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HARMONIC MAJORIZATION OF SUBHARMONIC FUNCTIONS IN UNBOUNDED DOMAINS

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1. Introduction

Let Ω be a domain in Euclidean space $\mathbf{R}^n$, where $n \geqq 2$. Its closure and boundary in $\mathbf{R}^n$ are respectively denoted by $\bar{\Omega}$ and $\partial\Omega$. We define a class of functions on $\bar{\Omega}$ by saying that $s \in \mathscr{S}(\bar{\Omega})$ if

(i) s is defined on $\bar{\Omega}$,
(ii) s is subharmonic in Ω,
(iii) $\limsup_{\substack{X \to Y \\ X \in \Omega}} s(X) = s(Y) < +\infty \quad (Y \in \partial\Omega)$.

We note immediately that if $s \in \mathscr{S}(\bar{\Omega})$, then $s < +\infty$ on $\bar{\Omega}$, s is upper semicontinuous (u.s.c.) on $\bar{\Omega}$ and so locally bounded above thereon.

First consider the case where Ω is bounded. (We will use the notation of Helms [12, Chapter 8] for concepts related to the Perron—Wiener—Brelot solution of the generalized Dirichlet problem.) If $s \in \mathscr{S}(\bar{\Omega})$ then, since s is bounded above in $\bar{\Omega}$, it follows that $s \in \mathscr{L}_s$ and that $\mathscr{U}_s$ contains a finite constant function. Thus

$$s \leqq \underline{H}_s^\Omega \leqq \overline{H}_s^\Omega < +\infty$$

in Ω. But, since the restriction of s to $\partial\Omega$ is u.s.c., $\overline{H}_s^\Omega = \underline{H}_s^\Omega$ from [12, Theorem 8.13], and so H_s^Ω exists and is a harmonic majorant of s in Ω.

However, if Ω is unbounded and $s \in \mathscr{S}(\bar{\Omega})$, it need no longer be the case that s has a harmonic majorant in Ω. An obvious step is to take an expanding sequence (Ω_m) of bounded subdomains of Ω such that $\cup \Omega_m = \Omega$, and to consider the limit of the increasing sequence of harmonic functions $(H_s^{\Omega_m})$. Unfortunately, except in the very simplest of cases, the expression for the harmonic measure of Ω_m will be too complicated to give a criterion for harmonic majorization in Ω that could be considered either elegant or useful. Instead, research in the half-space (Kuran [13] and Armitage [3]) and infinite strip (Brawn [8] and Armitage and Fugard [5]) has

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shown that special "means" over the boundaries of suitable subdomains of Ω can be used. The general outline of the methods in [5] may be employed in other domains but each case would require separate treatment. The purpose of this paper is to develop a fairly general theory, of which the above examples are actually special cases. The general results are not difficult to prove once a suitable expression for the mean has been identified (see $\mathcal{M}(s, x)$ below). The great benefit is that, having established these, applications to individual domains can be made very quickly — for example, contrast § 8 with [3] or § 9 with [5].

2. General results

2.1. We denote points of $\mathbf{R}^n$ by X, Y, Z, P or Q. When appropriate, X will be written in terms of its co-ordinates

$$X = (x_1, \ldots, x_n) = (X', x_n)$$

where $X' \in \mathbf{R}^{n-1}$. We recall that a bounded domain $\omega \in \mathbf{R}^n$ is called a Lipschitz domain if $\partial\omega$ can be covered by right circular cylinders whose bases have positive distance from $\partial\omega$, and corresponding to each cylinder L, there is a co-ordinate system $(\tilde{X}', \tilde{x}_n)$ with $\tilde{x}_n$-axis parallel to the axis of L, a function $\varphi: \mathbf{R}^{n-1} \to \mathbf{R}$ and a real number c such that

$$|\varphi(\tilde{X}') - \varphi(\tilde{Y}')| \leqq c|\tilde{X}' - \tilde{Y}'|$$

for all $\tilde{X}', \tilde{Y}' \in \mathbf{R}^{n-1}$,

$$L \cap \omega = \{X \in L : \tilde{x}_n > \varphi(\tilde{X}')\},$$

and

$$L \cap \partial\omega = \{X \in L : \tilde{x}_n = \varphi(\tilde{X}')\}.$$

From now on Ω will always denote an unbounded domain. Let ν be a fixed Radon measure with compact support $E \subset \bar{\Omega}$, and suppose that for each positive real number x there is a corresponding open set W_x whose intersection Ω_x with Ω is a bounded Lipschitz domain satisfying

(a) $E \subseteqq \cap \bar{\Omega}_x$,

(b) $\bar{\Omega} = \cup \bar{\Omega}_x$,

(c) $x < y \Rightarrow \bar{\Omega}_x \subset W_y \cap \bar{\Omega}$.

The sets $\partial\Omega_x \cap \Omega$ and $\partial\Omega_x \cap \partial\Omega$ will be abbreviated to σ_x and τ_x, and $\mu_{x,X}$ will be used to denote harmonic measure on $\partial\Omega_x$ with respect to $X \in \Omega_x$.

If $s \in \mathcal{S}(\bar{\Omega})$, then, for each positive real number x, let

$$H_{s,x}(X) = H_s^{\Omega_x}(X) - \int_{\tau_1} s(X)\, d\mu_{1,X}(Y),$$

$$I_{s,x}(X) = \int_{\sigma_x} s(Y)\, d\mu_{x,X}(Y),$$

and

$$K_{s,x}(X) = \int_{\tau_x \setminus \tau_1} s(Y)\, d\mu_{x,X}(Y),$$

which are clearly harmonic in $\Omega_{\min(x,1)}$. Let h_* denote a strictly positive harmonic function in Ω_1 which vanishes on τ_1 (clearly such functions exist, for example the Green function for Ω_2 with its pole in $\Omega_2 \setminus \Omega_1$; this is a consequence of the regularity of Ω_2, which follows from the Zaremba cone criterion, [12, Theorem 8.27]).

Lemma 1. *The quotients* $H_{s,x}/h_*$, $I_{s,x}/h_*$ *and* $K_{s,x}/h_*$ *can be continuously extended to* $W_y \cap \bar{\Omega}$, *where* $y = \min(x, 1)$.

The extended functions of Lemma 1 will be denoted respectively by $\mathscr{H}_{s,x}$, $\mathscr{I}_{s,x}$ and $\mathscr{K}_{s,x}$ and are used to define

$$\mathscr{M}(s, x) = \int_E \mathscr{H}_{s,x}(X)\, d\nu(X),$$

$$\mathscr{N}(s, x) = \int_E \mathscr{I}_{s,x}(X)\, d\nu(X),$$

$$\mathscr{P}(s, x) = \int_E \mathscr{K}_{s,x}(X)\, d\nu(X).$$

2.2. The following result is now almost immediate.

Theorem 1. (i) *If* $s \in \mathscr{S}(\bar{\Omega})$, *then* $\mathscr{M}(s, x)$ *is an increasing, real-valued function of* x.

(ii) *If also* $s \leqq 0$ *on* $\partial\Omega$, *then the same is true of* $\mathscr{N}(s, x)$.

(iii) *If* h *is harmonic in* Ω *and continuous on* $\bar{\Omega}$, *then* $\mathscr{M}(h, x)$ *is a constant function of* x.

Theorem 2. *If* $s \in \mathscr{S}(\bar{\Omega})$ *and* $\mathscr{M}(s, x)$ *is bounded above on* $(0, +\infty)$, *then* s *has a harmonic majorant in* Ω.

It will be shown later that the converse of Theorem 2 is false. However, defining $\mathscr{T}(\bar{\Omega})$ to be the class of functions subharmonic in an open set containing $\bar{\Omega}$, we obtain the following.

Theorem 3. *Let* $s \in \mathscr{T}(\bar{\Omega})$. *Then* s *has a harmonic majorant in* Ω *if and only if* $\mathscr{M}(s, x)$ *is bounded above on* $(0, +\infty)$.

Corollary. *Let* $s \in \mathscr{T}(\bar{\Omega})$ *and* $s \geqq 0$ *in* Ω. *Then* s *has a harmonic majorant in* Ω *if and only if* $\mathscr{N}(s, x)$ *and* $\mathscr{P}(s, x)$ *each tend to a finite limit as* $x \to +\infty$.

Finally we give two results which are very simple to prove.

Theorem 4. *If* h *is non-negative and harmonic in* Ω *and continuous in* $\bar{\Omega}$, *then*

$$\lim_{x \to +\infty} \mathscr{P}(h, x) < +\infty.$$

Theorem 5. *If* $s \in \mathcal{S}(\bar{\Omega})$, $s \leqq 0$ *on* $\partial\Omega$ *and*

$$\liminf_{x\to+\infty} \mathcal{N}(s^+, x) = 0, \tag{1}$$

then $s \leqq 0$ *in* Ω.

Theorem 5 is a generalization of a Phragmén—Lindelöf type of result (for example, see [1, Corollary to Theorem 1] and [14, Theorem 4]). One generalization of this type of theorem has already been given in [4] for arbitrary unbounded domains. There the mean employed is given by

$$\lambda_m(s^+) = \liminf_{X\to P} I_{s,m}(X)/G_{\Omega_1}(Q, X),$$

where (Ω_m) is an expanding sequence of bounded domains such that $\cup\Omega_m = \Omega$, and where we denote by P either a point of Ω_1 or a minimal Martin boundary point of Ω_1, by Q a fixed point in $\Omega_1 \setminus \{P\}$, and by G_{Ω_1} the Green kernel of Ω_1. It will later be clear that for certain domains Ω such as the whole space, half-space, and infinite cone, where the natural choice of E is a singleton, $\lambda_m(s^+)$ and $\mathcal{N}(s^+, m)$ can coincide. However, for other domains such as the infinite strip and infinite cylinder, these means differ significantly, and $\mathcal{N}(s^+, x)$ appears to be more natural, coinciding (in the case of the strip) with means that have previously been studied, [5].

We point out that if $E = \{P\} \subset \Omega$ and one of the subdomains, which we label as Ω_1, satisfies $\bar{\Omega}_1 \subset \Omega$, then the mean $\mathcal{M}(s, x)$ is merely a multiple of $H_s^{\Omega_x}(P)$.

The results of this section will be proved in §§ 3—6.

3. Proofs of Lemma 1 and Theorem 1

3.1. We will make use of the following results.

Theorem A. *Let* Ω' *be a bounded Lipschitz domain and* f *be a resolutive boundary function on* $\partial\Omega'$. *If* f *is continuous (in the extended sense) at a point* $Y \in \partial\Omega'$, *then*

$$\lim_{X\to Y} H_f^{\Omega'}(X) = f(Y).$$

Theorem B. *If* h_1 *and* h_2 *are positive harmonic functions on a bounded Lipschitz domain* Ω' *vanishing on a relatively open subset* A *of* $\partial\Omega'$, *then* h_1/h_2 *can be continuously extended to a strictly positive function defined on* $\Omega' \cup A$.

Theorem A is due to Armitage [2, Theorem 2]. Note that it asserts more than the regularity of Ω'. Theorem B is a slight modification of [6, Theorem 2].

3.2. We now prove Lemma 1. First we show that $H_{s,x}$ vanishes on $W_y \cap \partial\Omega$ If $x \geqq 1$, then define

$$F(X) = \begin{cases} H_s^{\Omega_x}(X) & \text{if } X \in \Omega_x \\ s(X) & \text{if } X \in \partial\Omega_x. \end{cases}$$

From [9, p. 98 (ε)] it follows that $H_F^{\Omega_1}=H_s^{\Omega_x}$ in Ω_1, and so

(2) $$H_{s,x}(X) = H_s^{\Omega_x}(X) - \int_{\tau_1} s(Y)\,d\mu_{1,X}(Y) = \int_{\sigma_1} F(Y)\,d\mu_{1,X}(Y).$$

Alternatively, if $x<1$, then similarly

(3) $$H_{s,x}(X) = \int_{\sigma_x} \Big\{s(Y) - \int_{\tau_1} s(Z)\,d\mu_{1,Y}(Z)\Big\}\,d\mu_{x,X}(Y).$$

Thus, in either case, $H_{s,x}$ is a harmonic function in Ω_y which, by Theorem A, vanishes on $W_y \cap \partial\Omega$.

If we now rewrite (2) as

$$H_{s,x}(X) = \int_{\sigma_1} F^+(Y)\,d\mu_{1,X}(Y) - \int_{\sigma_1} F^-(Y)\,d\mu_{1,X}(Y),$$

and treat (3) analogously, it follows by two applications of Theorem B that $H_{s,x}/h_*$ can be continuously extended to $W_y \cap \bar{\Omega}$.

In the case of the quotients $I_{s,x}/h_*$ and $K_{s,x}/h_*$, only the latter part of the above argument is required.

3.3. Theorem 1 is straightforward to prove. If $w<x$, then, with F as in § 3.2,

$$H_s^{\Omega_w}(X) \leqq H_F^{\Omega_w}(X) = F(X) \quad (X \in \Omega_w)$$

and so $\mathcal{M}(s,x)$ is an increasing function of x. If also $s \leqq 0$ on $\partial\Omega$, then

$$\int_{\tau_x} s(Y)\,d\mu_{x,X}(Y) \leqq \int_{\tau_w} s(Y)\,d\mu_{w,X}(Y)$$

since $\mu_{x,X} \geqq \mu_{w,X}$ on τ_w, and so $I_{s,x} \geqq I_{s,w}$ in Ω_w, proving (ii). Part (iii) is trivial.

4. Proof of Theorem 2

4.1. The following results will be required.

Theorem C. *Let Ω' be a bounded Lipschitz domain of which P is a fixed point, A be a relatively open subset of $\partial\Omega'$, and W' be a subdomain of Ω' satisfying $\partial\Omega' \cap \partial W' \subseteqq A$. Then there is a constant c such that, if h_1 and h_2 are two positive harmonic functions in Ω' vanishing on A and $h_1(P)=h_2(P)$, then $h_1(X) \leqq c h_2(X)$ for all $X \in W'$.*

Lemma 2. *Let $s \in \mathcal{S}(\bar{\Omega})$ and define*

$$s_x(X) = \begin{cases} H_s^{\Omega_x}(X) & \text{if } X \in \Omega_x \\ s(X) & \text{if } X \in \bar{\Omega} \setminus \Omega_x. \end{cases}$$

Then $s_x \in \mathcal{S}(\bar{\Omega})$.

Theorem C can be found in [10, Theorem 4] or [17, Theorem 1]. The proof of Lemma 2 is very similar to that of [5, Lemma 2], but is given below for the sake of completeness.

To establish the subharmonicity of s_x in Ω, it suffices to show upper semi-continuity and the mean-value inequality at points of σ_x. Since Ω_x is regular and the restriction of s to $\partial\Omega_x$ is u.s.c. and bounded above,

(4) $$\limsup_{X\to Y} H_s^{\Omega_x}(X) \leqq s(Y) \quad (Y\in\partial\Omega_x)$$

[12, Lemma 8.20, Theorem 8.22]. Upper semicontinuity is now proved, and the mean-value inequality is an immediate consequence of the fact that $s\leqq H_s^{\Omega_x}$ in Ω_x. In fact, combining the latter inequality with (4) yields

$$\limsup_{X\to Y} s_x(X) = s(Y) = s_x(Y) \quad (Y\in\partial\Omega),$$

whence $s_x\in\mathcal{S}(\bar{\Omega})$.

4.2. We will now prove Theorem 2. For each positive integer k let s_k be as in Lemma 2. From Theorem 1 (i) and the hypothesis of Theorem 2 it follows that

(5) $$\mathcal{M}(s_k, 1) \leqq \mathcal{M}(s_k, k) = \mathcal{M}(s, k) \leqq \sup_k \mathcal{M}(s, k) < +\infty.$$

Since, for each k, $(s_i)_{i\geqq k}$ is an increasing sequence of harmonic functions in Ω_k, either $\lim s_i$ is harmonic in Ω or is identically equal to $+\infty$. The theorem will follow if the latter case is shown not to hold.

Suppose $\lim s_i\equiv+\infty$. Then there exists k_0 such that $s_{k_0}>s_2$ in Ω_2. Fixing $P\in\Omega_1$, Theorem C shows that there is a positive constant c such that, for $k\geqq k_0$ and $X\in\Omega_{1/2}$,

$$H_{s_k,1}(X)-H_{s_2,1}(X) \geqq c\{h_*(X)/h_*(P)\}\{H_{s_k,1}(P)-H_{s_2,1}(P)\}$$

and so

$$\mathcal{M}(s_k, 1)-\mathcal{M}(s_2, 1) \geqq c'\int_{\sigma_1}\{s_k(Y)-s_2(Y)\}\,d\mu_{1,P}(Y) \to +\infty \quad (k\to+\infty).$$

This contradicts (5) and so the result follows.

5. Proof of Theorem 3 and Corollary

5.1. The "if" part is contained in Theorem 2 since $\mathcal{T}(\bar{\Omega})\subseteq\mathcal{S}(\bar{\Omega})$.

5.2. Conversely, s is subharmonic in an open set W containing $\bar{\Omega}$ and has a harmonic majorant in Ω. It follows (compare [5, Lemma 3]) that the function s_0, equal in Ω to the least harmonic majorant of s there, and equal in $W\setminus\Omega$ to s, is also subharmonic in W. From [9, Chapter IX, § 6],

$$H_s^{\Omega_x}(X) \leqq H_{s_0}^{\Omega_x}(X) = s_0(X) \quad (X\in\Omega_x),$$

whence

$$\mathscr{M}(s, x) \leqq \mathscr{M}(s_0, x) = \mathscr{M}(s_0, 1) \quad (x \in (0, +\infty)),$$

as required.

5.3. We come now to the proof of the Corollary. If $x \geqq 1$, define

$$L_{s,x}(X) = \int_{\tau_1} s(Y)\{d\mu_{x,X}(Y) - d\mu_{1,X}(Y)\},$$

and observe that $L_{s,x}/h_*$ may be continuously extended to $W_1 \cap \bar{\Omega}$ (compare Lemma 1). Denoting the integral of this function with respect to ν by $\mathscr{Q}(s, x)$, it follows that

$$(6) \qquad \mathscr{M}(s, x) = \mathscr{N}(s, x) + \mathscr{P}(s, x) + \mathscr{Q}(s, x).$$

Since $s \geqq 0$ and $\mu_{y,X} \geqq \mu_{x,X}$ on τ_x whenever $y > x$, it is easily seen that both $\mathscr{P}(s, x)$ and $\mathscr{Q}(s, x)$ increase with x. Also, s has a positive upper bound, c say, on $\bar{\Omega}_1$, whence

$$L_{s,x}(X) \leqq c - c\mu_{1,X}(\tau_1) = c\mu_{1,X}(\sigma_1) \leqq c' h_*(X) \quad (X \in \Omega_{1/2})$$

by Theorem C. Hence $\mathscr{Q}(s, x)$ is bounded above for all x and so has a finite limit as $x \to +\infty$.

Now suppose that $\mathscr{N}(s, x)$ and $\mathscr{P}(s, x)$ each tend to a finite limit as $x \to +\infty$. Since this is also true of $\mathscr{Q}(s, x)$, it follows from (6) that $\mathscr{M}(s, x)$ is bounded above and so, by Theorem 3, s has a harmonic majorant in Ω.

Conversely, suppose that s has a harmonic majorant in Ω. Then, using Theorems 1 and 3 and equation (6), it can be seen that $\mathscr{N}(s, x) + \mathscr{P}(s, x)$ tends to a finite limit as $x \to +\infty$. Since $\mathscr{P}(s, x) \geqq 0$ and $\mathscr{P}(s, x)$ is increasing, both $\mathscr{N}(s, x)$ and $\mathscr{P}(s, x)$ tend to a finite limit as $x \to +\infty$.

6. Proofs of Theorems 4 and 5

6.1. To prove Theorem 4, we note from Theorem 1 (iii) that $\mathscr{M}(h, x)$ is independent of x. Since $h \geqq 0$, it follows from (6) that $\mathscr{P}(h, x)$ is (increasing and) bounded above, whence the result.

6.2. To prove Theorem 5, first observe that $s^+ \in \mathscr{S}(\bar{\Omega})$ and $s^+ = 0$ on $\partial\Omega$. Thus, from Theorem 1 (ii) and (1), $\mathscr{N}(s^+, x) = 0$ for all x. In view of Theorem B, it follows that $H_{s^+}^{\Omega_x} \equiv 0$ in Ω_x for all x. Since $s^+ \leqq H_{s^+}^{\Omega_x}$ in Ω_x, we have $s^+ \equiv 0$ in Ω, as required.

7. Applications to $\mathbf{R}^n$

Let $\Omega=\mathbf{R}^n$ $(n\geqq 2)$. The results in this case are elementary, but are given for the purpose of illustration. Let $E=\{O\}$, ν be the Dirac measure at the origin, $\Omega_x=B(x)$, the ball of radius x centred at the origin, and take $h_*\equiv 1$ in $\mathbf{R}^n$. Clearly the hypotheses of § 2.1 are satisfied. Since τ_1 is empty, it is immediate that

$$\mathcal{M}(s, x) = \mathcal{N}(s, x) = \{c_n x^{n-1}\}^{-1} \int_{\partial B(x)} s(Y)\,d\sigma(Y),$$

where c_n denotes the surface area of the unit sphere in $\mathbf{R}^n$, and σ denotes surface area measure. From § 2 we give immediately the following well-known results.

Theorem 6. *Let s be subharmonic in $\mathbf{R}^n$ $(n\geqq 2)$.*

(i) *The mean $\mathcal{M}(s, x)$ is an increasing function of $x\in(0, +\infty)$;*

(ii) *s has a harmonic majorant in $\mathbf{R}^n$ if and only if $\mathcal{M}(s, x)$ is bounded above on $(0, +\infty)$;*

(iii) *if $\liminf_{x\to+\infty} \mathcal{M}(s^+, x)=0$, then $s\leqq 0$ in $\mathbf{R}^n$.*

8. Applications to the half-space

Let $\Omega=\mathbf{R}^{n-1}\times(0, +\infty)$ $(n\geqq 2)$ and let E and ν be as in § 7, $\Omega_x=B(x)\cap\Omega$ and $h_*(X)=x_n$ in Ω. Again the hypotheses of § 2.1 are satisfied.

In order to find $\mathcal{M}(s, x)$ we recall (see (6)) that, if $x\geqq 1$,

$$\mathcal{M}(s, x) = \mathcal{N}(s, x)+\mathcal{P}(s, x)+\mathcal{Q}(s, x).$$

As in [4, Example 2] it can be shown that

$$\mathcal{N}(s, x) = (2n/c_n)x^{-n-1} \int_{\sigma_x} y_n s(Y)\,d\sigma(Y).$$

From the expression for the half-ball Poisson kernel given in [13, p. 615] it follows that

$$\mathcal{P}(s, x) = \lim_{X\to O} (2/c_n) \int_{\tau_x\setminus\tau_1} s(Y)\{|X-Y|^{-n}-x^n|Y|^{-n}|X-Y_x|^{-n}\}\,d\sigma(Y),$$

and

$$\mathcal{Q}(s, x) = \lim_{X\to O}(2/c_n) \int_{\tau_1} s(Y)\{|Y|^{-n}|X-Y_1|^{-n}-x^n|Y|^{-n}|X-Y_x|^{-n}\}\,d\sigma(Y),$$

where Y_x denotes the image of Y under the inversion of centre O and radius x. If $s\in\mathcal{S}(\bar{\Omega})$, then s is locally integrable on $\partial\Omega$ (see [14, Theorem 1 (ii)]), and it is easy to check that the convergence of both integrands above is dominated by an integrable function. Hence

$$\mathscr{P}(s, x)+\mathscr{Q}(s, x) = (2/c_n)\Big\{ \int_{\tau_x\setminus\tau_1} s(Y)\{|Y|^{-n}-x^{-n}\}d\sigma(Y)+\int_{\tau_1} s(Y)\{1-x^{-n}\}d\sigma(Y)\Big\}$$

$$= (2n/c_n)\int_1^x t^{-n-1}\int_{\tau_t} s(Y)\,d\sigma(Y)\,dt,$$

using integration by parts (see [14, p. 314]). Thus

$$\mathscr{M}(s, x) = (2n/c_n)\Big\{x^{-n-1}\int_{\sigma_x} y_n s(Y)\,d\sigma(Y)+\int_1^x t^{-n-1}\int_{\tau_t} s(Y)\,d\sigma(Y)\,dt\Big\},$$

and the same formula may similarly be established for $0<x<1$.

Further, if $s\geqq 0$ in $\bar{\Omega}$ then the statement $\lim_{x\to+\infty}\mathscr{P}(s, x)<+\infty$ holds if and only if

$$\int_{\partial\Omega}(1+|Y|^2)^{-n/2}s(Y)\,d\sigma(Y) < +\infty.$$

To see this,

$$\frac{1}{2}c_n\lim_{x\to+\infty}\mathscr{P}(s, x) = \lim_{x\to+\infty}\int_{\tau_x\setminus\tau_1}(|Y|^{-n}-x^{-n})s(Y)\,d\sigma(Y)$$

$$\geqq \lim_{x\to+\infty}(1-2^{-n})\int_{\tau_{x/2}\setminus\tau_1}|Y|^{-n}s(Y)\,d\sigma(Y) \geqq (1-2^{-n})\int_{\partial\Omega\setminus\tau_1}(1+|Y|^2)^{-n/2}s(Y)\,d\sigma(Y),$$

proving the "only if" case. The converse is even more elementary.

In this context, Theorems 1 (i), (iii), 4 and 5 are given in [14], the Corollary to Theorem 3 is part of [13, Theorem 3], and Theorems 2, 3 are due to Armitage [3], who also provides a counterexample to the converse of Theorem 2, [3, § 6].

9. Applications to the infinite cylinder

Recently results similar to those of § 2.2 have been obtained in the infinite strip, [5], and the infinite cone, [11]. These can be shown to be special cases of our general theorems. However, for the sake of originality, we will deduce previously unpublished results for the infinite cylinder. Modifications of our methods can be employed in the above-mentioned cases.

Let $\Omega=\{X=(x_1, \dots, x_n)=(X', x_n)\colon |X'|<1\}$, $(n\geqq 2)$. We will employ the Bessel function $J_{(n-3)/2}$ defined in Watson [16, pp. 40—42]. The least positive zero of this function will be denoted by a_n and we write

$$\psi(t) = t^{(3-n)/2}J_{(n-3)/2}(a_n t) \quad (t>0)$$

and

$$b_n = a_n J_{(n-1)/2}(a_n) > 0,$$

(see [16, p. 45 (4) and p. 479 § 15.22]). We will require the following

Lemma 3. *The functions* $\psi(|X'|)\exp(\pm a_n x_n)$ *are positive and harmonic in* Ω, *and vanish on* $\partial\Omega$.

The proof of Lemma 3 is analogous to that of [7, Lemma 1].

Now let

$$E=\{X\in\mathbf{R}^n : |X'|\leqq 1, \quad x_n=0\},$$

$$dv(X)=2a_n\{\psi(|X'|)\}^2 dX' d\delta_0(x_n) \quad (X\in E)$$

and

$$h_*(X)=\psi(|X'|)\cosh(a_n x_n),$$

where δ_0 denotes the Dirac measure at the origin of $\mathbf{R}$. It follows from Lemma 3 that h_* satisfies the hypotheses of § 2.1. Also, let $\Omega_x=\{X\in\Omega : |x_n|<x\}$. If $s\in\mathscr{S}(\bar{\Omega})$, then

$$\mathscr{N}(s,x)=2a_n\int_{\{|Y'|<1\}}\Big\{\int_{\sigma_x} s(X)\,d\mu_{x,(Y',0)}(X)\Big\}\psi(|Y'|)\,dY'$$

$$=2a_n\gamma_n\int_{\sigma_x} s(X)\Big\{\int_{\{|Y'|<1\}}\psi(|Y'|)\frac{\partial}{\partial n_X}G_{\Omega_x}(X,(Y',0))\,dY'\Big\}\,d\sigma(X),$$

where $\gamma_2=(2\pi)^{-1}$, $\gamma_n=\{(n-2)c_n\}^{-1}$ $(n\geqq 3)$ (see [10, Theorem 3] or [15, Theorem C]), and n_X denotes the inward unit normal at X with respect to Ω_x. The interchange in order of integration is justified because s is integrable with respect to harmonic measure on the boundary of Ω_x. Now, if $x_n=x$,

$$(\partial/\partial n_x)G_{\Omega_x}(X,(Y',0))=\lim_{t\to x-}G_{\Omega_x}((X',t),(Y',0))/(x-t),$$

this convergence being dominated by a constant for all $|Y'|<1$ (see Theorem C). A similar argument (with a change of sign) holds for $x_n=-x$, and so

$$\mathscr{N}(s,x)=\int_{\sigma_x} s(X)(\partial/\partial n_x)G_{\Omega_x}\mu(X)\,d\sigma(X), \tag{7}$$

where

$$d\mu(Y)=2a_n\gamma_n\psi(|Y'|)\,dY'\,d\delta_0(y_n) \quad (Y\in E).$$

Now consider the function

$$u_x(X)=\sinh a_n(x-|x_n|)\,\mathrm{sech}(a_n x)\,\psi(|X'|)$$

which, as is easy to check from Lemma 3, is positive and superharmonic in Ω_x, harmonic in $\Omega_x\setminus\{x_n=0\}$ and continuously vanishes on $\partial\Omega_x$. It follows that u_x is a potential in Ω_x, and its corresponding measure is given by $\mu'=-\gamma_n\Delta u_x$, where Δu_x is the distributional Laplacian of u_x.

Now let Ψ be a C^∞ function with compact support in Ω_x. From Green's theorem it follows that

$$(\Delta u_x)(\Psi) = \int_{\Omega_x} u_x(X)\Delta\Psi(X)\,dX$$

$$= \lim_{\varepsilon\to 0+}\left\{\int_{\{|X'|<1,\,x_n=\varepsilon\}}\left\{\Psi(X)\frac{\partial u_x}{\partial x_n}(X)-u_x(X)\frac{\partial\Psi}{\partial x_n}(X)\right\}d\sigma(X)\right.$$

$$\left.+\int_{\{|X'|<1,\,x_n=-\varepsilon\}}\left\{u_x(X)\frac{\partial\Psi}{\partial x_n}(X)-\Psi(X)\frac{\partial u_x}{\partial x_n}(X)\right\}d\sigma(X)\right\}$$

$$= -2a_n\int_{\{|X'|<1\}}\Psi(X',0)\,\psi(|X'|)\,dX',$$

whence $\mu=\mu'$. Thus, from (7),

$$\mathcal{N}(s,x) = a_n\,\mathrm{sech}\,(a_n x)\int_{\sigma_x}\psi(|X'|)\,s(X)\,d\sigma(X).$$

The determination of the remaining part of $\mathcal{M}(s,x)$ is similar, and involves calculating the normal derivative of u_x at points of $\tau_x\setminus\{|x_n|=0 \text{ or } x\}$ (noting that the interchange of differentiation and integration to obtain (7) is valid except on the set $\tau_x\cap\{|x_n|=0 \text{ or } x\}$ which has surface area measure zero), performing the corresponding calculation for u_1, and integrating by parts (as in § 8) to obtain

$$\mathcal{M}(s,x) = \mathcal{N}(s,x)+a_n b_n\int_1^x \mathrm{sech}^2(a_n t)\int_{\tau_t} s(X)\cosh(a_n x_n)\,d\sigma(X)\,dt.$$

The results of § 2.2 may now be applied to subharmonic functions in the infinite cylinder. In the case of Theorem 4, it is a routine matter to check that the conclusion is equivalent to the simple condition

$$\int_{\partial\Omega}\exp(-a_n|x_n|)\,h(X)\,d\sigma(X) < +\infty.$$

The details are left to the reader.

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