Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 7, 1982, 349—367

FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC

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1. Introduction

We shall introduce a new notion of functions of uniformly bounded characteristic in the disk in terms of the Shimizu-Ahlfors characteristic function.

Let f be a function meromorphic in the disk $D=\{|z|<1\}$ in the complex plane $C=\{|z|<\infty\}$. Let $f^{\#}=|f'|/(1+|f|^2)$, $0<r<1$, and $z=x+iy$. Set

$$S(r, f) = (1/\pi) \iint_{|z|<r} f^{\#}(z)^2 dxdy.$$

The Shimizu-Ahlfors characteristic function of f,

$$T(r, f) = \int_0^r t^{-1} S(t, f)dt,$$

is a non-decreasing function of $r, 0<r<1$, so that

$$T(1, f) = \lim_{r\to 1} T(r, f) \leqq \infty,$$

exists.

Let BC be the family of f meromorphic in D with $T(1,f)<\infty$. Then, g meromorphic in D is of bounded (Nevanlinna) characteristic in D if and only if $g\in$BC. Letting $w\in D$ as a parameter we set

$$\varphi_w(z) = (z+w)/(1+\bar{w}z), \quad z\in D.$$

The inverse map of φ_w is then φ_{-w}. We set $f_w(z)=f(\varphi_w(z))$, $z\in D$. If $f\in$BC, then $f_w\in$BC for all $w\in D$.

Definition. *A meromorphic function f in D is said to be of uniformly bounded characteristic in D if and only if*

$$\sup_{w\in D} T(1, f_w) < \infty.$$

Denote by UBC the family of meromorphic functions in D of uniformly bounded characteristic in D. By UBC_0 we mean the family of functions f meromorphic in D

doi:10.5186/aasfm.1982.0733

such that

$$\lim_{|w|\to 1} T(1, f_w) = 0.$$

Then $UBC \subset BC$. However, the inclusion formula $UBC_0 \subset UBC$ is never obvious and needs a proof (Lemma 2.1.).

In Section 2 we propose a criterion (Theorem 2.2) for a meromorphic f to belong to UBC or UBC_0 in terms of the Green function of D.

In Section 3 we show that UBC is a subfamily of the family N of meromorphic functions normal in D in the sense of O. Lehto and K. I. Virtanen [5]; an analogue: $UBC_0 \subset N_0$, is also considered (Theorem 3.1). Use is made of J. Dufresnoy's lemma [1, p. 218], from which a criterion for f to be of N or of N_0 is obtained in terms of the spherical areas of the Riemannian images of the non-Euclidean disks (Lemma 3.2). We believe that this criterion itself is novel.

In Section 4 we consider Blaschke products

$$b(z) = z^k \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$$

$$(k \geqq 0 \text{ integer}; \ \sum (1 - |a_n|) < \infty).$$

If $f \in UBC$ is not identically zero, then f, as a member of BC, has the decomposition $b_1 g/b_2$, where $g \in BC$ is pole- and zero-free, and b_1 and b_2 are Blaschke products without common zeros. We observe that $g \in UBC$. One of the essential differences of UBC from BC is that UBC is not closed for summation and multiplication. This is a consequence of Theorem 4.2. For the proof, Blaschke products play fundamental roles.

In Section 5 holomorphic functions f in D are considered. A criterion for $f \in UBC$ or $f \in UBC_0$ is obtained in terms of the harmonic majorants (Theorem 5.1). In Theorem 5.2 we claim that if the image $f(D)$ is contained in a domain in $\boldsymbol{C}$ of a certain type, then $f \in UBC$.

If f is holomorphic and bounded in D, then $f \in UBC$. In Section 6 we show that if a meromorphic f satisfies the condition

$$\iint_D f^{\#}(z)^2 \, dx dy < \infty,$$

then $f \in UBC$. Thus, if f is "bounded" in a natural sense, then $f \in UBC$.

In the final section, Section 7, we consider BMOA and VMOA functions. These are, roughly speaking, holomorphic functions in D whose boundary values are of bounded or vanishing mean oscillation on the circle $\{|z|=1\}$ in the sense of F. John and L. Nirenberg [4] or of D. Sarason [7], respectively. The main result is that $BMOA \subset UBC$ and $VMOA \subset UBC_0$.

To extend the notion of UBC and UBC_0 (as well as BMOA and VMOA) to Riemann surfaces R is possible. Some arguments in D are also available on R. We hope we can publish a systematic study of UBC and UBC_0 on R in the near future.

2. Criteria

First we show, as was promised in Section 1, that $\mathrm{UBC}_0 \subset \mathrm{UBC}$; for the proof, use is made of

Theorem 2.1. *If* $f \in \mathrm{BC}$, *then for each* $\varrho, 0<\varrho<1$,

$$\sup_{|w|<\varrho} T(1, f_w) < \infty.$$

Proof. Set for $w \in D$ and for $\lambda, 0<\lambda<1$,

$$\Delta(w, \lambda) = \{z \in D;\ |w-z|/|1-\bar{w}z| < \lambda\};$$

this is the non-Euclidean disk of the non-Euclidean center w and the non-Euclidean radius $(1/2)\log[(1+\lambda)/(1-\lambda)]$. The change of variable $\zeta=\xi+i\eta=\varphi_w(z)$ then yields that

$$S(\lambda, f_w) = (1/\pi) \iint_{|z|<\lambda} f_w^{\#}(z)^2 dx dy = (1/\pi) \iint_{\Delta(w,\lambda)} f^{\#}(\zeta)^2 d\xi d\eta; \tag{2.1}$$

hereafter, $(f_w)^{\#}=f_w^{\#}$ and $(\varphi_w)'=\varphi_w'$ for short.

Fix $\varrho, 0<\varrho<1$, and then let w satisfy $|w|<\varrho$. For $r_0 \equiv 1/2<r<1$, we shall estimate upwards the characteristic function

$$T(r, f_w) = T(r_0, f_w) + \int_{r_0}^{r} t^{-1} S(t, f_w)\, dt \equiv \alpha+\beta$$

by a constant independent of r and w.

For the α-part we note that

$$|z| < r_0 \Rightarrow |\varphi_w(z)| \leqq (|w|+|z|)/(1+|zw|) < R_0 \equiv (r_0+\varrho)/(1+r_0\varrho).$$

Then, for $|z|<r_0$,

$$f_w^{\#}(z) = f^{\#}(\varphi_w(z))|\varphi_w'(z)| \leqq \big[\max_{|\zeta|\leqq R_0} f^{\#}(\zeta)\big](1-\varrho r_0)^{-2} \equiv K < \infty$$

by the continuity of $f^{\#}$. Consequently,

$$f_w^{\#}(z) \leqq K \quad \text{for} \quad |z| < t < r_0,$$

so that the inequality $S(t, f_w) \leqq K^2 t^2$ yields

$$\alpha \leqq K^2/8. \tag{2.2}$$

To estimate β we notice that, for $0<t<1$,

$$\Delta(w, t) \subset \{|z| < u\}, \quad u \equiv (t+\varrho)/(1+\varrho t).$$

By (2.1), together with $R \equiv (r+\varrho)/(1+r\varrho) > R_0$, we obtain

$$\beta \leqq \int_{r_0}^{r} t^{-1} S(u, f)\, dt = \int_{R_0}^{R} C(u, \varrho) u^{-1} S(u, f)\, du,$$

where

$$C(u, \varrho) = \frac{u(1-\varrho^2)}{(u-\varrho)(1-\varrho u)} \leqq 2/(R_0-\varrho)$$

because $\varrho<R_0<u<1$ for $r_0<t<r$. Therefore

$$\beta \leqq 2T(R, f)/(R_0-\varrho) \leqq 2T(1, f)/(R_0-\varrho),$$

which, together with (2.2), completes the proof.

Lemma 2.1. $\mathrm{UBC}_0 \subset \mathrm{UBC}$.

Proof. For $f \in \mathrm{UBC}_0$ there exists δ, $0<\delta<1$, such that $T(1, f_w)<1$ in $\{\delta<|w|<1\}$. Then $f \in \mathrm{BC}$ because f is the composed function $f=f_\varrho \circ \varphi_{-\varrho}$ for $\varrho=(1+\delta)/2$ with $f_\varrho \in \mathrm{BC}$. It now follows from Theorem 2.1 that

$$K \equiv \sup_{|w|<\varrho} T(1, f_w) < \infty,$$

whence

$$\sup_{w \in D} T(1, f_w) \leqq K+1.$$

Remark. Theorem 2.1 also yields:
For f meromorphic in D to be of UBC *it is necessary and sufficient that*

$$\limsup_{|w| \to 1} T(1, f_w) < \infty.$$

The Green function of D with pole at $w \in D$ is given by

$$G(z, w) = \log |(1-\bar{w}z)/(z-w)| = -\log |\varphi_{-w}(z)|, \quad z \in D.$$

We now propose the main result in the present section.

Theorem 2.2. *Let f be meromorphic in D. Then the following propositions hold.*
(I) $f \in \mathrm{UBC}$ *if and only if*

$$\sup_{w \in D} \iint_D f^{\#}(z)^2 G(z, w)\,dxdy < \infty. \tag{2.3}$$

(II) $f \in \mathrm{UBC}_0$ *if and only if*

$$\lim_{|w| \to 1} \iint_D f^{\#}(z)^2 G(z, w)\,dxdy = 0. \tag{2.4}$$

For the proof we need

Lemma 2.2. *For f meromorphic in D and for* $0<r\leqq 1$ *we have*

$$T(r, f) = (1/\pi) \iint_{|z|<r} f^{\#}(z)^2 \log (r/|z|)\,dxdy. \tag{2.5}$$

Proof. For $0<r<1$, we let X_r be the characteristic function of the disk $\{|z|<r\}$, namely, $X_r(z)=1$ for $|z|<r$, $X_r(z)=0$ for $r \leqq |z|<1$.

It suffices to prove (2.5) for $0<r<1$. For, if (2.5) is true for $0<r<1$, then

$$T(r, f) = (1/\pi)\iint_D f^{\#}(z)^2 X_r(z)\log(r/|z|)\,dxdy.$$

Since $0 \leqq X_r(z)\log(r/|z|) \nearrow \log(1/|z|)$ as $r\to 1$ at each $z\in D$, (2.5) for $r=1$ follows.

Now, for $0<r<1$,

$$\int_0^r t^{-1}X_t(z)\,dt = \log(r/|z|) \quad \text{if} \quad |z|<r,$$
$$= 0 \quad \text{if} \quad r \leqq |z| < 1,$$

so that (2.5) is a consequence of

$$T(r, f) = (1/\pi)\iint_D f^{\#}(z)^2 \Big[\int_0^r t^{-1}X_t(z)\,dt\Big]\,dxdy.$$

Proof of Theorem 2.2. Since $f_w^{\#} = (f^{\#}\circ\varphi_w)|\varphi_w'|$, it follows from Lemma 2.2, together with the change of variable $\zeta=\varphi_w(z)$, that

$$T(1, f_w) = (1/\pi)\iint_D f^{\#}(\zeta)^2 \log(1/|\varphi_{-w}(\zeta)|)\,d\xi d\eta. \tag{2.6}$$

This completes the proof of Theorem 2.2.

Remark. For $f\in\mathrm{BC}$, the function $T(1, f_w)$ of $w\in D$ is well defined. The identity (2.6) shows that $T(1, f_w)$ is lower semicontinuous with respect to $w\in D$. Actually, $T(1, f_w)$ is a Green's potential in D of the measure in the differential form

$$(1/\pi)f^{\#}(\zeta)^2\,d\xi\,d\eta.$$

3. Normal meromorphic functions

Let N be the family of meromorphic functions f in D such that

$$\sup_{z\in D}(1-|z|^2)f^{\#}(z) < \infty,$$

and let N_0 be the family of meromorphic functions f in D such that

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0.$$

Each $f\in\mathrm{N}$ is normal in D in the sense of Lehto and Virtanen [5], and vice versa. By the continuity of $f^{\#}$, the inclusion formula $\mathrm{N}_0\subset\mathrm{N}$ is easily established.

Theorem 3.1. *The following inclusion formulae hold:*

$$\mathrm{UBC}\subset\mathrm{N} \quad \textit{and} \quad \mathrm{UBC}_0\subset\mathrm{N}_0;$$

both are shown to be sharp.

We begin with Dufresnoy's result.

LEMMA 3.1 [1, Lemma, p. 218] (See [3, Theorem 6.1, p. 152].). *Suppose that f is meromorphic in D and that there exists r, $0<r<1$, such that $S(r,f)<1$. Then*

$$f^{\#}(0)^2 \leqq S(r,f)r^{-2}[1-S(r,f)]^{-1}.$$

Note that our Riemann sphere is of radius 1/2, touching $\boldsymbol{C}$ from above at 0, while Dufresnoy considered the sphere of radius 1 bisected by $\boldsymbol{C}$.

LEMMA 3.2. *Let f be meromorphic in D. Then the following propositions hold.* (I) $f\in \mathrm{N}$ *if and only if there exists* $r, 0<r<1$, *such that*

$$\sup_{w\in D} S(r, f_w) = (1/\pi)\sup_{w\in D} \iint_{\Delta(w,r)} f^{\#}(z)^2\,dxdy < 1. \tag{3.1}$$

(II) $f\in \mathrm{N}_0$ *if and only if there exists r, $0<r<1$, such that*

$$\lim_{|w|\to 1} S(r, f_w) = \lim_{|w|\to 1} \iint_{\Delta(w,r)} f^{\#}(z)^2\,dxdy = 0. \tag{3.2}$$

In the proof of Theorem 3.1, the "if" parts of (I) and (II) are needed. Lemma 3.2 (I) gives a new criterion for f to be normal in D.

There exist a nonnormal holomorphic function f and $r>0$ for which $S(r,f_w)<1$ for each $w\in D$; see [12, Remark, p. 226]. This function f must satisfy

$$\sup_{w\in D} S(r, f_w) = 1.$$

Proof of Lemma 3.2. For the proof of (I) we first assume that $f\in \mathrm{N}$ with

$$(1-|z|^2)f^{\#}(z) \leqq K < \infty \quad \text{for all} \quad z\in D.$$

Then, for each $w\in D$,

$$(1-|z|^2)f_w^{\#}(z) = (1-|\varphi_w(z)|^2)f^{\#}(\varphi_w(z)) \leqq K, \quad z\in D.$$

Therefore, for a small r, $0<r<1$, with $K^2r^2/(1-r^2)<1$,

$$\pi S(r, f_w) = \iint_{|z|<r} f_w^{\#}(z)^2\,dxdy \leqq 2\pi K^2 \int_0^r \varrho(1-\varrho^2)^{-2}d\varrho = \pi K^2 r^2/(1-r^2),$$

whence (3.1) follows. Conversely, let the supremum in (3.1) be S. Then, by Lemma 3.1, together with $x/(1-x)\nearrow$ as $0\leqq x\nearrow 1$,

$$(1-|w|^2)^2 f^{\#}(w)^2 = f_w^{\#}(0)^2 \leqq r^{-2}S(1-S)^{-1}$$

for all $w\in D$, whence $f\in \mathrm{N}$.

To prove (II) we first suppose that $f\in \mathrm{N}_0$. Then, for each $\varepsilon>0$, there exists δ, $0<\delta<1$, such that

$$\delta < |z| < 1 \Rightarrow (1-|z|^2)f^{\#}(z) < \varepsilon^{1/2}. \tag{3.3}$$

Choose r such that $0<r<\delta$ and $r^2/(1-r^2)<1$. Then

(3.4) $$\delta < (r+\delta)/(1+r\delta) < |w| < 1 \Rightarrow \Delta(w, r) \subset \{\delta < |z| < 1\}$$

because

$$\delta < (|w|-r)/(1-r|w|) < |z| \quad \text{for} \quad z \in \Delta(w, r).$$

The formula (2.1), together with (3.3) and (3.4), yields that

$$\pi S(r, f_w) = \iint_{\Delta(w,r)} f^{\#}(z)^2 dxdy \leqq \varepsilon\pi r^2/(1-r^2);$$

in fact, the non-Euclidean area of $\Delta(w, r)$ is $\pi r^2/(1-r^2)$. Therefore,

$$S(r, f_w) < \varepsilon \quad \text{for} \quad (r+\delta)/(1+r\delta) < |w| < 1.$$

Conversely, suppose that (3.2) holds. Then, for each $\varepsilon>0$, there exists δ, $0<\delta<1$, such that

$$S(r, f_w) < \varrho \quad \text{for} \quad \delta < |w| < 1,$$

where $0<\varrho<1$ and $\varrho r^{-2}(1-\varrho)^{-1}<\varepsilon/2$. By Lemma 3.1,

$$(1-|w|^2)^2 f^{\#}(w)^2 = f^{\#}_w(0)^2 < \varepsilon \quad \text{for} \quad \delta < |w| < 1,$$

which completes the proof.

Remark. The condition (3.1) can be replaced by

$$\limsup_{|w|\to 1} S(r, f_w) < 1.$$

Proof of Theorem 3.1. Suppose that $f \in \mathrm{UBC}$. Then (2.3) of Theorem 2.2 holds; we denote by A the supremum in (2.3). Choose r, $0<r<1$, such that

(3.5) $$A/[\pi \log(1/r)] < 1.$$

Since, for each $w \in D$, the formula (2.1) yields that

$$A \geqq \iint_{\Delta(w,r)} f^{\#}(z)^2 G(z, w)\, dxdy \geqq \pi \log(1/r) S(r, f_w),$$

it follows from Lemma 3.2, (I), together with (3.5), that $f \in \mathrm{N}$. Therefore $\mathrm{UBC} \subset \mathrm{N}$. The proof of $\mathrm{UBC}_0 \subset \mathrm{N}_0$ is similar.

To prove the sharpness it suffices to observe the existence of $f \in \mathrm{N}_0 - \mathrm{BC}$. Then $f \in \mathrm{N}_0 - \mathrm{UBC}_0$ and $f \in \mathrm{N} - \mathrm{UBC}$. Consider the gap series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in D,$$

where the sequence $\{n_k\}$ of positive integers satisfies $n_{k+1}/n_k \geqq q > 1$ for all $k \geqq 1$. Suppose that

$$\sum_{k=1}^{\infty} |a_k|^2 = \infty \quad \text{and} \quad \lim_{k\to\infty} |a_k| = 0.$$

Then it is known (see [10, Corollary, p. 34]) that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)|=0$$

and f does not have finite radial limit a.e. on $\{|z|=1\}$. Therefore, $f\in N_0$, yet $f\notin BC$.

4. Blaschke products

First of all we prove

Lemma 4.1. *Suppose that* $f\in UBC$ *and that* g *is a rational function. Then* $g\circ f\in UBC$.

Proof. There exists $K>0$ such that

$$g^{\#}(z)\leqq K/(1+|z|^2) \quad \text{for all} \quad z\in C.$$

Since $(g\circ f)_w=g\circ f_w$, it follows that

$$(g\circ f)_w^{\#}=(g\circ f_w)^{\#}=(g^{\#}\circ f_w)|f_w'|\leqq Kf_w^{\#}.$$

Consequently,

$$T(1,(g\circ f)_w)\leqq K^2T(1,f_w),$$

which shows that $g\circ f\in UBC$.

As we shall observe later in Theorem 4.2, UBC is not closed for summation and multiplication. The family UBC resembles N at this point. However, a decisive difference between UBC and N is that, each non-zero $f\in UBC$, as a member of BC, admits the decomposition

$$f=b_1g/b_2, \tag{4.1}$$

where $g\in BC$ has neither pole nor zero in D, and b_1 (b_2, respectively) is the Blaschke product whose zeros are precisely the zeros (poles, respectively) of f, the multiplicity being counted. For simplicity we shall call b_2 the polar Blaschke product of f. If f is pole-free, then $b_2\equiv 1$.

We shall show that g of (4.1) is a member of UBC if $f\in UBC$ as a corollary of

Theorem 4.1. *Let* $f\in UBC$, *and let* b *be the polar Blaschke product of* f. *Then* $bf\in UBC$.

For the proof of Theorem 4.1, we first deduce the formula (4.4) in Lemma 4.2 by making use of a precise description of the first step in the Nevanlinna theory. The adjective "precise" in the preceding sentence means that there is no Landau's notation $O(1)$.

Let

$$I(r,f)=(1/4\pi)\int_0^{2\pi}\log(1+|f(re^{it})|^2)\,dt,$$

and let $n(r,f)$ $(n^*(r,f))$ be the number of the poles of f in the disk $\{|z|<r\}$ (on the circle $\{|z|=r\}$), the multiplicity being counted, $0<r<1$. Delete from $\{|z|<r\}$ the closed disks, with poles on the closed disk $\{|z|\leqq r\}$ as centers, and with common small radii $\varepsilon>0$, apply the Green formula to $\log(1+|f|^2)$ in the resulting domain, and, finally, let $\varepsilon\to 0$. Then, for $0<r<1$, the identity $\Delta\log(1+|f|^2)=4f^{\#2}$ (except for poles of f) yields

$$r(d/dr)I(r,f) = S(r,f)-n(r,f)-(1/2)n^*(r,f). \tag{4.2}$$

Arrange $r>0$ with $n^*(r,f)\neq 0$ as

$$0<r_0<\ldots<r_j<r_{j+1}<\ldots<1.$$

For each $R, r_0\leqq R<1$, there is a j such that $r_j\leqq R<r_{j+1}$. Divide both sides of (4.2) by r, and integrate from $\varepsilon, 0<\varepsilon<r_0$, to R, to obtain

$$I(R,f)-I(\varepsilon,f) = \int_\varepsilon^R r^{-1}S(r,f)\,dr-\int_\varepsilon^R r^{-1}n(r,f)\,dr, \tag{4.3}$$

where

$$\int_\varepsilon^R = \int_\varepsilon^{r_0}+\Big(\sum_{p=1}^{j}\int_{r_{p-1}}^{r_p}\Big)+\int_{r_j}^R .$$

Lemma 4.2. *Let b be the polar Blaschke product of $f\in$ BC. Then,*

$$T(1,f) = I(1,f)-(1/2)\log\big[|b(0)|^2+\lim_{z\to 0}|b(z)f(z)|^2\big], \tag{4.4}$$

where

$$I(1,f) = \lim_{r\to 1} I(r,f).$$

Proof. Suppose that 0 is a pole of order $k\geqq 0$. Then

$$\int_\varepsilon^{r_0} r^{-1}n(r,f)\,dr = k(\log r_0-\log\varepsilon)$$

and, in case $k=0$,

$$I(\varepsilon,f)\to(1/2)\log\big(1+|f(0)|^2\big),$$

as $\varepsilon\to 0$, while in case $k>0$,

$$I(\varepsilon,f)\sim -k\log\varepsilon+\log A$$

as $\varepsilon\to 0$, where

$$A = \lim_{z\to 0}|z^k f(z)|.$$

Therefore, $\varepsilon\to 0$, and then $R\to 1$ in (4.3) yield

$$T(1,f) = I(1,f)-(1/2)\log\big(1+|f(0)|^2\big)-\log|b(0)|$$

if $k=0$, while if $k>0$, then

$$T(1, f) = I(1, f) - \log A - \log \left[\lim_{z\to 0} |z^{-k} b(z)|\right]$$
$$= I(1, f) - \log \left[\lim_{z\to 0} |b(z) f(z)|\right],$$

which completes the proof.

As an immediate consequence of (4.4) in Lemma 4.2 we obtain

Lemma 4.3. *If f is holomorphic and bounded, $|f| \leqq K$, in D, then*

$$T(1, f_w) \leqq I(1, f_w) \leqq (1/2) \log (1+K^2) \quad \textit{for all} \quad w \in D.$$

Therefore $f \in \mathrm{UBC}$.

Lemma 4.4. *Let b be the polar Blaschke product of $f \in \mathrm{BC}$. Then for each constant α, $|\alpha|=1$,*

$$T(1, \alpha b f) \leqq T(1, f) + (1/2) \log 2. \tag{4.5}$$

Proof. By (4.4) in Lemma 4.2, applied to f with $g=bf$, we obtain

$$T(1, f) = I(1, f) - (1/2) \log \left(|b(0)|^2 + |g(0)|\right)^2,$$

and it is apparent that $(\alpha g)^{\#} = g^{\#}$. Therefore,

$$T(1, \alpha b f) = T(1, g) = I(1, g) - (1/2) \log \left(1 + |g(0)|^2\right)$$
$$\leqq I(1, b) + I(1, f) - (1/2) \log \left(1 + |g(0)|^2\right) \leqq (1/2) \log 2 + T(1, f) + (1/2) \log A,$$

where

$$A = \left(|b(0)|^2 + |g(0)|^2\right) / \left(1 + |g(0)|^2\right) \leqq 1.$$

We thus obtain (4.5).

Proof of Theorem 4.1. Let b^w be the polar Blaschke product of f_w. Then $|b^w| = |b_w|$ in D. Actually, defining

$$\psi(z, a) = |z-a| / |1 - \bar{a} z|, \quad z \in D,$$

for $a \in D$, one obtains

$$\psi(z, \varphi_{-w}(a)) = \psi(\varphi_w(z), a).$$

Since $a \in D$ is a pole of order $k \geqq 1$ of f if and only if $\varphi_{-w}(a)$ is a pole of order $k \geqq 1$ of f_w, it follows from the expression

$$|b(z)| = \prod_{j=1}^{\infty} \psi(z, a_j)$$

that

$$|b^w(z)| = \prod_{j=1}^{\infty} \psi(z, \varphi_{-w}(a_j)) = |b \circ \varphi_w(z)|$$

for all $z \in D$.

Now, there is a constant α, $|\alpha|=1$, such that $b_w=\alpha b^w$. Set $g=bf$. Then $g_w=b_wf_w=\alpha b^w f_w$. It follows from (4.5) in Lemma 4.4, applied to f_w, that

$$T(1, g_w) \leqq T(1, f_w)+(1/2)\log 2 \quad \text{for all} \quad w\in D.$$

Consequently, $g\in$ UBC.

Corollary 4.1. *If* $f\in$ UBC *with* (4.1), *then* $g\in$ UBC. *The converse is false.*

Proof. By Theorem 4.1, $b_1g=b_2f\in$ UBC. By Lemma 4.1,

$$h \equiv 1/(b_1 g) = (1/g)/b_1 \in \text{UBC}.$$

Again, by Theorem 4.1, $1/g=b_1h\in$ UBC, whence, by Lemma 4.1 once more, $g\in$ UBC. To prove that the converse is false we remember that there exist Blaschke products b_1 and b_2 with no common zero in D such that the quotient b_1/b_2 is not normal in D; see, for example, [11] and [13]. Therefore, $g\equiv 1\in$ UBC, yet $f\equiv b_1g/b_2\notin$ UBC because $f\notin$ N.

Finally in this section we prove

Theorem 4.2.

(I) *There exist* $f\in$ UBC *and* $g\in$ UBC *such that* $fg\notin$ N.

(II) *There exist* $f\in$ UBC *and* $g\in$ UBC *such that* $f+g\notin$ N.

Combined with the inclusion formula UBC$\subset$N, Theorem 4.2 asserts that UBC is not closed for the product and the sum.

Lemma 4.5. *Let* $f\in$ UBC, *and let* g *be a holomorphic function bounded from below and above in* D:

$$0 < m \leqq |g| \leqq M < \infty.$$

Then $fg\in$ UBC.

Proof. By Lemma 4.3, $g\in$ UBC. Set

$$K = (1+M^2)/\min(1, m^2).$$

Then,

$$1+|fg|^2 \geqq K^{-1}(1+|f|^2)(1+|g|^2),$$

whence

$$(fg)^{\#2} \leqq \frac{|f'g|^2+2|ff'gg'|+|fg'|^2}{K^{-2}(1+|f|^2)^2(1+|g|^2)^2} \leqq K^2(f^{\#2}+2f^{\#}g^{\#}+g^{\#2}). \tag{4.6}$$

On the other hand, the Cauchy inequality, together with (2.1), yields

$$\Big[\iint_{\Delta(w,r)} f^{\#}(z)g^{\#}(z)\,dx\,dy\Big]^2 \leqq \pi^2 S(r, f_w)\, S(r, g_w)$$

for all $w\in D$ and all r, $0<r<1$. Consequently, by (2.1), together with (4,6), we obtain

$$\begin{aligned}\pi S(r, (fg)_w) &\leqq \pi K^2\{S(r, f_w)+S(r, g_w)+2[S(r, f_w)S(r, g_w)]^{1/2}\}\\ &\leqq 2\pi K^2[S(r, f_w)+S(r, g_w)].\end{aligned}$$

Therefore

$$T(1, (fg)_w) \leqq 2K^2[T(1, f_w)+T(1, g_w)],$$

whence $fg \in \mathrm{UBC}$.

Proof of Theorem 4.2. Again we consider the Blaschke products b_1 and b_2 such that b_1/b_2 is not normal. To prove (I), set $f=b_1$ and $g=1/b_2$. Then $f\in\mathrm{UBC}$ and $g\in\mathrm{UBC}$, yet $fg\notin\mathrm{N}$. To prove (II) we set $f=2/b_2$ and $g=(b_1-2)/b_2$. Then $f\in\mathrm{UBC}$. Since $1<|b_1-2|<3$ and $1/b_2\in\mathrm{UBC}$, it follows from Lemma 4.5 that $g\in\mathrm{UBC}$. However, $f+g=b_1/b_2\notin\mathrm{N}$.

5. Harmonic majorant

Let $u\not\equiv-\infty$ be a subharmonic function in a domain $\mathscr{D}\subset C$. We call h a harmonic majorant of u in $\mathscr{D}$ if h is harmonic and $u\leqq h$ in $\mathscr{D}$. If u has a harmonic majorant in $\mathscr{D}$, then u has the least harmonic majorant $u^\wedge$ in $\mathscr{D}$, that is, $u^\wedge$ is a harmonic majorant of u in $\mathscr{D}$ and $u^\wedge\leqq h$ for each harmonic majorant h of u in $\mathscr{D}$. In the special case $\mathscr{D}=D$, $u^\wedge$ is given by the limiting function

$$u^\wedge(z) = \lim_{r\to 1}(1/2\pi)\int_0^{2\pi} u(re^{it})\frac{r^2-|z|^2}{|re^{it}-z|^2}\,dt, \quad z\in D.$$

Theorem 5.1. *Let f be holomorphic in D. Then the following criteria hold for the subharmonic function $F=(1/2)\log(1+|f|^2)$ in D.*

(I) $f\in\mathrm{UBC}$ *if and only if*

$$\sup_{w\in D}(F^\wedge(w)-F(w)) < \infty.$$

(II) $f\in\mathrm{UBC}_0$ *if and only if*

$$\lim_{|w|\to 1}(F^\wedge(w)-F(w)) = 0.$$

Lemma 5.1. *Suppose that a subharmonic function u in D has a harmonic majorant in D. Then $(u\circ\varphi_w)^\wedge=u^\wedge\circ\varphi_w$ for each $w\in D$.*

Proof. Since $u^\wedge\circ\varphi_w$ is a harmonic majorant of $u\circ\varphi_w$ for each $w\in D$, it follows that

$$(u\circ\varphi_w)^\wedge \leqq u^\wedge\circ\varphi_w. \tag{5.1}$$

Apply (5.1) to $v=u\circ\varphi_w$ and φ_{-w} instead of u and φ_w, respectively. Then

$$u^\wedge = (v\circ\varphi_{-w})^\wedge \leqq v^\wedge\circ\varphi_{-w},$$

whence

$$u^\wedge\circ\varphi_w \leqq v^\wedge = (u\circ\varphi_w)^\wedge.$$

Combining this with (5.1) we have the equality.

Proof of Theorem 5.1. (I) There exists $K>0$ for $f\in$UBC such that $K\geqq T(1,f_w)$ for all $w\in D$. On the other hand, by Lemma 5.1,

$$I(1,f_w)=(F\circ\varphi_w)\hat{}(0)=F\hat{}\circ\varphi_w(0)=F\hat{}(w),$$

whence

$$K\geqq T(1,f_w)=I(1,f_w)-(1/2)\log(1+|f_w(0)|^2)=F\hat{}(w)-F(w)\quad\text{for all}\quad w\in D.$$

The converse is also true, so that (I) is established. The proof of (II) is similar.

Remarks. (a) We may replace F in the UBC criterion (I) by $\log^+|f|=\max(\log|f|,0)$ because

$$\log^+|f|\leqq F\leqq\log^+|f|+(1/2)\log 2.$$

(b) Suppose that $f\in$BC is pole-free. Since $F\hat{}$ exists and since the identity

$$T(1,f_w)=F\hat{}(w)-F(w),\quad w\in D,$$

is also true for the present f,

$$F(w)=F\hat{}(w)-T(1,f_w),\quad w\in D,$$

represents the Riesz decomposition of the subharmonic function F which has a harmonic majorant in D. The potential $T(1,f_w)$ is continuous in the present case because the same is true of F and $F\hat{}$. The problem is that $T(1,f_w)$ is or is not continuous depending on whether f admits poles in D. If $T(1,f_w)$ is proved to be continuous in D for each meromorphic $f\in$BC, then Theorem 2.1 is immediate.

A subdomain $\mathscr{D}$ of $\boldsymbol{C}$ is called a UBC domain if each holomorphic function f in D which assumes only the values in $\mathscr{D}$ is of UBC. We next consider a criterion for a holomorphic f in D to be of UBC.

Theorem 5.2. *Suppose that the function* $H(z)=(1/2)\log(1+|z|^2)$ *has a harmonic majorant in* $\mathscr{D}\subset\boldsymbol{C}$, *and suppose that* $H\hat{}-H$ *is bounded in* $\mathscr{D}$. *Then* $\mathscr{D}$ *is a* UBC *domain. The converse is true under the condition that the universal covering surface of* $\mathscr{D}$ *is conformally equivalent to* D.

Proof. Let $F=(1/2)\log(1+|f|^2)$ for a holomorphic $f\colon D\to\mathscr{D}$. The first half follows from $F=H\circ f$, $F\hat{}\leqq H\hat{}\circ f$ and Theorem 5.1 (I). To prove the converse we let p be the projection of the universal covering surface $\mathscr{D}^\infty$ of $\mathscr{D}$ onto $\mathscr{D}$, and let q be a conformal homeomorphism from D onto $\mathscr{D}^\infty$. Then $f=p\circ q\in$UBC. Since $F=(1/2)\log(1+|f|^2)$ and $F\hat{}$ both are automorphic with respect to the covering transformations, namely, automorphic with respect to a group of conformal homeomorphisms from D onto D, $H\hat{}(z)=F\hat{}(f^{-1}(z))$ is well-defined in $\mathscr{D}$. Consequently,

$$F\hat{}-F\leqq K\quad\text{in}\quad D\quad\text{by Theorem 5.1 (I)}$$

implies

$$H\hat{}-H\leqq K\quad\text{in}\quad\mathscr{D}.$$

6. Riemannian image of finite spherical area

In this short section we prove

Theorem 6.1. *Suppose that a meromorphic function f in D satisfies*

$$\iint_D f^{\#}(z)^2 dxdy < \infty.$$

Then $f \in \mathrm{UBC} \cap \mathrm{N}_0$.

See the remark at the end of the next section.

Proof of Theorem 6.1. For the proof of $f \in \mathrm{N}_0$ we set

$$A = \iint_D f^{\#}(z)^2 dxdy,$$

and we fix r, $0<r<1$, arbitrarily. Since

$$\lim_{\delta \to 1} \iint_{\delta<|z|<1} f^{\#}(z)^2 dxdy = 0,$$

it follows that, for each $\varepsilon>0$, there exists δ, $0<\delta<1$, such that

$$\iint_{\delta<|z|<1} f^{\#}(z)^2 dxdy < \pi\varepsilon.$$

Since

$$(r<)(r+\delta)/(1+\delta r) < |w| < 1 \Rightarrow \Delta(w, r) \subset \{\delta < |z| < 1\},$$

it follows that

$$\pi S(r, f_w) = \iint_{\Delta(w,r)} f^{\#}(z)^2 dxdy < \pi\varepsilon,$$

or $S(r, f_w) < \varepsilon$. By Lemma 3.2 (II), f is a member of N_0.

For the proof of $f \in \mathrm{UBC}$, we first note that

$$(1-|z|^2) f_w^{\#}(z) = (1-|\varphi_w(z)|^2) f^{\#}(\varphi_w(z)) \leqq K$$

for all z and w in D, because $f \in \mathrm{N}$. Fix R, $0<R<1$, and then let $R<r<1$. We have then

$$T(r, f_w) = T(R, f_w) + \int_R^r t^{-1} S(t, f_w) dt \equiv \alpha + \beta.$$

By (2.5) in Lemma 2.2,

(6.1)

$$\pi\alpha = \iint_{|z|<R} f_w^{\#}(z)^2 \log(R/|z|) dxdy \leqq 2\pi K^2 \int_0^R \varrho(1-\varrho^2)^{-2} \log(R/\varrho) d\varrho \equiv C_1(R) < \infty.$$

On the other hand, since

$$\pi t^{-1} S(t, f_w) \leqq R^{-1} A \quad \text{for} \quad R < t < r,$$

it follows that

$$\pi\beta \leqq (1-R) R^{-1} A \equiv C_2(R) < \infty,$$

which, together with (6.1), yields that

$$\pi \sup_{w\in D} T(1, f_w) \leqq C_1(R)+C_2(R).$$

This completes the proof of Theorem 6.1.

Remark. There exists a holomorphic function f in D such that $f\notin \mathrm{N}$, yet

$$\iint_D |f'(z)|^p\,dxdy < \infty \quad \text{for all} \quad p,\ 0<p<2;$$

see [9]. Therefore $f\notin \mathrm{UBC}$, yet

$$\iint_D f^{\#}(z)^p\,dxdy < \infty \quad \text{for all} \quad p,\ 0<p<2. \tag{6.2}$$

In other words, condition (6.2) for meromorphic f does not necessarily assure that $f\in \mathrm{UBC}$.

7. BMOA and VMOA

Let $|J|$ be the linear Lebesgue measure of a subarc J of the circle $\Gamma=\{|z|=1\}$. For each f of complex $L^1(\Gamma)$ we set

$$J(f) = (1/|J|)\int_J f(e^{it})\,dt,$$

called the mean of f on J. Then f is said to have bounded mean oscillation on Γ, in notation, $f\in \mathrm{BMO}(\Gamma)$, if and only if the mean oscillation $J(|f-J(f)|)$ of f on J, the mean of $|f-J(f)|$ on J, remains bounded as J ranges over all subarcs of Γ. Furthermore, f is said to have vanishing mean oscillation on Γ, in notation, $f\in \mathrm{VMO}(\Gamma)$, if and only if $f\in \mathrm{BMO}(\Gamma)$ and for each $\varepsilon>0$ there exists $\delta>0$ such that

$$|J|<\delta \Rightarrow J(|f-J(f)|)<\varepsilon.$$

For the properties of $\mathrm{BMO}(\Gamma)$ and $\mathrm{VMO}(\Gamma)$, see [6] and [8].

Let H^p be the Hardy class consisting of f holomorphic in D such that $|f|^p$ has a harmonic majorant in D, where $0<p<\infty$. Each $f\in H^p$ has a boundary value $f(e^{it})\in \boldsymbol{C}$, being the angular limit, at a.e. point $e^{it}\in\Gamma$ and $f(e^{it})$ is of $L^p(\Gamma)$. For $f\in H^p$, the norm $\|f\|_p\geqq 0$ is defined by

$$\|f\|_p^p = (|f|^p)\hat{\ }(0) = (1/2\pi)\int_0^{2\pi} |f(e^{it})|^p\,dt.$$

By definition ([8, p. 90]; see also [2, Theorem 3.1, p. 34]),

$$\mathrm{BMOA} = \{f\in H^1;\ f(e^{it})\in \mathrm{BMO}(\Gamma)\},$$

$$\mathrm{VMOA} = \{f\in H^1;\ f(e^{it})\in \mathrm{VMO}(\Gamma)\}.$$

It is known (see [8, Theorem, p. 36]) that if $f\in\mathrm{BMOA}$, then for each p, $1\leqq p<\infty$,

$$\sup_{w\in D}(|f-f(w)|^p)\hat{\ }(w) < \infty. \tag{7.1}$$

An immediate consequence of (7.1) is that $f\in H^p$ for all p, because, for $p\geqq 1$,

$$(|f|^p)^\frown \leqq 2^{p-1}(|f-f(0)|^p)^\frown+2^{p-1}|f(0)|^p, \tag{7.2}$$

where $(|f-f(0)|^p)^\frown$ exists by (7.1), namely,

$$(|f-f(0)|^p)^\frown(0)<\infty.$$

Conversely, if $f\in H^1$ and if (7.1) is valid for a certain p, $1\leqq p<\infty$, then $f\in$BMOA.

Therefore, a holomorphic function f in D is of BMOA if and only if

$$\sup_{w\in D}\|f_w-f(w)\|_2<\infty. \tag{7.3}$$

Actually, setting $g=f-f(w)$ and considering Lemma 5.1, one calculates that

$$\|f_w-f(w)\|_2^2=(|g\circ\varphi_w|^2)^\frown(0)=(|g|^2\circ\varphi_w)^\frown(0)$$
$$=(|g|^2)^\frown\circ\varphi_w(0)=(|g|^2)^\frown(w)=(|f-f(w)|^2)^\frown(w).$$

A straightforward modification of the proof of [8, Theorem, p. 36] yields the VMOA version:

If $f\in$VMOA, then for each p, $1\leqq p<\infty$,

$$\lim_{|w|\to 1}(|f-f(w)|^p)^\frown(w)=0. \tag{7.4}$$

Conversely, if $f\in$BMOA and (7.4) for a certain p, $1\leqq p<\infty$, holds, then $f\in$VMOA.

However, it must be emphasized that the condition $f\in$BMOA in the preceding sentence can be dropped. Namely, if a holomorphic f in D satisfies (7.4) for a p, $1\leqq p<\infty$, then $f\in$VMOA. To ascertain this it suffices to show that $f\in$BMOA under the condition (7.4). First, there exists δ, $0<\delta<1$, such that

$$\delta<|w|<1\Rightarrow(|f-f(w)|^p)^\frown(w)<1. \tag{7.5}$$

On replacing 0 in (7.2) by $r_0=(1+\delta)/2$, we observe that $f\in H^p$. Now, for w, $|w|\leqq r_0$,

$$(|f-f(w)|^p)^\frown(w)\leqq 2^{p-1}(|f|^p)^\frown(w)+2^{p-1}|f(w)|^p.$$

The right-hand side is apparently bounded for $|w|\leqq r_0$, which, together with (7.5), shows that (7.1) is valid. Consequently, $f\in$BMOA.

By the observation in the preceding paragraph we can now conclude that a holomorphic function f in D is of VMOA if and only if

$$\lim_{|w|\to 1}\|f_w-f(w)\|_2=0, \tag{7.6}$$

a VMOA counterpart of (7.3).

We propose

Theorem 7.1. *The inclusion formulae*

$$\text{BMOA}\subset\text{UBC}\quad\textit{and}\quad\text{VMOA}\subset\text{UBC}_0$$

hold.

For the proof we first consider the holomorphic analogue $T^*(r,f)$ of the Shimizu-Ahlfors characteristic function basing on the identity

$$\Delta(|f|^2) = 4|f'|^2 \tag{7.7}$$

for f holomorphic in D instead of $\Delta \log(1+|f|^2)=4f^{\#2}$.

For f holomorphic in D we set

$$M(r,f) = \left[(1/2\pi)\int_0^{2\pi} |f(re^{it})|^2 dt\right]^{1/2}, \quad 0 < r \leqq 1,$$

where $M(1,f)=\lim_{r\to 1} M(r,f)$. If $f\in H^2$, then $\|f\|_2=M(1,f)$. Since (7.7) holds, the Green formula yields

$$r(d/dr)[M(r,f)^2] = A(r,f),$$

where

$$A(r,f) = (2/\pi)\iint_{|z|<r} |f'(z)|^2 dxdy$$

is the holomorphic analogue of $S(r,f)$. Setting

$$T^*(r,f) = \int_0^r t^{-1}A(t,f)dt, \quad 0 < r \leqq 1,$$

one obtains the formula

$$M(r,f)^2 - |f(0)|^2 = T^*(r,f), \quad 0 < r \leqq 1. \tag{7.8}$$

Applying (7.8) to $g=f_w-f(w)$ $(g(0)=0)$, one observes from (7.3) and (7.6), together with

$$T^*(r,g) = T^*(r,f_w)$$

that

$$f\in \mathrm{BMOA} \quad \textit{if and only if} \quad \sup_{w\in D} T^*(1,f_w) < \infty,$$

while

$$f\in \mathrm{VMOA} \quad \textit{if and only if} \quad \lim_{|w|\to 1} T^*(1,f_w) = 0.$$

Since

$$T^*(r,f) = (2/\pi)\iint_{|z|<r} |f'(z)|^2 \log(r/|z|)dxdy$$

for f holomorphic in D and for $0<r\leqq 1$, the analogue of (2.5) holds, and it is now an easy exercise to obtain the following holomorphic counterpart of Theorem 2.2.

Lemma 7.1. *Let f be holomorphic in D. Then the following propositions hold.*
(I) $f\in$BMOA *if and only if*

$$\sup_{w\in D}\iint_D |f'(z)|^2 G(z,w)dxdy < \infty.$$

(II) $f \in \mathrm{VMOA}$ *if and only if*

$$\lim_{|w|\to 1} \iint_D |f'(z)|^2 G(z, w)\,dxdy = 0.$$

Lemma 7.1 (I) is known [6, Proposition 7.2.13, p. 85]. Theorem 7.1 now follows from Theorem 2.2 and Lemma 7.1, because $|f'| \geqq f^{\#}$ for f holomorphic in D.

Remark. At this point we remark that if f is holomorphic in D and if

$$\iint_D |f'(z)|^2\,dxdy < \infty,$$

then $f \in \mathrm{VMOA}$. By the theorem at the bottom of [8, p. 50] it suffices to show that

$$\lim_{|J|\to 0} \mu_f(R(J))/|J| = 0,$$

where $|J| < \pi$, and $R(J)$ is the annular trapezoid

$$\{z \in D;\ z/|z| \in J,\ 1-|z| \leqq |J|/(2\pi)\},$$

and

$$\mu_f(R(J)) = \iint_{R(J)} (1-|z|)|f'(z)|^2\,dxdy.$$

Since $1-|z| \leqq |J|(2\pi)$, it follows that

$$\mu_f(R(J)) \leqq [|J|/(2\pi)] \iint_{R(J)} |f'(z)|^2\,dxdy \leqq [|J|/(2\pi)] \iint_{1-|J|/(2\pi)<|z|<1} |f'(z)|^2\,dxdy.$$

Therefore $\mu_f(R(J))/|J| \to 0$ as $|J| \to 0$.

A natural question then arises: Can the conclusion in Theorem 6.1 be replaced by $f \in \mathrm{UBC}_0$?

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Received 1 June 1982