|\varphi|\,(k_0=(K_0-1)/(K_0+1))$ and $F(D-T)$ is a domain which is the unit disk slit along some subarcs of vertical trajectories and connected subsets of the vertical critical graph of $\varphi$ (this is defined to be the union of all vertical critical trajectories and zeroes of $\varphi$). These subarcs then correspond to the set $T$. We restrict ourself to the case where at most denumerably many components of $D-F(D-T)$ are points. With this slight restriction the necessary conditions turn out to be sufficient for $F$ to be absolutely extremal. Moreover $F$ then is uniquely absolutely extremal.

## 2. The general necessary condition for absolute extremality

If $F$ is as above, then the complex dilatation $\varkappa$ of $f=F^{-1}$ is a measurable function in $F(D-T)$. We extend $\varkappa$ by setting

$$\varkappa(z)=0 \quad \text{for} \quad z\in D-F(D-T)$$

to get a measurable function in $D$. We prove the

Theorem 1. *If $F$ is absolutely extremal in $Q_F$, then*

$$\sup_{\substack{\varphi\in\mathscr{B}\\ \|\varphi\|=1}}\left|\iint_D \varkappa\varphi\,dx\,dy\right| = \|\varkappa\|_\infty. \tag{1}$$

*Proof.* We apply a technique employed by Krushkal [3] and elaborated by Reich [5]:

Let $k_0=\|\varkappa\|_\infty$. If $k_0=0$ nothing has to be shown. We assume $k_0>0$. If (1) does not hold, then

$$\sup_{\substack{\varphi\in\mathscr{B}\\ \|\varphi\|=1}}\left|\iint_D \varkappa\varphi\,dx\,dy\right| = a < k_0.$$

By the Hahn—Banach and Riesz representation theorems there exist a complex valued measurable function $\alpha(z)$ with

$$\iint_D \varkappa\varphi\,dx\,dy=\iint_D \alpha\varphi\,dx\,dy, \quad \text{for every} \quad \varphi\in\mathscr{B},$$

and $\|\alpha\|_\infty=a$.

We form $v(z)=\varkappa(z)-\alpha(z)$, $z\in D$. For $0<t<1/\|v\|_\infty$, let $g$ denote the quasiconformal selfmapping of $D$ with complex dilatation $tv$ and with $g(1)=1$, $g(i)=i$, $g(-1)=-1$. Then we put $h=f\circ g^{-1}$ and have

$$\frac{h_{\bar\zeta}}{h_\zeta}=\mu_h(\zeta)=\frac{\varkappa(z)-tv(z)}{1-t\bar{v}(z)\varkappa(z)}\,\frac{g_z(z)}{\overline{g_z(z)}}, \quad \zeta=g(z). \tag{2}$$

By the Fundamental variational lemma ([5], p. 107) there exists a $(1+c)/(1-c)$ quasiconformal mapping $g^*$ of $D$ onto itself whose boundary values agree with those of $g$ at $F(\Gamma)$ such that

$$c \leqq \frac{t^2\|v\|_\infty^2}{1-t\|v\|_\infty+t^2\|v\|_\infty^2}.$$

We put $\tilde{f}=h\circ g^*=f\circ g^{-1}\circ g^*$. Then $\tilde{f}$ is defined in $g^{*-1}\circ g\circ F(D-T)$ and maps this domain onto $D-T$ with boundary values of $f$ on $F(\Gamma)$. Therefore $\tilde{f}^{-1}$ belongs to $Q_F$. We want to show that

$$K[\tilde{f}]<K[F] \quad \text{for} \quad t>0, \quad \text{sufficiently small.}$$

This would contradict the absolute extremality of $F$.

Let

$$V_1=\left\{z\in F(D-T) \,\middle|\, |\varkappa(z)|\leqq \frac{k_0+a}{2}\right\},$$

$$V_2=\left\{z\in F(D-T) \,\middle|\, \frac{k_0+a}{2}<|\varkappa(z)|\leqq k_0\right\}.$$

Since $k_0>0$, it is immediately clear from (2) that there exist $\delta_1>0$, $t_1>0$, such that

$$|\mu_h(\zeta)|\leqq k_0-\delta_1 t, \quad \text{if} \quad 0\leqq t\leqq t_1, \quad \zeta\in g(V_1).$$

Expanding (2) we obtain for $\zeta\in g(V_2)$ and $\zeta=g(z)$,

$$|\mu_h(\zeta)| = |\varkappa(z)|-t\frac{1-|\varkappa(z)|^2}{|\varkappa(z)|}\operatorname{Re}\big(v(z)\bar{\varkappa}(z)\big)+O(t)^2.$$

Here $O(t^2)$ is uniform with respect to $z$ in $V_2$. We have $v\bar{\varkappa}=|\varkappa|^2-\alpha\bar{\varkappa}$ and therefore $\operatorname{Re} v\bar{\varkappa}\geqq|\varkappa|^2-|\alpha|\,|\varkappa|\geqq|\varkappa|(|\varkappa|-a)\geqq|\varkappa|(k_0-a)/2$, hence

$$\frac{1-|\varkappa|^2}{|\varkappa|}\operatorname{Re} v\bar{\varkappa}\geqq(1-|\varkappa|^2)\frac{k_0-a}{2}\geqq(1-k_0^2)\frac{k_0-a}{2}>0.$$

Therefore there exist $\delta_2>0$, $t_2>0$ such that

$$|\mu_h(\zeta)|\leqq k_0-\delta_2 t, \quad 0\leqq t\leqq t_2, \quad \zeta\in g(V_2).$$

We consider the effect of $g^*$, using $c=O(t^2)$, and obtain

$$K[\tilde{f}]<K[f]=K_0$$

for $t>0$, sufficiently small.

### 3. The existence of a substantial boundary point

Let $\zeta \in \Gamma$. Then the local dilatation $H_\zeta^\Gamma$ of $F|_{\partial D}$ in $\zeta$ with respect to $\Gamma$ is defined by

$$H_\zeta^\Gamma = \inf \{K[G] | G: U(\zeta) \xrightarrow{\text{qc}} U(F(\zeta)), \quad G|_{\Gamma \cap U(\zeta)} = F|_{\Gamma \cap U(\zeta)}\},$$

where the inf is taken over all such mappings $G$ and all open neighbourhoods $U(\zeta)$ of $\zeta$ and $U(F(\zeta))$ of $F(\zeta)$ in $\bar{D}$. For this definition and notation we refer to [4, 8, 1]. If $K_0 = H_\zeta^\Gamma$, then $\zeta$ is called a substantial boundary point. Since the function $\zeta \to H_\zeta^\Gamma$ is upper semicontinuous, there always exists a point $\zeta_0 \in \Gamma$ with $H_{\zeta_0}^\Gamma = \text{Max}_{\zeta \in \Gamma} H_\zeta^\Gamma$.

We assume now that $F$ fulfills the general necessary condition (1). It is possible that there is a sequence $\varphi_n \in \mathscr{B}$ $\|\varphi_n\| = 1$ such that $\varphi_n$ tends to zero locally uniformly in $D$ and

$$\left|\iint_D \varkappa \varphi_n \, dxdy\right| \to \|\varkappa\|_\infty, \quad n \to \infty.$$

We recall that $\varkappa = f_{\bar{z}}/f_z$ in $F(D-T)$ and $\varkappa = 0$ in $D - F(D-T)$. Let $f^*$ be the quasiconformal selfmapping of $D$ with complex dilatation $\varkappa$ in $D$ and $f^*(1) = 1$, $f^*(i) = i$, $f^*(-1) = -1$. By [1] we conclude that $f^*$ has a substantial boundary point on $F(\Gamma)$ and is hence extremal for its boundary values on $F(\Gamma)$. Since $f$ and $f^*$ have the same complex dilatation in $F(D-T)$, there is a conformal mapping $h$ in $D-T$ such that $h \circ f = f^*$ in $F(D-T)$. But because $K[f] = K[f^*]$ and because local dilatations are conformally invariant, this point is a substantial boundary point for $f$ too, and we conclude that $F$ has a substantial boundary point on $\Gamma$, i.e. $K[F] = \text{Max}_{\zeta \in \Gamma} H_\zeta^\Gamma$. We remark, that this forces $F$ to be absolutely extremal, since $K_0 \geqq \text{Max}_{\zeta \in \Gamma} H_\zeta^\Gamma$ clearly holds, hence $K_0 = K[F]$. Therefore the general necessary condition (1) together with the existence of a degenerating sequence $\varphi_n$ as above is a sufficient condition for $F$ to be absolutely extremal.

### 4. A second necessary condition in the case without substantial boundary point

We consider the case where $K_0 > \text{Max}_{\zeta \in \Gamma} H_\zeta^\Gamma$ and $F$ is absolutely extremal, hence the condition (1) is fulfilled. Since there is no substantial boundary point, every sequence $\varphi_n \in \mathscr{B}$, $\|\varphi_n\| = 1$ with

$$\iint_D \varkappa \varphi_n \, dxdy \to \|\varkappa\|_\infty = k_0, \quad n \to \infty,$$

contains a subsequence which tends to a function $\varphi \in \mathscr{B}$, $0 < \|\varphi\| \leqq 1$. It is known that then

$$\iint_D \varkappa \varphi \, dxdy = k_0 \|\varphi\|.$$

Therefore we conclude that $\varkappa = k_0\bar{\varphi}/|\varphi|$ a.e. in $D$. Since $k_0 > 0$ we conclude that the measure of $D - F(D-T)$ is zero (because $\varkappa = 0$ there) and that $F$ is a Teichmüller mapping.

But as can be seen by an example these conditions are not sufficient. We derive a second necessary condition: We claim that every component $\Lambda$ of $D \setminus F(D-T)$ is a subarc of a vertical trajectory of $\varphi$ or a connected subset of the vertical critical graph of $\varphi$. To prove this we may assume that $\Lambda$ contains at least two points. We consider again the quasiconformal mapping $f^*: D \to D$ with complex dilatation $\varkappa^* = k_0\bar{\varphi}/|\varphi|$ in $D$ and $f^*(1)=1$, $f^*(i)=i$, $f^*(-1)=-1$. The mapping $f^* \circ F$ is conformal in $D-T$. Let us consider the class of mappings from $D-\Lambda$ onto $D-f^*(\Lambda)$ with the boundary values of $f^*$ on $F(\Gamma)$. The mapping $f^*$ must be extremal in this class, otherwise we could replace $F^* := f^{*-1}$ by a quasiconformal mapping $G^*$: $D-f^*(\Lambda) \to D-\Lambda$ with $K[G^*] < K_0$ and boundary values as $f^{*-1}$. The mapping $G^* \circ f^* \circ F$ would contradict the absolute extremality of $F$.

Therefore $f^*$ is extremal in this class of mappings between these two ring domains. We have already seen that the lack of a substantial boundary point on $\Gamma$ for $F$ implies a lack of a substantial boundary point on $F(\Gamma)$ for $f^*$. We conclude by [1, 2] that $\varphi$ is real along $\Lambda$, i.e. by transformation of the ring domains $D-\Lambda$ and $D-f^*(\Lambda)$ onto annuli, the induced quadratic differential must be real along the interior boundary component. But $\Phi = \int \sqrt{\varphi}\, dz$ is conformal in neighbourhoods of points $z_0 \in \Lambda$ where $\varphi(z_0) \neq 0$, therefore $\Lambda$ must consist of horizontal and vertical arcs of $\varphi$ including zeroes. We show that horizontal arcs do not occur by using the following lemma.

Lemma. *Let $R$ be the square $\{x+iy \mid -1/2 < x < 1/2,\ 0 < y < 1\}$, $s$ its horizontal side $\{y=0\}$ and $A_K$ denote the affine stretch $A_K(x+iy) = Kx+iy$, $K>1$. Then there is a quasiconformal mapping $F_0$ defined in $A_K(R)$, such that $F_0(A_K(R)) \subset R$, $F_0$ agrees with $A_K^{-1}$ on $\delta A_K(R) - A_K(s)$, and the maximal dilatation of $F_0$ is less than $K$.*

*Proof.* We consider the right half $R^+ = R \cap \{x+iy \mid x>0\}$ of the square $R$. We choose $\Gamma = \delta A_K(R^+) \cap \{x+iy \mid x = K/2 \text{ or } y=1\}$. Let $F_0$ be an extremal mapping from $A_K(R^+)$ onto $R^+$ which agrees with $A_K^{-1}$ on $\Gamma$. If $F_0$ has a substantial boundary point with respect to $\Gamma$, then its maximal dilatation is less than $K$. Otherwise, if there is no substantial boundary point, then $F_0$ is uniquely determined and it is a Teichmüller mapping with associated quadratic differential of finite norm, which is real along $\delta A_K(R^+) - \Gamma$. Therefore $A_K^{-1} \neq F_0$ since the quadratic differential $-1$ of $A_K^{-1}$ has a pole of first order at the corner (0,0). We conclude that the maximal dilatation of $F_0$ is less than $K$.

Next we consider $F_0(0)$. This point must be on the interior of the vertical side $\{iy \mid 0<y<1\}$ of $R^+$, since otherwise the ratio of the moduli of the rectangles $A_K(R^+)(0, K/2, K/2+i, i)$ and $R^+(F_0(0), 1/2, 1/2+i, i)$ would be larger or equal to $K$. By reflection we extend $F_0$ to a mapping from $A_K(R)$ onto the slit rectangle $R - \{iy \mid 0 < y < \operatorname{Im} F_0(0)\}$, and this lemma is proved.

We assume now that $\Lambda$ contains a horizontal arc of $\varphi$. Since $f^*$ is locally equal to a conformal mapping $\Phi$ followed by an affine stretch $A_{K_0}$ and again by a conformal mapping $\Psi^{-1}$, we can choose a $\varphi$-rectangle $R^*$, which is mapped by $\Phi = \int \sqrt{\varphi}\, dz$ onto a square $R = \{x+iy \mid -1/2 < x < 1/2,\ 0 < y < 1\}$ such that a horizontal arc of $\varphi$ on $\Lambda$ is mapped onto $-1/2 < x < 1/2$. We apply the Lemma on $R$ and $A_{K_0}$. Therefore we can replace $F^* = f^{*-1}$ in $f^*(R^*)$ by $\Phi^{-1} \circ F_0 \circ \Psi$, i.e. we define

$$\overline{F^*} = \begin{cases} F^* & \text{in } f^*(D - \Lambda - R^*), \\ \Phi^{-1} \circ F_0 \circ \Psi & \text{in } f^*(R^*). \end{cases}$$

Because $F^*$ and $\Phi^{-1} \circ F_0 \circ \Psi$ agree on those three sides of $f^*(R^*)$ which are contained in $f^*(D - \Lambda)$, $\overline{F^*}$ is well-defined in $f^*(D - \Lambda)$, $K_0$-quasiconformal and not a Teichmüller mapping since its dilatation is not constant. In the class $Q_F$ the mapping $\overline{F^*} \circ f^* \circ F$ is absolutely extremal but not a Teichmüller mapping.[1)] This contradicts the first conclusion of §4, that an absolutely extremal mapping without substantial boundary point must be a Teichmüller mapping. We have proved the

Theorem 2. *If* $K_0 > \mathrm{Max}_{\zeta \in \Gamma} H_\zeta^\Gamma$, *then an absolutely extremal mapping* $F$ *is a Teichmüller mapping and the holomorphic quadratic differential* $\varphi$ *of the inverse mapping* $f = F^{-1}$ *is in* $\mathcal{B} = \mathcal{B}_{F(\Gamma)}$. *The set* $D - F(D - T)$ *has area-measure zero and each component of it is a subarc of a vertical trajectory of* $\varphi$ *or a connected subset of the vertical critical graph of* $\varphi$.

## 5. Sufficient conditions

We have already seen that condition (1) together with the existence of a degenerating sequence $\varphi_n$ as described in §3 is a sufficient condition for $F$ to be absolutely extremal. Now we want to show that with a slight restriction the necessary conditions of Theorem 2 are sufficient for absolute extremality.

Theorem 3. *With* $F$ *given as above, let* $f = F^{-1}$ *be a Teichmüller mapping with associated quadratic differential* $\varphi \in \mathcal{B} = \mathcal{B}_{F(\Gamma)}$ *and let the following conditions be fulfilled:*

a) $D - F(D - T)$ *has area-measure zero,*

b) *the components of* $D - F(D - T)$ *are subarcs of vertical trajectories of* $\varphi$ *or connected subsets of the vertical critical graph of* $\varphi$,

c) *at most denumerably many components of* $D - F(D - T)$ *are points. Then* $F$ *is uniquely absolutely extremal.*

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[1)] In the set of positive measure which is mapped by $F$ onto $F(D - T) \cap R^*$, the maximal dilatation is less than $K_0$.

This theorem is a consequence of the Main Inequality of Reich and Strebel [7], stated in the following form:

*Main Inequality. Let $\varphi \in \mathcal{B}_\Gamma$ and $L$ be a compact set in $D$, such that $D-L$ is a domain and each component $\Lambda$ of $L$ is a subarc of a vertical trajectory of $\varphi$ or a connected subset of the vertical critical graph of $\varphi$. Furthermore, suppose that the set of all vertical trajectories of $\varphi$ in $D-L$ which meet $L$ has area-measure zero. Then a quasiconformal mapping $g$ with complex dilatation $\varkappa$ which maps $D-L$ into $D$, $\delta D$ onto itself and keeps the points of $\Gamma$ pointwise fixed fulfills the inequality*

$$\|\varphi\| \leqq \iint_D |\varphi| \frac{\left|1-\varkappa \frac{\varphi}{|\varphi|}\right|^2}{1-|\varkappa|^2} \, dxdy. \tag{3}$$

*Proof.* We consider non-critical vertical trajectories $\beta$ of $\varphi$. If $\beta$ is contained in $D-L$, we have the length-inequality (see [9])

$$\int_\beta |\varphi(z)|^{1/2} |dz| \leqq \int_{g(\beta)} |\varphi(z)|^{1/2} |dz|. \tag{4}$$

As in the proof of the Main Inequality in [7], we consider each vertical strip $S$ of $\varphi$ in $D$. By our assumption all vertical trajectories $\beta$ of $\varphi$ up to a set of area-measure zero fulfill the length-inequality (4). Therefore the length-area method applied to each strip and then summed up yields

$$\Big(\iint_{D-L} |\varphi(z)| dxdy\Big)^2 \leqq \iint_{g(D-L)} |\varphi(z)| dxdy \cdot \iint_{D-L} |\varphi(z)| \frac{\left|1-\varkappa(z)\frac{\varphi(z)}{|\varphi(z)|}\right|^2}{1-|\varkappa(z)|^2} \, dxdy.$$

Using $g(D-L) \subset D$ and the fact that $L$ necessarily has area-measure zero gives (3). (The intersection of $L$ with each strip must have $\varphi$-area zero!).

## Proof of Theorem 3

Let $G$ be a mapping in $QF$. We apply inequality (3) for $g = G \circ F^{-1} = G \circ f$, where $\Gamma$ is replaced by $F(\Gamma)$. The mapping $G \circ f$ is defined in $F(D-T)$ and keeps $F(\Gamma)$ pointwise fixed, and $L = D - F(D-T)$. By assumption c), there are at most denumerably many vertical trajectories $\beta$ which meet components of $L$ which are points, so these trajectories cover only a set of area-measure zero. $L$ has area-measure zero by assumption a), so the vertical trajectories $\beta$ which meet components of $L$ which are vertical subarcs of positive length can only cover a horizontal length-measure zero in each strip. Therefore these trajectories cover only a set of area-measure zero too, and we may apply the Main Inequality:

$$\|\varphi\| \leqq \iint_D |\varphi| \frac{\left|1-\varkappa\frac{\varphi}{|\varphi|}\right|^2}{1-|\varkappa|^2} E[\varphi, F, G]\,dxdy. \tag{5}$$

Here $E[\varphi, F, G](z) = \dfrac{\left|1+\varkappa(z)\dfrac{\varkappa_1(w)}{\hat{\varkappa}(w)}\dfrac{\varphi(z)}{|\varphi(z)|}\left[\dfrac{|1-\bar{\varkappa}(z)\bar{\varphi}(z)|\,|\varphi(z)|}{|1-\varkappa(z)\varphi(z)|\,|\varphi(z)|}\right]\right|^2}{1-|\varkappa_1(w)|^2}$,

and $w=f(z)$, $\varkappa_1=G_{\bar{w}}/G_w$, $\hat{\varkappa}=F_{\bar{w}}/F_w$, $\varkappa=f_{\bar{z}}/f_z$. Then we use $\varkappa=k\bar{\varphi}/|\varphi|$ and $E\leqq K[G]$ and get

$$K[F] \leqq K[G],$$

i.e., $F$ is absolutely extremal.

By the procedure of [6] one concludes from (5) that if $K[G]=K[F]$, then

$$\varkappa_1 = \hat{\varkappa} \quad \text{a.e. in} \quad D-T,$$

and therefore $G\circ f$ is conformal.

Because $G\circ f$ keeps $F(\Gamma)$ pointwise fixed, we conclude at once: If $\Gamma$ does not only consist of single points, then $G\circ f$ is the identity. So then $F$ is uniquely absolutely extremal. We can see this also in the general case where $\Gamma$ contains at least three points.[2)] We assume $F$ and $G$ to be absolutely extremal mappings and necessarily $f=F^{-1}$ and $g=G^{-1}$ to be Teichmüller mappings with quadratic differentials $\varphi$ and $\psi$ in $\mathscr{B}$. So the conformal mapping $G\circ f$ consists of two Teichmüller mappings, and there is a quadratic differential $\varphi_0$ of finite norm in $D-T$, such that

$$\varkappa_1 = \hat{\varkappa} = k_0 \frac{\varphi_0}{|\bar{\varphi}_0|}.$$

We consider a component $\Lambda$ of $D-F(D-T)$, which necessarily is a subarc of a vertical trajectory or a connected subset of the vertical graph of $\varphi$. We map $D\setminus f(\Lambda)$ conformally onto an annulus (without loss of generality a punctured disk can be excluded), and in the conformal image of $D-T$ we get an induced quadratic differential $\tilde{\varphi}_0$. Along one boundary circle of this annulus $\tilde{\varphi}_0$ is real and the zeroes of $\varphi$ and the $\varphi$-length of the subarcs of $\Lambda$ determine the zeroes of $\tilde{\varphi}_0$ on this circle. But the same can be done with the corresponding component of $D-G(D-T)$ and the quadratic differentials $\psi$ and $\varphi_0$. Therefore, corresponding slits of $D\setminus F(D-T)$ and $D-G(D-T)$ have the same length in the metric of the quadratic differentials and the same configuration. Hence the conformal mapping $G\circ f$ can be extended homeomorphically in all of $D$. Therefore it must be the identity.

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2) We exclude the conformal case where non-uniqueness may occur.

## 6. An application: $H = \text{Max}\, H_\zeta$

We consider the special case, where $\Gamma = \delta D$, $T = \{w \mid |w| \leqq r\}$ for some $r$, $0 \leqq r < 1$, and the boundary homeomorphism of $\delta D$ onto itself is called $h$. The local dilatations of $h$ are denoted by $H_\zeta$. If $\text{Max}_{\zeta \in \partial D} H_\zeta$ is finite, it is known that $h$ is quasisymmetric, i.e. quasiconformally extendable in $D$ [1]. Then the dilatation $H$ of $h$ is defined in [8] by

$$H = \inf \{K[G] \mid G : U(\partial D) \xrightarrow{\text{qc}} U'(\partial D),\ G|_{\partial D} = h\}$$

where the inf is taken over all such mappings $G$ and all open neighborhoods $U(\delta D)$ and $U'(\delta D)$ of $\delta D$ in $\bar{D}$. The class of all extensions of $h$ in $D - T$ which map $D - T$ into $D$ is denoted by $Q_r$, in view of the dependence on $r$. For the same reason we call the absolutely extremal mapping in this class $F_r$ and its maximal dilatation $K_r$. Evidently we then have $H = \lim_{r \to 1} K_r$.

The function $r \to K_r$ is strictly monotonic decreasing as long as $K_r > \text{Max}_{\zeta \in \partial D} H_\zeta$. This can be seen by the preceding result: For each number $r$, $0 < r < 1$, where $K_r > \text{Max}\, H_\zeta$, $F_r$ is a uniquely determined Teichmüller mapping, and the associated quadratic differential $\varphi_r$ of its inverse mapping $f_r = F_r^{-1}$ is defined in all of $D$. Moreover $D \setminus F_r(D_r)$ $(D_r = \{w \mid r < |w| < 1\})$ consists of a subarc of a vertical trajectory of $\varphi_r$ or of a connected subset of the vertical critical graph of $\varphi_r$.

We remark that the Main Inequality (5) holds for $F = F_r$, $\varphi = \varphi_r$ and $G$, if $G$ is an extension of $h$ and if it is defined in a domain which contains $D_r$. We prove the

Theorem 4. *The dilatation $H$ of a quasisymmetric mapping $h$: $\delta D \to \delta D$ is equal to* $\text{Max}_{\zeta \in \partial D} H_\zeta$.

*Proof.* We may assume $K_r > \text{Max}_{\zeta \in \partial D} H_\zeta$, $0 < r < 1$. Let $\varphi_r$ and $\psi_r$ denote the quadratic differentials associated with the Teichmüller mapping $f_r$: $F_r(D_r) \to D_r$, which are normalised by

$$\|\varphi_r\| = \iint_D |\varphi_r(z)|\, dx\, dy = 1, \quad \|\psi_r\|_{D_r} = \iint_{D_r} |\psi_r(w)|\, du\, dv = K_r \quad (w = u + iv).$$

Then we have locally

$$f_r = \Psi_r^{-1} \circ A_{K_r} \circ \Phi_r,$$

where $\Phi_r(z) = \int \sqrt{\varphi_r(z)}\, dz$, $\Psi_r(w) = \int \sqrt{\psi_r(w)}\, dw$ and $A_{K_r}(\xi + i\eta) = K_r \xi + i\eta$.

For every measurable subset $E \subset F_r(D_r)$ we have

(6) $$K_r \iint_E |\varphi_r(z)|\, dx\, dy = \iint_{f_r(E)} |\psi_r(w)|\, du\, dv.$$

We extend the functions $\psi_r$ in $T$ by putting

$$\psi_r(w) = 0 \quad w \in D - D_r.$$

Then $\psi_r$ are measurable functions in $D$ with finite $L_1$-norm $\|\psi_r\| = \iint_D |\psi_r(w)|\, du\, dv$, i.e. $\psi_r \in L_1(D)$. If $r \to 1$, then $\psi_r$ tend to zero locally uniformly in $D$ and their

$L_1$-norms $\|\psi_r\| = K_r$ are bounded ($K_r \to H$!), hence the assumptions of Lemma 4.1 of [1] are fulfilled.

Let $I$ denote an open interval on $\delta D$ with endpoints $w_1$, $w_2$ and $\arg w_1 < \arg w < \arg w_2$ for $w \in I$. We then write $I = \overline{w_1 w_2}$, and define

$$S_I := \{w \in D \mid \arg w_1 < \arg w < \arg w_2\}.$$

For a given sequence $r_n \to 1$ we put

$$\theta(I) := \varlimsup_{n\to\infty} \iint_{S_I} |\psi_{r_n}(w)|\, du dv$$

and finally

$$\theta(\zeta) := \inf \{\theta(I) \mid \zeta \in I, \quad I \text{ an open interval on } \partial D\}$$

for every $\zeta \in \delta D$.

By Lemma 4.1 of [1] we can choose for given numbers $\varepsilon > 0$, $l > 0$ a subdivison $\{w_1, \dots, w_N\}$ of $\delta D$ and a sequence $r_n \to 1$ such that

$$\sum_{i=1}^{N} \theta(w_i) < \varepsilon \quad \text{and} \quad |\overline{w_{i-1} w_i}| < l.$$

(Here $|\overline{w_{i-1} w_i}|$ denotes the arc length of $\overline{w_{i-1} w_i}$.)

We apply the method for the proof of Theorem 4.1 in [1]. Let $H' > \operatorname{Max}_{\zeta \in \partial D} H_\zeta$. There is an $l > 0$ such that the restriction of $h$ to an arbitrary interval on $\delta D$ with length less than $l$ can be extended $H'$-quasiconformally in a neighbourhood in $\bar{D}$ of this interval. We apply Lemma 4.1 of [1] on $\varepsilon$ and $l$ and Corollary 4.1 of [1] on $\varepsilon$, $l$ and the sequence $\psi_{r_n}$. Therefore we can cut off some neighborhoods $G_i (i \leqq N)$ of subintervals in $\overline{w_{i-1} w_i}$ by Jordan arcs $\gamma_i$, such that $h$ can be extended $H'$-quasiconformally in the $G_i$, and for $D_\varepsilon = D - \bigcup_{i=1}^{N} G_i$ we have

$$\varlimsup_{n\to\infty} \iint_{D_\varepsilon} |\psi_{r_n}(w)|\, du dv \leqq \varepsilon. \tag{7}$$

The construction in [1] yields a quasiconformal extension $h_\varepsilon$ of $h$ in $\{|w| < 1\}$ which is $H'$-quasiconformal in $\bigcup_i G_i$ and $\hat{H}$-quasiconformal in a neighborhood $U(\delta D)$ of $\delta D$, where $\hat{H}$ does not depend on $\varepsilon$ (only on $\operatorname{Max}_{\zeta \in \partial D} H_\zeta$!).

We apply the Main Inequality (5) for $F = F_r$, $\varphi = \varphi_r$ and $G = h_\varepsilon$. Then

$$\frac{\left|1 - \varkappa \dfrac{\varphi}{|\varphi|}\right|^2}{1 - |\varkappa|^2} = \frac{1}{K_r},$$

and $E[\varphi_r, F_r, h_\varepsilon](z) \leqq D_{h_\varepsilon}(f_r(z))$, where $D_{h_\varepsilon}$ denotes the dilatation of $h_\varepsilon$. If $r$ is close to one, the image of $f_r$ is contained in $U(\delta D)$. Hence $D_{h_\varepsilon}$ is bounded by $\hat{H}$, and in $G_i' := f_r^{-1}(G_i)$ we have $D_{h_\varepsilon} \leqq H'$. Therefore, (5) yields

$$K_r \leqq H' \iint_{\bigcup_{i=1}^{N} G_i'} |\varphi_r(z)|\, dx dy + \hat{H} \iint_{D - \bigcup_{i=1}^{N} G_i'} |\varphi_r(z)|\, dx dy.$$

Because $f_r(F_r(D_r)-\bigcup_{i=1}^N G_i')=D_r-\bigcup_{i=1}^N G_i$, we have by (6) and the fact that $D-F_r(D_r)$ has measure zero,

$$K_r \iint_{D-\bigcup_{i=1}^N G_i'} |\varphi_r(z)|\,dxdy = \iint_{D_r-\bigcup_{i=1}^N G_i} |\psi_r(w)|\,dudv.$$

Since $D_r-\bigcup_{i=1}^N G_i\subset D_\varepsilon$ and $\iint_{\bigcup_{i=1}^N G_i'}|\varphi_r(z)|\,dx\,dy\leqq\|\varphi_r\|=1$, it follows that

$$K_r \leqq H'+\frac{\hat{H}}{K_r}\iint_{D_\varepsilon} |\psi_r(w)|\,dudv.$$

Putting $r=r_n$ and letting $r_n$ tend to one we get because of (7)

$$H \leqq H'+\frac{\hat{H}}{H}\varepsilon.$$

Since $\hat{H}$ does not depend on $\varepsilon$, we conclude that $H\leqq H'$. $H'$ was arbitrarily close to $\operatorname{Max}_{\zeta\in\partial D} H_\zeta$, and so

$$H \leqq \operatorname*{Max}_{\zeta\in\partial D} H_\zeta,$$

which finishes the proof.

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University of Minnesota
School of Mathematics
Minneapolis, Minnesota 55455
USA

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