

ON THE PROBLEM OF CLASSIFYING INFINITE CHAINS IN PROJECTIVE AND ORTHOGONAL GEOMETRY

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Introduction

In [8] a classification of subforms of diagonal forms (symmetric bilinear over an algebraically closed field or else alternate) in dimensions less than \aleph_{ω_0} was obtained. Trying to push the classification up to dimension \aleph_{ω_0} , and beyond, one discovers that the recursive methods developed for dimensions below \aleph_{ω_0} get stuck.

The reason is the following. The classification (up to isometry) of subforms of diagonal forms Φ is equivalent to the classification of certain descending chains of subspaces in the vector space E that carries Φ (the second classification is modulo the action of the orthogonal group of the quadratic space (E, Φ)). Now, if $\dim E \cong \aleph_{\omega_0}$ then these chains are infinite and, with infinite *descending* chains, a variety of new difficulties makes its appearance. Some of these difficulties are of a metric nature, others are of a projective kind, and still others are of an entirely topological character. We shall show that these aspects are largely independent one from the other.

A somewhat similar situation is encountered in the classification theory of abelian groups (Ulm's Theorem) in the uncountable case. However, the analogy does not carry very far. The group structure is trivial in the vector space case; on the other hand there are various additional structures originating with the form.

The classification of subforms of diagonal forms is a special case of a mapping problem which has been the source of many new results during recent years (refer to [5, 8, 10, 15, 17]). We begin our paper by formulating a projective version of this mapping problem and by giving a solution (Theorem 1) of it when no infinite descending chains interfere.

AMS (MOS) subject classification (1970). Primary 10 C 01, 15 A 63, 51 D 25, 51 F 20. Secondary 20 E 15, 20 H 20, 51 F 25. Key phrase: infinite chains of subspaces.

^{*)} This work was done while the second author was a Fellow of the Schweizerische Nationalfonds (Grant Nr. 2. 627—0.80).

Next we construct a series of examples which illustrate the various kinds of difficulties met in the classification of infinite descending chains \mathcal{C} under the action of $\text{GL}(E)$ and certain of its subgroups. The examples are such that the orbits of \mathcal{C} split into proper suborbits as we pass from one classification to the next.

We then treat cases where the mapping problem in the metric setting can be solved even in the presence of infinite descending chains (Theorems 6 and 7). We apply the results to the classification of subforms of diagonal forms ("prediagonal forms"). We mention in particular Theorem 11 which gives a topological characterization of the isometry types of prediagonal spaces of small uncountable dimensions.

In order to make our constructions as perspicuous as possible we stick to the simplest cases suitable for our purposes. This means restriction to the consideration of ω_0 -chains. We are aware of the fact that more gadgets can be thought up for longer chains; however, a firm grasp of the ω_0 -chains is pre-eminent. We shall consider chains of the sort

$$(0) \quad \mathcal{C} = (V_i)_{i < \omega_0}, \quad V_i \supseteq V_{i+1}, \quad \dim V_i/V_{i+1} = 1, \quad i < \omega_0$$

where the V_i are linear subspaces in a vector space E of dimension \aleph_1 (2^{\aleph_0} whenever more convenient) in which case we shall assume that $\bigcap_i V_i = (0)$ ("reduced chain"), or else in a space E of dimension \aleph_{ω_0} in which case we set $V_\infty := \bigcap_i V_i$. Furthermore, we shall use the following *notations*:

E	vector space over a commutative field of arbitrary characteristic,
$\mathcal{L}(E)$	lattice of all linear subspaces of E ,
\mathcal{V}	sublattice of $\mathcal{L}(E)$,
\mathcal{C}	descending chain in $\mathcal{L}(E)$,
DCC	descending chain condition,
\mathcal{G}	filter in $\mathcal{L}(E)$,
$\varrho(\mathcal{G}), \varrho(\mathcal{C})$	the linear topology on E with zero-neighbourhood filter basis \mathcal{G}, \mathcal{C} , respectively,
ϱX	closure of $X \subset E$ with respect to the topology ϱ ,
(E, Φ)	Φ symmetric or antisymmetric bilinear form on E ,
$d \cdot E$	external orthogonal sum of d copies of (E, Φ) , d any cardinal,
$\tau_\gamma(\Phi), \sigma_\gamma(\Phi)$	certain linear topologies on E , related by $\sigma_{\gamma+1}(\Phi) = \tau_\gamma(\Phi)$, γ any ordinal,
$\tau_\gamma X, \sigma_\gamma X$	closure of $X \subset E$ with respect to $\tau_\gamma(\Phi), \sigma_\gamma(\Phi)$, respectively,
\tilde{E}	completion of the uniform space $(E, \tau_0(\Phi)) = (E, \sigma_1(\Phi))$.

Refer to [9] for terminology and basic facts on infinite dimensional sesquilinear forms.

I. A mapping problem of projective geometry

I. 1. *The Problem* (henceforth referred to as “the mapping problem”). Let E be an infinite dimensional vector space and

$$\eta: \mathcal{V} \rightarrow \mathcal{V}'$$

a lattice isomorphism between sublattices of $\mathcal{L}(E)$. When is η induced by an element of $GL(E)$ (the group of linear automorphisms of E)? An obvious requirement is that η preserve rank, i.e. all “indices” $\dim Y/X$ for $X \subset Y$ in \mathcal{V} ; cf. Remark 8. It is furthermore no restriction to assume $\mathcal{V}, \mathcal{V}'$ at the outset to be complete as sublattices of $\mathcal{L}(E)$. Quite often we shall assume \mathcal{V}' to be a subset in $\mathcal{L}(E')$, E' a vector space isomorphic to E which makes for more systematic notations.

I. 2. *The lattices admitted.* We shall study the mapping problem for various classes of lattices. In order to concentrate on the difficulties with descending chains we shall always restrict ourselves to the consideration of *distributive* lattices \mathcal{V} ; they will be subjected to one more condition below (namely (4)). We assume furthermore that $(0) \in \mathcal{V}$ and $E \in \mathcal{V}$.

I. 3. *The recursive setup for the solution.* It is natural to visualize the required mapping $T: E \rightarrow E'$ as the union of partial mappings. We therefore introduce the family $\mathcal{F}_0 = \mathcal{F}_0(\eta)$ of all linear isomorphisms $\varphi: X \rightarrow X', X \in \mathcal{L}(E), X' \in \mathcal{L}(E')$, which satisfy the condition

$$(1) \quad \varphi(X \cap A) = X' \cap \eta A \quad \text{for all } A \in \mathcal{V}.$$

One then looks for chains $(\varphi_\gamma: X_\gamma \rightarrow X'_\gamma)_{\gamma \in I}$ in \mathcal{F}_0 with

$$(2) \quad \bigcup_I X_\gamma = E, \quad \bigcup_I X'_\gamma = E'.$$

The limit map of such a chain solves the mapping problem.

I. 4. *Solution of the problem for lattices with the descending chain condition (DCC).* In the case of certain lattices it is possible to restrict oneself to the subfamily $\mathcal{F}_1(\eta)$ of all $(\varphi: X \rightarrow X') \in \mathcal{F}_0(\eta)$ with the two properties:

$$(3) \quad (X+A) \cap (X+B) = X + (A \cap B); \quad A, B \in \mathcal{V}$$

$$(3') \quad (X'+A') \cap (X'+B') = X' + (A' \cap B'); \quad A', B' \in \mathcal{V}'$$

This is shown by

Lemma 1. *Let \mathcal{V} be a complete and distributive sublattice of $\mathcal{L}(E)$ ($\dim E = \omega_\alpha$) which satisfies DCC. Then there exists a set of generators $(x_\nu)_{\nu < \omega_\alpha}$ of E with the following property: If $X_\alpha := \text{span} \{x_\nu | \nu < \alpha\}$ then for all $\alpha < \omega_\alpha$ and all $A, B \in \mathcal{V}$ we have $(X_\alpha + A) \cap (X_\alpha + B) = X_\alpha + (A \cap B)$.*

Thus, if there is a solution to the mapping problem at all then it can be obtained as a limit map of a chain in $\mathcal{F}_1(\eta)$.

Proof of Lemma 1. Let $(e_\lambda)_{\lambda < \omega_\alpha}$ be a basis of E . Assume that for $v < \alpha$ x_v is defined such that (3) holds for all X_μ , $\mu \leq \alpha$. Let λ be the smallest index with $e_\lambda \notin X_\alpha$. The set

$$\mathcal{M} = \mathcal{M}(e_\lambda, X_\alpha) := \{A \in \mathcal{V} \mid e_\lambda \in X_\alpha + A\}$$

is a filter by the induction assumption, hence a principal filter by DCC, say $\mathcal{M} = (D)$. First case: D is $+$ -irreducible so \mathcal{M} is a prime filter). We decompose e_λ , $e_\lambda = d + x$ where $d \in D$ and $x \in X_\alpha$. Set $x_\alpha := d$; one verifies just as in [9, p. 118] that (3) holds again for $X = X_{\alpha+1}$. Second case: D is $+$ -reducible. It is here that we need a consequence of DCC, namely

(4) compact elements are joins of join-irreducibles*

The case is now reduced to the former as in [9, p. 119]: one finds $x_\alpha, x_{\alpha+1}, \dots, x_{\alpha+m}$ such that the finitely many $X_{\alpha+1}, \dots, X_{\alpha+m+1}$ satisfy (3) and such that $e_\lambda \in X_{\alpha+m+1}$.

Conversely, we now show that a solution of the mapping problem always exists under the provisos stated:

Theorem 1. *Let $\mathcal{V}, \mathcal{V}'$ be complete, distributive sublattices of $\mathcal{L}(E)$ and $\mathcal{L}(E')$ satisfying DCC and (4); here E, E' are vector spaces of arbitrary (equal) dimension over the same field. Each rank preserving lattice isomorphism $\eta: \mathcal{V} \rightarrow \mathcal{V}'$ comes from a projectivity, i.e. is induced by a linear bijection $T: E \rightarrow E'$.*

Proof. The union of any ascending chain of elements in $\mathcal{F}_1(\eta)$ belongs to $\mathcal{F}_1(\eta)$ again (“chain-property”). Furthermore, $\mathcal{F}_1(\eta)$ enjoys the “Ping Pong-property” (PP): for each $(\varphi: X \rightarrow X') \in \mathcal{F}_1(\eta)$ and each $x \in E$ ($x' \in E'$) there exist $(\varphi_1: X_1 \rightarrow X'_1) \in \mathcal{F}_1(\eta)$ with $x \in X_1$ ($x' \in X'_1$) and $\varphi_1 \supseteq \varphi$. Starting with the zero map $(\varphi_0: (0) \rightarrow (0)) \in \mathcal{F}_1(\eta)$ we may therefore, by transfinite recursion, obtain a chain $(\varphi_\gamma: X_\gamma \rightarrow X'_\gamma)_{\gamma \in I}$ in $\mathcal{F}_1(\eta)$ that satisfies (2) just as in the denumerable case [9, Chapter IV].

Remark 1. The core of the preceding proof is the (PP) property of $\mathcal{F}_1(\eta)$: one has to extend a partial mapping $\varphi: X \rightarrow X'$ such that the extension satisfies (1). According to [9, Chapter IV] this is always possible if the filter

(5) $\mathcal{M}(x, X) := \{A \in \mathcal{V} \mid x \in X + A\}$

happens to be a *principal* filter. If, in the following, we also succeed with the recursive setup (I.3) in some cases where the lattices do not satisfy DCC then, invariably, it is because we have discovered a way to monitor the course of recursion

*) For complete sublattices of $\mathcal{L}(E)$ distributivity together with (4) is equivalent to complete distributivity. A proof is outlined in [9, Chapter IV, Section 2].

in such a manner that, at each step, the arising filters \mathcal{M} turn out to be principal filters.

I. 5. *Infinite descending chains lead to new invariants.* Let $\mathcal{C} \subseteq \mathcal{L}(E)$ be a descending ω_0 -chain and $\varrho(\mathcal{C})$ the linear topology on E with \mathcal{C} as a zero neighbourhood basis. Assume that the lattice \mathcal{V} in our mapping problem contains such a \mathcal{C} ; then by (1) each element in $\mathcal{F}_0(\eta)$ is a homeomorphism with respect to $\varrho(\mathcal{C})$ and $\varrho(\eta\mathcal{C})$; likewise, each linear isomorphism $T: E \rightarrow E'$ which induces η is a homeomorphism $(E, \varrho(\mathcal{C})) \cong (E', \varrho(\eta\mathcal{C}))$. Hence the importance of the topologies $\varrho(\mathcal{C})$ (for all $\mathcal{C} \subset \mathcal{V}$) for our mapping problem.

We first convince ourselves that there are enough such chains in $\mathcal{L}(E)$ for our discussions.

Lemma 2. *Let the k -space E have dimension $a := |k|^{\aleph_0}$ and let $0 \neq r < \omega_0$ be fixed. In $\mathcal{L}(E)$ there are precisely $|k|^a$ chains \mathcal{C} with property (0) and such that (i) the spaces $(E, \varrho(\mathcal{C}))$ are mutually non-homeomorphic, (ii) if \bar{E} is the completion of the uniform space $(E, \varrho(\mathcal{C}))$ then $\dim \bar{E}/E = r$ for all these \mathcal{C} .*

Proof. We begin by estimating the number of \mathcal{C}' such that $(E, \varrho(\mathcal{C}'))$ is homeomorphic to a fixed $(E, \varrho(\mathcal{C}))$. Let $(h_i)_{i \geq 1}$ be a basis of a \aleph_0 -dimensional k -space H . In the algebraic dual H^* pick a subspace E such that the chain $\mathcal{C}_E = \mathcal{C} := (V_i)_{i \in \omega_0}$, $V_i := \{e \in E \mid \langle e, h_1 \rangle = \dots = \langle e, h_i \rangle = 0\}$ satisfies (0). Let $E' \subset H^*$ and $\mathcal{C}' = \mathcal{C}'_{E'}$ be analogous objects. The weak linear topology $\sigma(E, H)$ of the pairing $\langle E, H \rangle$ is precisely $\varrho(\mathcal{C})$. One now proves with the usual arguments that each linear homeomorphism

$$(7) \quad f: (E, \varrho(\mathcal{C})) \rightarrow (E', \varrho(\mathcal{C}')) \text{ has the form } f = g^*|_E, \\ \text{where } g^* \in GL(H^*) \text{ is the adjoint of a } g \in GL(H).$$

The adjoint $g^{**}: H^{**} \rightarrow H^{**}$ then maps the r -dimensional orthogonals $E^\perp \subset H^{**}$, $E'^\perp \subset H^{**}$ onto each other. Hence the number of $(E, \varrho(\mathcal{C}'))$, homeomorphic to a fixed $(E, \varrho(\mathcal{C}))$, is less or equal to $|GL(H)| = a$. On the other hand there are precisely $|k|^a \cong 2^a > a$ subspaces $Y \subset H^{**}$ of fixed dimension r . If we think of H^* as a subspace of H^{**} , then we can also pick $|k|^a$ such Y which, in addition, satisfy $Y \cap H = (0)$. These Y appear as orthogonals E^\perp of suitable subspaces $E \subset H$, i.e. E such that the chains \mathcal{C}_E which they define all satisfy (0). Hence the assertion of the lemma.

I. 6. *A linear homeomorphism $(E, \varrho(\mathcal{C})) \cong (E, \varrho(\mathcal{C}'))$ does not imply that \mathcal{C} and \mathcal{C}' are in the same orbit under $GL(E)$.* In other words, the projective classification of the chains $\mathcal{C} \subset \mathcal{L}(E)$ is finer than the classification via their associated topologies $\varrho(\mathcal{C})$. We shall prove

Theorem 2. *Let E and \bar{E} be as in Lemma 2 and $\dim \bar{E}/E = 1$. We can specify $\mathcal{C}, \mathcal{C}'$ with property (0) such that $(E, \varrho(\mathcal{C}))$ and $(E, \varrho(\mathcal{C}'))$ are linearly homeomorphic but no $T \in GL(E)$ can map \mathcal{C} onto \mathcal{C}' .*

Proof. Let $(h_{2i})_{i \geq 1}, (h_{2i-1})_{i \geq 1}$ be bases of two \aleph_0 -dimensional k -spaces H_+, H_- . We set $H := H_- \oplus H_+$ and choose in its algebraic dual $H^* \cong H_-^* \oplus H_+^*$ a hyperplane $E := H_-^* \oplus L, L \subset H_+^*$ in such a way that the chain $\mathcal{C} = (V_i)_{i \geq 0}$ with $V_i := \{e \in E \mid \langle e, h_1 \rangle = \dots = \langle e, h_i \rangle = 0\}$ satisfies (0). Let $g \in \text{GL}(H)$ be the operation which interchanges h_{2i} and h_{2i-1} . We set $E' := g^*(E)$,

$$E' = M \oplus H_+^*, \quad M = g^*(L) \subset H_-^*.$$

Obviously, the chain $\mathcal{C}' := (V'_i)_{i \geq 0}$, defined in analogy to \mathcal{C} above, satisfies (0) as well and, according to the criterion (7), we have a linear homeomorphism $(E, \varrho(\mathcal{C})) \cong (E', \varrho(\mathcal{C}'))$. Next we show:

(8) If there is a linear bijection $f : E \rightarrow E'$ with $f(V_i) = V'_i$ ($i \geq 0$) then it is of the form $f = l^*|_E$, where $l^* \in \text{GL}(H^*)$ is the adjoint of a $l \in \text{GL}(H)$ and the matrix of l relative to $(h_i)_{i \geq 1}$ is triangular.

Indeed, let $v_i \in V_{i-1} \setminus V_i$ ($i \geq 1$). Construct a dual basis $(w_i)_{i \geq 1}$ of H by induction, $\langle v_i, w_j \rangle = \delta_{ij}$ ($i, j \geq 1$). We have $w_n = \alpha_1 h_1 + \alpha_2 h_2 + \dots + \alpha_n h_n$ with $0 \neq \alpha_n \in k$. If now $f(V_i) = V'_i$ ($i \geq 0$) for some linear bijection $f : E \rightarrow E'$ then $v'_i := f(v_i) \in V'_{i-1} \setminus V'_i$. Let $(w'_i)_{i \geq 1}$ be the dual basis of $(v'_i)_{i \geq 1}$ in H . Let $l \in \text{GL}(H)$ be the map with $w'_i \rightarrow w_i$ ($i \geq 1$). One readily verifies that $f = l^*|_E$ and that the matrix of l has the desired diagonal shape.

By way of contradiction assume that there exists an f enjoying (8). Let $p : H \rightarrow H_-$ be the projection along $H_+, q : H^* \rightarrow H_-^*$ the projection along $H_+^*, t := p \circ l|_{H^*}$. Then the adjoint of t is $t^* = q \circ l^*|_{H_-^*}$. Now $t^* : H_-^* \rightarrow H_-^*$ is bijective, for the matrix of t (relative to $(h_{2i-1})_{i \geq 1}$) is obtained from the matrix of l by cancelling rows and columns of even index and, therefore, is a triangular matrix with nonzero diagonal coefficients. Ergo t and t^* are invertible. We now obtain a contradiction as follows. Pick $y \in H_-^* \setminus M$. There is $z \in H_-^*$ with $t^*(z) = y$ by the invertibility of t^* ; thus $z \in E$. Furthermore $y = q \circ l^*(z)$ so that $l^*(z) = y + x$ for some $x \in H_+^*$. Therefore $f(z) = l^*(z) \notin E'$, a contradiction.

Remark 2. By abuse let us say that $X \subset (E, \varrho(\mathcal{C}))$ is *closed* if and only if the image of X under the projection $E \rightarrow E/\mathcal{C}$ is closed in the associated (hausdorff) quotient topology $\hat{\varrho}$ of $\varrho(\mathcal{C})$. We see that X is closed if and only if the filters $\mathcal{M}(x, X)$ in (5) are principal for all $x \in E$. Unfortunately, there are no “nice” cases $(E, \varrho(\mathcal{C}))$ where all subspaces X would be closed; e.g. $S := \bigoplus_{i \geq 1} S_i$, where $V_i \oplus S_i = V_{i-1}$ ($i \geq 1$), is always dense in $(E, \varrho(\mathcal{C}))$. Now, in the case of countable $\dim E$ one never runs into trouble with $\varrho(\mathcal{C})$ because such spaces E can be exhausted by finite dimensional X which are, of course, always closed. (In lieu of our distributivity (3), (3') above it was possible in [9, p. 114] to postulate a stronger distributivity which required in particular ϱ -closedness for the domains of all partial mappings considered [18].)

II. The mapping problem in orthogonal geometry

II. 1. *On the problem.* We now assume the vector space E to be equipped with a nondegenerate (orthosymmetric) sesquilinear form $\Phi: E \times E \rightarrow k$. Let $\mathcal{O} = \mathcal{O}(E, \Phi) \subset GL(E)$ be the subgroup of elements with respect Φ (orthogonal group or group of isometries $E \rightarrow E$). We are now looking for a $T \in \mathcal{O}$ which induces the given lattice isomorphism $\eta: \mathcal{V} \rightarrow \mathcal{V}'$ of I.1.

In order to make evident the novelty of geometric situations encountered in dimensions $> \aleph_0$ we repeat a fact mentioned in [8]: Whereas in each dimension $\cong \aleph_0$ there is precisely one isometry class of nondegenerate symmetric bilinear forms over \mathbb{C} there are $2^{\aleph_1} \cong \aleph_2$ isometry classes of such forms in dimension \aleph_1 . Similar results hold for nondegenerate alternating forms which, in dimensions $\cong \aleph_0$, are considered to be without gestalt.

In particular, the abundance of isometry classes of alternate forms in dimensions $> \aleph_0$ testifies to the extent of geometric complications that have nothing to do with the arithmetic of the base field.

Everything that follows could be phrased for symmetric forms over any field k which has the property of allowing for only one isometry class of nondegenerate forms in dimension \aleph_0 . In [9, Appendix 1 to Chapter II] many classes of such k are listed. However, in order to stress the new geometric features of the uncountable we stay mostly with the simplest case, to wit, alternate forms. A few definitions will be stated for arbitrary forms. (In [5] and [17] the lattice method is applied to anisotropic forms; in [5] the lattices are infinite. [10] and [15] treat of bilinear forms in characteristic 2. Dimensions are invariably \aleph_0 .)

II. 2. *The prediagonal forms.* Let (E, Φ) be a sesquilinear space and γ an ordinal. We call (E, Φ) γ -*diagonal* if there exists an orthogonal decomposition $E = \bigoplus_i^\perp E_i$ with $\dim E_i < \aleph_\gamma$ for all i ; a sesquilinear space which is isometric to a subspace of a γ -diagonal space is called γ -*prediagonal*; 1-diagonal spaces are also 0-diagonal and, trivially, the other way round; hence for $\gamma \in \{0, 1\}$ we simply speak of *diagonal* and *prediagonal* forms or spaces. A diagonal space is always an orthogonal sum of lines and planes [9, p. 63]. The basic theory about γ -diagonal and prediagonal forms can be found in [4].

The classification of prediagonal spaces is solved in [8] for dimensions $< \aleph_{\omega_0}$. Theorems 8, 10, 11 below will treat of certain cases of larger dimensions.

II. 3. *The diagonal hull.* Let (E, Φ) be a sesquilinear space, nondegenerate and of dimension \aleph_α ($\alpha > 0$), α an ordinal. The linear topology $\sigma_\gamma(\Phi)$ on E is defined by the 0-neighbourhood filter basis

$$\{X^\perp \mid X \in \mathcal{L}(E) \text{ and } \dim X < \aleph_\gamma\} \quad (\gamma \cong 0),$$

whereas the linear topology $\tau_\gamma(\Phi)$ on E is defined by the basis

$$\{X^\perp \mid X \in \mathcal{L}(E) \text{ and } \dim X \cong \aleph_\gamma\} \quad (\gamma \cong 0).$$

We have $\tau_\gamma(\Phi) = \sigma_{\gamma+1}(\Phi)$; for the closure $\sigma_\gamma X$ of a subspace $X \subset (E, \sigma_\gamma(\Phi))$ we have, in the case of a limit number,

$$(9) \quad \lim(\lambda) \Rightarrow \sigma_\lambda X = \bigcap_{\gamma < \lambda} \sigma_\gamma X = \bigcap_{\gamma < \lambda} \tau_\gamma X.$$

If Φ is nondegenerate then $\sigma_\gamma(\Phi), \tau_\gamma(\Phi)$ are hausdorff; $\sigma_\gamma(\Phi)$ (or $\tau_\gamma(\Phi)$) is discrete when $\gamma > \alpha$ ($\gamma \cong \alpha$, respectively). Each $(\gamma+1)$ -diagonal space is complete in the topology $\tau_\gamma(\Phi)$. Conversely, we have the beautiful *Theorem* [16, p. 245]: Let $\gamma \cong 0$ be a fixed ordinal and (E, Φ) a nondegenerate $(\gamma+1)$ -prediagonal space. Then there exists, uniquely determined up to isometry, a smallest $(\gamma+1)$ -diagonal overspace $(\tilde{E}, \tilde{\Phi})$; \tilde{E} is the completion of the space $(E, \tau_\gamma(\Phi))$; the completion topology $\tilde{\tau}_\gamma(\Phi)$ coincides with $\tau_\gamma(\tilde{\Phi})$ and $\dim \tilde{E} = \dim E$.

II. 4. *Further conditions on the lattices.* Each $T \in \mathcal{O}(E, \Phi)$ operates on $\mathcal{L}(E)$ and commutes with the operations \perp (taking the orthogonal) and τ_γ (forming the $\tau_\gamma(\Phi)$ -closure) for all $\gamma \cong 0$. It is therefore appropriate to consider in the mapping problem II. 1 but lattices \mathcal{V} that are stable under these operations (cf. [11]). For $V \in \mathcal{L}(E)$ we define $\mathcal{V}(V)$ to be the *complete* sublattice of $\mathcal{L}(E)$, stably generated by $(0), V, E$ under the operations $+, \cap, \perp, \tau_\gamma(\gamma \cong 0)$.

Examples. Let E be diagonal, $E^1 = (0)$, $\dim E = \aleph_\alpha$. In the cases $\alpha = 0, 1, 2$ the lattice $\mathcal{V} = \mathcal{V}(V)$ is distributive and finite, $|\mathcal{V}| \cong 14$ if $\alpha = 0$ [12, p. 11], $|\mathcal{V}| \cong 34$ if $\alpha = 1$ ([1], [2]), $|\mathcal{V}| \cong 88$ if $\alpha = 2$ ([8]); the bounds indicated are reached for suitable V . *The lattices $\mathcal{V}(V)$ are unknown for $\alpha \cong 3$.*^{*)} (See Problems 1, 2 in II. 7.) However, $\mathcal{V}(V)$ is known for arbitrary α when V is $\tau_0(\Phi)$ -dense in E , it is the $\alpha+2$ chain

$$(10) \quad E = \tilde{V} = \sigma_1 V \supseteq \sigma_2 V \supseteq \dots \supseteq \sigma_i V \supseteq \dots \supseteq \sigma_{\omega_0} V \supseteq \sigma_{\omega_0+1} V \supseteq \dots \supseteq \sigma_\alpha V \supseteq V \supseteq (0).$$

II. 5. *How can one classify prediagonal forms?* The uniqueness of the diagonal hull easily yields the following fundamental fact: If V is isometric to a prediagonal V' then there exists an isometry T from $E = \tilde{V}$ onto $E' = \tilde{V}'$ which maps the chain $\mathcal{V}(V)$ onto $\mathcal{V}(V')$, i.e. T induces a lattice isomorphism $\eta: \mathcal{V}(V) \rightarrow \mathcal{V}(V')$ which, obviously, preserves rank and commutes with \perp and all closure operators $\tau_\gamma, \sigma_\gamma$. Thus we see that *the classification of \aleph_α -dimensional prediagonal forms is equivalent to solving our mapping problem II. 1 for $\mathcal{V}, \mathcal{V}'$ descending chains of the kind (10).* (Cf. Remark 11.) These chains are infinite when $\alpha \cong \omega_0$. As already mentioned in the Introduction we shall discuss only the typical case $\alpha = \omega_0$. The case $\alpha < \omega_0$ is solved in [8]. We begin by showing that the chain (10) can be entirely arbitrary:

II. 6. *The chain (10) of closures of a prediagonal V has no special properties.*

^{*)} Note added in proof. When $\alpha = 3$ the lattice is known and $|\mathcal{V}| \cong 957$ ([20]). If $\alpha \cong 4$ we know that $|\mathcal{V}| = \infty$ ([19]); the lattices are not known.

For the sake of clarity in the description of our construction we restrict ourselves to proving the following theorem only. It can easily be generalized.

Theorem 3. *Let (E, Φ) be nondegenerate and of dimension \aleph_{ω_0} , Φ symmetric and admitting an orthogonal basis. Assume that we are given an orthogonal decomposition $E = W \oplus H$, $\dim W = \aleph_1$, and an ω_0 -chain in W , $W = W_0 \supset W_1 \supset W_2 \supset \dots$ with $\dim W_n/W_{n+1} = 1$, $\bigcap_{n < \omega_0} W_n = (0)$. In E there exists a subspace V such that its $\tau_n(\Phi)$ -closures are*

$$\tau_n V = V \oplus W_n \quad (n \leq \omega_0).$$

Remark 3. From Lemma 2 and Theorem 3 one gathers that the cardinals obtained from (10) by taking dimensions of quotients are a long way from pinning down the isometry type of V when $\alpha \leq \omega_0$.

II. 7. Proof of Theorem 3. It will be divided into several paragraphs.

7.1. In each W_n we choose a basis $\{w_{n,\gamma} | 0 \leq \gamma < \omega_1\}$ such that $w_{n,\gamma} \in W_{n+\gamma} \setminus W_{n+\gamma+1}$ for all $\gamma < \omega_0$.

7.2. H is spanned by an orthogonal basis [4, p. 1578 Theorem 4]. We introduce orthogonal decompositions: $H = \bigoplus^\perp \{H_n | n < \omega_0\}$, $\dim H_n = \aleph_{n+1}$, $H_n = \bigoplus^\perp \{H_{n,\gamma} | \gamma < \omega_1\}$, $\dim H_{n,\gamma} = \aleph_{n+1}$. In each $H_{n,\gamma}$ we choose an orthogonal basis $\{e_{n,\gamma}(\lambda) | 0 \leq \lambda < \omega_{n+1}\}$.

7.3. Let V be the span of all

$$w_{n,\gamma} + e_{n,\gamma}(\lambda), \quad 0 \leq \lambda < \omega_{n+1}, \quad 0 \leq \gamma < \omega_1, \quad 0 \leq n < \omega_0.$$

7.4. We set $D_{n,\gamma} := \text{span} \{e_{n,\gamma}(\lambda) - e_{n,\gamma}(\mu) | \lambda, \mu < \omega_{n+1}\}$, $D_n := \bigoplus^\perp \{D_{n,\gamma} | \gamma < \omega_1\}$. Hence $D_n \subset H_n \cap V$.

7.5. We define spaces $C_{n,m} \subset H_n$ for all n, m with $0 \leq n \leq m < \omega_0$. To each linear combination $\sum_\gamma c_\gamma w_{n,\gamma}$ which falls into the space W_m we assign — with the same scalars c_γ — the element $\sum_\gamma c_\gamma e_{n,\gamma}(0) \in H_n$; $C_{n,m}$ is defined as the span of these assigned elements. It is evident that

$$C_{n,m} \supseteq C_{n,m+1} \supseteq \dots; \dim C_{n,m}/C_{n,m+1} = 1, \\ \bigcap_m C_{n,m} = (0), \quad C_{n,m} \cap D_n = (0).$$

7.6. We assert that the $\tau_n(\Phi)$ -closures of V are given by

$$\tau_0 V = W_0 \oplus H = E, \\ \tau_n V = W_n \oplus [C_{0,n} \oplus D_0] \oplus [C_{1,n} \oplus D_1] \oplus \dots \oplus [C_{n-1,n} \oplus D_{n-1}] \\ \oplus H_n \oplus H_{n+1} \oplus H_{n+2} \oplus \dots \\ \oplus \text{span} \{w_{0,0} + e_{0,0}(0), w_{0,1} + e_{0,1}(0), \dots, w_{0,n-1} + e_{0,n-1}(0), \\ w_{1,0} + e_{1,0}(0), \dots, w_{1,n-2} + e_{1,n-2}(0), \\ \vdots \\ w_{n-1,0} + e_{n-1,0}(0)\}.$$

7.7. In order to justify the *direct* sum between “span {...}” and the rest in the expression for $\tau_n V$ in 7.6 one makes systematic use of the normalization made in 7.1 for those $w_{n,\gamma}$ with $\gamma < \omega_0$.

7.8. On the inclusion \supseteq in 7.6: Because $\dim H_{n,\gamma} > \aleph_n$, each $w_{n,\gamma} \in W_n$ is a $\tau_n(\Phi)$ -clusterpoint of V , $W_n \subset \tau_n V$. Thus $H_n \subset \tau_n V$. Likewise, $H_m \subset \tau_n V$ for all $m > n$. Nontrivial are only the inclusions $C_{r,n} \subset \tau_n V$ for $r < n$. Let

$$x = \sum_{\gamma} c_{\gamma} e_{r,\gamma}(0) \in C_{r,n}, \quad \text{so} \quad \sum_{\gamma} c_{\gamma} w_{r,\gamma} \in W_n$$

by 7.5. Ergo

$$x - \sum_{\gamma} c_{\gamma} (w_{r,\gamma} + e_{r,\gamma}(0)) = - \sum_{\gamma} c_{\gamma} w_{r,\gamma} \in W_n, \quad x \in W_n + V \subset \tau_n V.$$

7.9. We show next that

$$W \cap V = (0), \quad W \cap \tau_n V = W_n \quad (n \geq 1).$$

The first assertion follows directly from the definition in 7.3. Let then $n \geq 1$ and $y \in W \cap \tau_n V$. If $X = W \oplus H_0 \oplus \dots \oplus H_{n-1}$ then $U := X^{\perp} = H_n \oplus H_{n+1} \oplus \dots$ is a $\tau_n(\Phi)$ -neighbourhood, $(y + U) \cap V \neq \emptyset$. Pick $u \in U$ with $y + u \in V$. By 7.3 there is $w \in W_n$ such that $w + u \in V$. Thus we obtain $y - w \in W \cap V = (0)$ so $y = w \in W_n$.

7.10. On the inclusion \subseteq in 7.6: Let $x \in \tau_n V$. By 7.8 we may modify x and assume $x = w + \sum_{j=0}^{n-1} \sum_{\gamma} c_{j,\gamma} e_{j,\gamma}(0)$ for certain scalars $c_{j,\gamma}$. Fix a pair (j, γ) . By the choice of the basis $\{w_{j,0}, \dots, w_{j,\gamma}, \dots\}$ in 7.1 there exist, for $\gamma \geq n-j$, scalars $d_{j,s}$ ($0 \leq s \leq n-j-1$) such that

$$w_{j,\gamma} - \sum_{s=0}^{n-j-1} d_{j,s} w_{j,s} \in W_n,$$

so

$$e_{j,\gamma}(0) - \sum_{s=0}^{n-j-1} d_{j,s} e_{j,s}(0) \in C_{j,n} \subseteq \tau_n V.$$

We may thus assume without loss of generality that $c_{j,\gamma} = 0$ if $\gamma \geq n-j$. In the space span $\{w_{0,0} + e_{0,0}(0), \dots, w_{n-1,0} + e_{n-1,0}(0)\} \subset V$ (compare 7.6) there exists therefore a vector y such that $x - y \in W$. Since $x - y \in \tau_n V$ also, we have $x - y \in W \cap \tau_n V = W_n$. This establishes the inclusion \subseteq in 7.6 and the proof of 7.6 is complete.

7.11. From 7.6 we obtain by counting that

$$\dim \tau_n V / \tau_{n+1} V = 1 \quad (n < \omega_0).$$

7.12. Since $V \oplus W_n \subseteq \tau_n V$ and $V \oplus W_0 = V \oplus W = E = \tau_0 V$ and because $\dim (V \oplus W_n) / (V \oplus W_{n+1}) = 1 = \dim \tau_n V / \tau_{n+1} V$ by 7.11, we obtain inductively that $V \oplus W_n = \tau_n V$. This terminates the proof of Theorem 3.

Remark 4. Let E be an orthogonal sum of \aleph_{ω_0} nondegenerate finite dimensional spaces $E_i \neq (0)$, \mathcal{B}_i a basis in E_i , $\mathcal{B} := \bigcup_i \mathcal{B}_i$. The hyperplane $V := \{\text{span } b - b' \mid b, b' \in \mathcal{B}\}$ has $\tau_n V = E$ for all $n < \omega_0$. By taking external orthogonal

sums of d copies of this example one obtains examples of prediagonal V with

$$\tilde{V} = \sigma_1 V = \sigma_{\omega_0} V \supset V \supset (0), \quad \dim \sigma_{\omega_0} V / V = d.$$

By using this example and Theorem 3 to form orthogonal sums one can obtain many new examples (Cf. III. 5).

Problem 1. Show that $\mathcal{V}(V)$ is infinite when $\alpha \cong 4$ (see II. 4 for the definition of $\mathcal{V}(V)$). A possible line of attack would be to use the technique of the proof of Theorem 3 to “produce” four previously fixed “independent” elements inside $\mathcal{V}(V)$; they would generate an infinite sublattice under $+$ and \cap alone. See also [13], [14].**)

Problem 2. $\mathcal{V}(V)$ when $\alpha < 3$.**)

II. 8. *The orbit of a chain under the group $GL(E)$ is larger than the orbit under $\mathcal{O}(E, \Phi)$.* Let (E, Φ) be an orthogonal sum of 2^{\aleph_0} hyperbolic planes and let the base field k have $|k| \cong 2^{\aleph_0}$. We are going to construct reduced chains $\mathcal{C} = (V_i)_{i < \omega_0}$, $\mathcal{C}' = (V'_i)_{i < \omega_0}$ in $\mathcal{L}(E)$ with property (0) and such that

(j) for each ordinal $\gamma \cong 0$ with $\aleph_\gamma < 2^{\aleph_0}$ the spaces V_i, V'_i are $\tau_\gamma(\Phi)$ -dense in E ($i < \omega_0$),

(jj) $(E, \varrho(\mathcal{C}))$ and $(E, \varrho(\mathcal{C}'))$ are complete.

The natural lattice isomorphism $\eta: \mathcal{C} \rightarrow \mathcal{C}'$ preserves rank, obviously, and commutes with all operations $\tau_\gamma, \sigma_\gamma$ (cf. II. 3). Furthermore, by (jj) it is clear that η is induced by an element $T \in GL(E)$. We assert

Theorem 4. *There are chains $\mathcal{C}, \mathcal{C}'$ with (0), (j), (jj) such that η is induced by an element $T \in GL(E)$ but not by any element of $\mathcal{O}(E, \Phi)$.*

Proof. For a fixed chain \mathcal{C} with (0) and (jj) we construct two forms Φ, Ψ on E such that (j) holds for either form and such that both (E, Φ) and (E, Ψ) are orthogonal sums of hyperbolic planes. Further, no $T \in GL(E)$ with $TV_i = V_i$ ($i < \omega_0$) will be an isometry $(E, \Phi) \rightarrow (E, \Psi)$. If $\varphi: (E, \Phi) \rightarrow (E, \Psi)$ is any fixed isometry and $\mathcal{C}' := \varphi\mathcal{C} = (\varphi V_i)_{i < \omega_0}$ then one obtains the assertion of the theorem.

The number of mappings $T \in GL(E)$ with $TV_i = V_i$ ($i < \omega_0$) is precisely $\aleph_\alpha := |k|^{\aleph_0} = 2^{\aleph_0}$; for these maps are describable by triangular $\aleph_0 \times \aleph_0$ -matrices (cf. the proof in I. 6). Let $\mathcal{E} = \{T_\xi \mid 0 \cong \xi < \omega_\alpha\}$ be an enumeration of these T .

We define recursively vectors $x_\xi, \bar{x}_\xi \in E$, $0 \cong \xi < \omega_\alpha$. To this end we subdivide the ω_0 -sections of ω_α into adjacent intervals of four elements each; we demand

- (i) $(x_\xi)_\xi, (\bar{x}_\xi)_\xi$ are linearly independent families,
- (ii) for all limit numbers $\lambda < \omega_\alpha$ we have $x_{\lambda+4n+r} \in V_n \setminus V_{n+1}$
(for all $n < \omega_0; r = 0, 1, 2, 3$),

***) Note added in proof. Problem 1 has been solved along the line indicated. For $\alpha > 3$ the lattices are infinite, in general, but their structure is not known ([19, 20]).

- (ii) for all limit numbers $\lambda < \omega_\alpha$ we have $\bar{x}_{\lambda+4n+r} \in V_n \setminus V_{n+1}$
 (for all $n < \omega_0$; $r = 0, 1, 2, 3$),
 (iii) for all limit numbers $\lambda < \omega_\alpha$ and all $n < \omega_0$ we have

$$T_{\lambda+n}: x_{\lambda+4n+r} \rightarrow \bar{x}_{\lambda+4n+r} \quad (r = 0, 1, 2, 3).$$

Assume that x_ξ, \bar{x}_ξ for $\xi < \lambda + 4n$ are already constructed, and let $F := \text{span} \{x_\xi, T_{\lambda+n}^{-1}(\bar{x}_\xi) \mid \xi < \lambda + 4n\}$, $\dim F < \dim E$. Because $\dim \text{span}(V_n \setminus V_{n+1}) = \aleph_\alpha$ linearly independent elements $y_0, y_1, y_2, y_3 \in V_n \setminus V_{n+1}$ can be found such that $F \cap \text{span} \{y_0, y_1, y_2, y_3\} = (0)$. Set $x_{\lambda+4n+r} := y_r$, $\bar{x}_{\lambda+4n+r} := T_{\lambda+n}(x_{\lambda+4n+r})$, ($r = 0, 1, 2, 3$). By construction (i), (ii), (iii) are satisfied; (ii) is satisfied because V_n, V_{n+1} are invariant under $T_{\lambda+n}$. On $X := \text{span}(x_\xi)_{\xi < \omega_\alpha}$ and $\bar{X} := \text{span}(\bar{x}_\xi)_{\xi < \omega_\alpha}$ we define forms Φ, Ψ according to the table

$$(iv) \quad \begin{array}{l|l} \Phi(x_{\lambda+4n}, x_{\lambda+4n+1}) = 1 & \Psi(\bar{x}_{\lambda+4n}, \bar{x}_{\lambda+4n+2}) = 1 \\ \Phi(x_{\lambda+4n+2}, x_{\lambda+4n+3}) = 1 & \Psi(\bar{x}_{\lambda+4n+1}, \bar{x}_{\lambda+4n+3}) = 1 \\ \text{all other products zero} & \text{all other products zero} \end{array}$$

One may always carry out the construction of the x_ξ in such a way that $\dim E/X, \dim E/\bar{X}$ are of equal infinite dimension; hence Φ and Ψ may each be extended to a nondegenerate symplectic form on all of E by letting X, \bar{X} be orthogonal summands.

It is evident that no $T \in \text{GL}(E)$ with $TV_i = V_i$ ($i < \omega_0$) can be an isometry $(E, \Phi) \rightarrow (E, \Psi)$.

In order to verify (j) let be given $\gamma < \alpha, V_n, z \in E$ and a $\tau_\gamma(\Phi)$ -neighbourhood $z + U^\perp$ ($\dim U \cong \aleph_\gamma$). There is a limit number $\lambda < \omega_\alpha$ such that $x_\xi \in U^\perp$ for all $\xi \cong \lambda$. By (ii) there is a linear combination

$$u = \alpha_0 x_\lambda + \alpha_1 x_{\lambda+4} + \dots + \alpha_{n-1} x_{\lambda+4(n-1)}$$

such that $z - u \in V_n$; so $(z + U^\perp) \cap V_n \neq \emptyset$ as was to be shown. The argument for $\tau_\gamma(\Psi)$ is mutatis mutandis the same.

II. 9. *The orbits under $\text{GL}(E)$ are larger than the orbits under the group $\mathcal{H}(E, \Phi)$ of all linear $\tau_0(\Phi)$ -homeomorphisms $E \rightarrow E$.* The technique used in II.8. can be further developed: we can show

Theorem 5. *Let k, E, Φ be as in II.8. There are chains $\mathcal{C}, \mathcal{C}'$ with (0), (j), (jj) such that η is induced by an element $T \in \text{GL}(E)$ but not by any element of $\mathcal{H}(E, \Phi)$.*

Proof. As in the foregoing proof we fix a given chain with properties (0) and (jj) and construct two symplectic forms Φ and Ψ on E .

Let $\omega_\alpha = \bigcup_{i < \omega_\alpha} P_i$ be a fixed partitioning of ω_α such that $|P_i| = \aleph_\alpha$ and each P_i is a union of some ω_0 -sections of ω_α . Let \mathcal{C} be as in II.8.

We then construct $(x_\xi)_\xi, (\bar{x}_\xi)_\xi$ with (i), (ii), (ii) just as before; instead of (iii) however we demand that

(iii*) for all limit numbers $\lambda < \omega_\alpha$ and all $n < \omega_0$ we have

$$T_\xi: X_{\lambda+4n+r} \rightarrow \bar{x}_{\lambda+4n+r} \quad (r = 0, 1, 2, 3)$$

where $P_\xi \ni \lambda + 4n + r$. In other words, whereas in the previous proof each $T_\xi \in \mathcal{E}$ was used to map one quadruple of x 's only, we will here use each T_ξ to map \aleph_α many quadruples.

The form Φ is defined on E as before (II.8 (iv)).

The form Ψ is defined as follows. Let $\mathcal{B}'_\xi := \{\bar{x}_{\lambda+4n+r} | r=0, 1; \lambda+4n+r \in P_\xi\}$, $\mathcal{B}''_\xi := \{\bar{x}_{\lambda+4n+r} | r=2, 3; \lambda+4n+r \in P_\xi\}$. Define Ψ'_ξ on $\text{span } \mathcal{B}'_\xi$ to be the transported form: $\Psi'_\xi(\bar{x}, \bar{y}) := \Phi(T_\xi^{-1}\bar{x}, T_\xi^{-1}\bar{y})$ for all $\xi < \omega_\alpha$. (This part is to guarantee that (j) will hold for Ψ as well.) In order to define Ψ''_ξ on $\text{span } \mathcal{B}''_\xi$, transform the \aleph_α elements of \mathcal{B}''_ξ by some invertible matrix that is row-finite but has non-denumerably many non-zero entries in some of its columns. Let $\bar{\mathcal{B}}_\xi = \{\bar{x} | \bar{x} \in \mathcal{B}''_\xi\}$ be the resulting basis of $\text{span } \mathcal{B}''_\xi$. The \bar{x} s naturally come in pairs (as transforms of pairs $\bar{x}_{\lambda+4n+2}, \bar{x}_{\lambda+4n+3}$). We declare these pairs of \bar{x} 's to form a symplectic basis of $\text{span } \mathcal{B}''_\xi$ thereby defining Ψ''_ξ . The upshot is the following: If Φ is transported under T_ξ onto $\text{span } \mathcal{B}''_\xi$ then this form will make the basis $\bar{\mathcal{B}}_\xi$ a non-continuous τ_0 -basis [3, p. 30 Satz 11].

Because $(\bar{x}_\xi)_{\xi < \omega_\alpha} = \bigcup_{\xi < \omega_\alpha} \mathcal{B}'_\xi \cup \mathcal{B}''_\xi$ is a disjoint union we can form the external orthogonal sum of all Ψ'_ξ, Ψ''_ξ and obtain the required symplectic form Ψ on all E as in the proof of Theorem 4.

By the very construction it is clear that no element $T \in \mathcal{E}$ is a homeomorphism $(E, \tau_0(\Phi)) \xrightarrow{\sim} (E, \tau_0(\Psi))$.

Problem 3. Study the orbits under $\mathcal{O}(E, \Phi), \mathcal{H}(E, \Phi)$ of the chain \mathcal{C} given in (10) in the special situation: $\alpha = \omega_0$ and $\dim \sigma_n V / \sigma_{n+1} V = 1$ ($n < \omega_0$) and $(E, \rho(\mathcal{C}))$ is complete. (The last assumption implies that all such chains \mathcal{C} form just one single orbit under $\text{GL}(E)$.)

Theorems 4 and 5 nicely demonstrate that the classification problems in the metric situation are not merely shadows of projective or affine problems; they certainly have their own status. The mapping problems I.1 and II.1 are of equal standing. The solution of one does not solve the other.

III. Solution of the mapping problem in special metric cases

III.1. On the lattice method in the presence of descending chains. The recursive method developed in [8] in order to solve the mapping problem II.1. for finite lattices is not, in general, adequate if the lattice contains infinite *descending* chains. This fact clearly manifests itself already in the simplest case when \mathcal{V} is a chain $\mathcal{C} = (V_i)_{i < \omega_0}$, reduced or not, with property (0) and $\dim E = \aleph_1$. For, the partial

maps φ whose limit ought to induce a lattice isomorphism

$$\eta: \mathcal{C} \rightarrow \mathcal{C}'$$

are homeomorphisms with respect to $q(\mathcal{C}), q(\mathcal{C}')$ by (1). Further, whatever course the recursion may take, after countably many steps supplements of V_{i+1} in V_i for all i will, by necessity, be contained in $\text{dom } \varphi$; i.e. $\text{dom } \varphi$ will be $q(\mathcal{C})$ -dense in E . Hence, the limit map, if such there is, will be uniquely determined modulo $\cap \mathcal{C}$ by this partial map φ .

Therefore, in order that the recursion will not be hopelessly stuck after the first few steps when $\dim E/\cap \mathcal{C} > \aleph_0$, additional assumptions on the chains are mandatory.

In [8] the class of finite (distributive) lattices was treated. By the same arguments the class of (distributive) *countable lattices with DCC* can be taken care of.

In the following we shall solve the mapping problem for two more classes of lattices. As these allow for infinite descending chains of a certain kind, the range of applicability of the results is greatly increased.

III.2. *Almost principal filters.* We say that a filter $\mathcal{G} \subset \mathcal{V}$ is almost a principal filter if

$$(11) \quad \min \{ \dim G/\cap, |G \in \mathcal{G} \} \cong \aleph_0.$$

In this section we treat of complete lattices $\mathcal{V} \subset \mathcal{L}(E)$ all of whose filters \mathcal{G} satisfy condition (11) and the well-known condition (4). We assert

Theorem 6. *Let (E, Φ) be a diagonal alternate nondegenerate space of dimension \aleph_α , $0 < \alpha < \omega_1$, \mathcal{V} a countable, complete, distributive sublattice of $\mathcal{L}(E)$ satisfying (4) and stable under the operations \perp and $\tau_\gamma (\gamma < \alpha)$. Let \mathcal{V}' be another lattice of the same kind in $\mathcal{L}(E)$ and*

$$\eta: \mathcal{V} \rightarrow \mathcal{V}'$$

a lattice isomorphism preserving rank and commuting with \perp and $\tau_\gamma (\gamma < \alpha)$. In order that η be induced by an isometry $T \in \mathcal{O}(E, \Phi)$ it is sufficient that all filters $\mathcal{G} \subset \mathcal{V}$ satisfy condition (11). Thus, in the class of complete distributive lattices with (4) and (11) described here, the orbits under the orthogonal group are characterized by cardinal invariants (namely the family of indices $\dim A/B, A \supset B$ in \mathcal{V}).

The proof of Theorem 6 follows essentially the line of the proof of [8, Theorem 2]. We refrain here from describing the details. We should rather like to stress some points of major interest.

III.3. *Remarks about Theorem 6.* 5. The reasons for the condition " $\alpha < \omega_1$ " are described in [8, Remarks 1, 9]. The discussion of this other great barrier \aleph_{ω_1} will not be taken up here.

Remark 6. (The role of condition (11).) One tries to follow the setup in I.4 and construct the required isometry T as a limit of partial maps $\varphi: X \rightarrow X'$ of a suitable subfamily $\mathcal{F}_2(\eta) \subset \mathcal{F}_1(\eta)$. Let $x \in E \setminus X$. The fact that the filter

$$\mathcal{M}(x, X) := \{A \in \mathcal{V} \mid x \in X + A\}$$

be principal is crucial for the possibility of extending φ to $X \oplus (x)$. That this will indeed be the case is taken care of as follows. In countably many steps we first construct a $(\varphi_*: X_* \rightarrow X'_*) \in \mathcal{F}_1(\eta)$ with the following properties:

(a) $\dim X_* \cong \aleph_0$, (b) if \mathcal{G} is a filter in \mathcal{V} and $D := \cap \mathcal{G}$ then $X_* + D \supseteq G$ for some $G \in \mathcal{G}$. It is then clear that for any extension $(\varphi: X \rightarrow X') \in \mathcal{F}_1$ of φ_* the filter $\mathcal{M}(x, X)$ is principal. Set $\mathcal{F}_2(\eta) := \{\varphi \in \mathcal{F}_1(\eta) \mid \varphi \supseteq \varphi_*\}$. We are left to show how φ_* is found.

For each filter $\mathcal{G} \subset \mathcal{V}$ let $D := \cap \mathcal{G}$; by (11) there is $S \in \mathcal{L}(E)$ with $\dim S \cong \aleph_0$ such that $G := D + S \in \mathcal{G}$. As $|\mathcal{V}| \cong \aleph_0$ the number of all possible pairs $(D, G) \in \mathcal{V} \times \mathcal{V}$ is countable. If, for each $\mathcal{G} \subseteq \mathcal{V}$, we pick such an S , then their sum S_* is of countable dimension. We require $S_* \subset \text{dom } \varphi_*$; φ_* can be obtained as the limit of a chain $(\varphi_i: X_i \rightarrow X'_i)_{i < \omega_0}$ in $\mathcal{F}_1(\eta)$ with $\dim X_i < \infty$ throughout. For finite $\dim X_i$ the filter $\mathcal{M}(x, X_i)$ is invariably principal.

III.4. *Compatible pairs* (\mathcal{V}, Φ) . Let (E, Φ) be a sesquilinear space and $\mathcal{V} \subset \mathcal{L}(E)$ a sublattice. \mathcal{V} and Φ are said to be compatible, if (E, Φ) admits at least one orthogonal decomposition $E = \bigoplus_I^\perp E_i$, $\dim E_i \cong \aleph_0$, such that the following condition is satisfied:

(12) If $E^+ = \bigoplus_J E_i$ with $J \subset I$ and $|J| < |I|$ and $\mathcal{G} \subset \mathcal{V}$ is a filter then the $\varrho(\mathcal{G})$ -closure of E^+ is $E^+ + \cap \mathcal{G}$ (i.e. the image of E^+ in $E/\cap \mathcal{G}$ is closed in the (hausdorff) quotient topology $\hat{\varrho}$ associated with $\varrho(\mathcal{G})$).

Remark 7. It is not true that, for a compatible pair (\mathcal{V}, Φ) , condition (12) holds for every orthogonal decomposition $E = \bigoplus E_i$ ($\dim E_i \cong \aleph_0$) of E ; for this would mean that each orthogonal summand F of E ($E = F + F^\perp$) would be ϱ -closed, and to this one can find counterexamples.

If (V, Φ) is a nondegenerate prediagonal space of a dimension \aleph_n , $n < \omega_0$, then the lattice (chain) $\mathcal{V}(V)$ in $\mathcal{L}(\tilde{V})$ is finite, hence $(\mathcal{V}(V), \tilde{\Phi})$ is a compatible pair. Less obvious examples will be considered in Sections III.5, III.6.

Theorem 7. Let (E, Φ) , $\mathcal{V}, \mathcal{V}'$ be as in Theorem 6 and let $\eta: \mathcal{V} \rightarrow \mathcal{V}'$ be a lattice isomorphism that commutes with \perp and τ_γ ($\gamma < \alpha$) and that preserves $\dim X/X_0$ for each compact join-irreducible $X \neq (0)$ in \mathcal{V} and its (unique) antecedent X_0 . In order that η be induced by an isometry $T \in \mathcal{O}(E, \Phi)$ it is sufficient that $\mathcal{V}, \mathcal{V}'$ are compatible with Φ .

The proof of Theorem 7 follows the general scheme detailed in [8]. Difficulties caused by the presence of infinite descending chains are offset by compatibility

which forces the crucial filters $\mathcal{M}(x, X)$ to be principal. Preservation of rank is needed only in Section 2.5 of [8] and only for pairs X, X_0 as indicated in Theorem 7.

Remark 8. Theorem 7 is an instance where preservation of rank follows from equality among certain selected indices by virtue of compatibility. Does this mean that, in the compatible case, well ordered descending chains $\mathcal{V} \subset \mathcal{L}(E)$ necessarily have additivity of supplements in the sense of the following definition?

Definition 1. The well ordered descending chain $\mathcal{V} = (X_i)_{0 \leq i < \gamma}$ in $\mathcal{L}(E)$ enjoys additivity of supplements if and only if there exists a family $(S_i)_{0 \leq i < \gamma}$ with

$$(13) \quad X_{i+1} \oplus S_i = X_i \quad (0 \leq i < \gamma)$$

and such that for arbitrary $v < \mu < \gamma$ we have

$$(14) \quad X_\mu \oplus \bigoplus_{v \leq i < \mu} S_i = X_v.$$

Remark 9. Given a family $(S_i)_i$ with (13) it is sometimes believed that the fact whether or not the equality (14) holds is a property of \mathcal{V} . This is of course not so: in dimension \aleph_0 each chain $(X_i)_{0 \leq i < \omega_0 + 1}$ admits some families $(S_i)_i$ with (13) that make (14) true and others with (13) that falsify (14).

Problem 5. Answer the question enunciated in Remark 8 (for the particular chains $\mathcal{V} = \mathcal{V}(V)$ considered in Theorem 8 below the answer is affirmative).

Problem 6. Classify compatible pairs (\mathcal{V}, Φ) .

III.5. *Examples of compatible pairs (orthogonal decompositions)*. We describe here a particularly simple brand of prediagonal spaces V that have the property that their lattices (chains) $\mathcal{V}(V)$ (10) are compatible with the form on the diagonal hull $\tilde{V} = E$. As we shall be able to prescribe indices Theorem 7 will be applicable: the examples offer themselves as normal forms (see the following section).

Let d_i ($1 \leq i \leq \alpha$) be arbitrary cardinals with $0 \leq d_i \leq \aleph_\alpha$; let furthermore U_γ be any prediagonal space of dimension \aleph_γ ($\gamma > 0$) with

$$(15) \quad \dim \tilde{U}_\gamma / U_\gamma = 1, \quad \tau_v U_\gamma = \tilde{U}_\gamma, \quad \text{for all } 0 \leq v < \gamma,$$

(there are such spaces by Remark 4). We form external orthogonal sums

$$(16) \quad V := \bigoplus_{1 \leq i \leq \alpha}^\perp d_i \cdot U_i.$$

We then read off:

$$(17) \quad \tau_v V = \sigma_{v+1} V = \bigoplus_{1 \leq i \leq v} d_i \cdot U_i \oplus \bigoplus_{v < i \leq \alpha} d_i \cdot \tilde{U}_i \quad (0 \leq v < \alpha),$$

$$(18) \quad \bigcap_{v < \lambda} \tau_v V = \sigma_\lambda V = \bigoplus_{1 \leq i < \lambda} d_i \cdot U_i \oplus \bigoplus_{\lambda \leq i \leq \alpha} d_i \cdot \tilde{U}_i \quad (\lim(\lambda) \lambda \leq \alpha),$$

$$(19) \quad \dim \sigma_v V / \sigma_{v+1} V = d_v \quad (2 \leq v+1 \leq \alpha),$$

$$(20) \quad \dim \sigma_\mu V / \sigma_v V = \sum_{\mu \leq i < v} d_i \quad (1 \leq \mu < v \leq \alpha),$$

$$(21) \quad \dim \sigma_\alpha V / V = d_\alpha.$$

It would be very easy, but cumbersome, to write the last three formulae in terms of the τ 's. We finish by establishing compatibility: As for $E = \bigoplus_{i < \omega_\alpha} E_i$ we choose a decomposition which is obtained by throwing together similar decompositions of the $\tau_0(\Phi)$ -completions of the single summands of V in (16), $\dim E_i \cong \aleph_0$. We consider a subspace $E^+ = \bigoplus F_i$ with $F_i \subset E_i$ and a limit number λ with $0 < \lambda \leq \alpha$. We shall show that

$$E^+ + \sigma_\lambda V \cong \bigcap_{v < \lambda} (E^+ + \sigma_v V).$$

Since the converse inclusion is trivial this tells that E^+ is "closed" under $\varrho((\sigma_v V)_{1 \leq v \leq \lambda})$. Let p be an element of the intersection. There is $i_0 < \lambda$ such that $p = p_1 + p_2$, $p_1 \in L := \bigoplus_{i \leq i_0} E_i$, $p_2 \in \bigoplus_{i \geq \lambda} E_i$. We have $p_2 \in \sigma_\lambda V$ by (18), hence p_1 is also an element of the intersection (ϱ -closure of E^+). Let $\pi: E \rightarrow L$ be the projection along L^\perp . As $F_i \subset E_i$ we have $\pi E^+ \subset E^+$, $\pi \tau_v V \subset \tau_v V$. For each $v < \lambda$ there is a decomposition $p_1 = x(v) + y(v)$ with $x(v) \in E^+$, $y(v) \in \sigma_v V$. Ergo $p_1 = \pi p_1 = \pi x(v) + \pi y(v) \in (E^+ \cap L) + (\sigma_v V \cap L)$. We fix a v , $i_0 < v < \lambda$, and find by (17) $\sigma_v V \cap L = \bigoplus_{i \leq i_0} d_i \cdot U_i$; thus $p_1 \in E^+ + V$. Altogether $p = p_1 + p_2 \in E^+ + \sigma_\lambda V$.

III.6. *Normal forms of prediagonal spaces.* Let E ($\dim E = \aleph_\alpha$) be any diagonal space; it can be written as an orthogonal sum of lines and planes

$$(22) \quad E_i = \bigoplus^\perp F_\xi, \quad 1 \leq \dim F_\xi \leq 2.$$

Choose any basis \mathcal{B}_ξ of F_ξ and set $\mathcal{B} := \cup \mathcal{B}_\xi$. If $\mathcal{L} > 0$ then the hyperplane

$$(23) \quad U_i := \text{span} \{b - b' \mid b, b' \in \mathcal{B}\}$$

is prediagonal and not diagonal [7, p. 104, Satz 1 and its obvious generalization to the case of hyperbolic planes]; it is the simplest specimen of its kind. Since it enjoys property (15) it can be used in (16). We obtain:

Theorem 8. *Let (V, Φ) be a nondegenerate prediagonal alternate space of dimension \aleph_α ($0 < \alpha < \omega_1$). The statements (i) and (ii) are equivalent:*

- (i) *The chain $\mathcal{V}(V) \subset \mathcal{L}(\tilde{V})$ (cf. (10)) is compatible with the form $\tilde{\Phi}$ on \tilde{V} .*
- (ii) *V has the form (16), with U_i as in (23), $\tilde{V} = \bigoplus_{i \leq \alpha} d_i \cdot E_i$, with uniquely determined cardinals d_i ($1 \leq i \leq \alpha$) given by (19) and (21).*

Proof. "(ii) \Rightarrow (i)" is the content of III.5. Conversely, given the chain $\mathcal{V}(V)$ we can make the d_i in (16) equal to the indices of corresponding neighbouring elements in $\mathcal{V}(V)$. An application of Theorem 7 yields "(i) \Rightarrow (ii)".

Corollary. *Let (V, Φ) be as in (i) of Theorem 8. Then we have additivity of supplements in the sense of Definition 1 (Remark 8); in particular we have "additivity of indices" in the sense of (20).*

Remark 10. By using Theorem 3 we can easily give examples of prediagonal V such that $\mathcal{V}(V)$ fails to have additivity. In these cases $\mathcal{V}(V)$ cannot be compatible with the form on \tilde{V} . By Theorem 8, these V cannot be orthogonally de-

composed into the building blocks U_i . Of course, $\alpha \cong \omega_0$ in these cases. In contrast, if $\alpha < \omega_0$, the requirement of compatibility is empty and we have

Theorem 9. *Every nondegenerate prediagonal alternate space V of dimension \aleph_n , $n < \omega_0$, is either diagonal or of the shape*

$$V = d_1 \cdot U_1 \oplus \dots \oplus d_n \cdot U_n \quad (\dim U_i = \aleph_i),$$

and its diagonal hull $E = \tilde{V}$ is

$$E = d_1 \cdot E_1 \oplus \dots \oplus d_n \cdot E_n,$$

where $E_i \supset U_i$ is given by (22), (23). Furthermore,

$$d_i = \dim \tau_{i-1}V / \tau_i V \quad (1 \leq i < n), \quad d_n = \dim \tau_{n-1}V / V;$$

the n cardinals $0 \leq d_i \leq \aleph_n$, together with $\dim V = \aleph_n$, form a complete set of independent invariants for the isometry class of V .

If $(\mathcal{V}(V), \tilde{\Phi})$ is compatible for V , a nondegenerate alternate prediagonal space of dimension \aleph_α ($\alpha < \omega_1$), then we learn from Theorem 8, (17), (18) and from the explicit shape of the U_i in (22), (23) that with (10) there is associated an ascending chain $(S_i)_{1 \leq i < \alpha+1}$ of totally isotropic ‘‘supplements’’ with

$$(24) \quad \sigma_i V_i \oplus S_i = \tilde{V}, \quad V \oplus S_{\alpha+1} = \tilde{V}, \quad S_i \text{ is } \sigma_0(\tilde{\Phi}) \text{ closed in } \tilde{V}$$

(and hence $\sigma_1(\tilde{\Phi})$ -closed and thus $\sigma_1(\tilde{\Phi})$ -complete), $1 \leq i \leq \alpha+1$.

Problem 6. Does each nondegenerate alternate prediagonal space admit a totally isotropic supplement in its diagonal hull?

A solution of Problem 6 is important for practical calculations; e.g., there is the following lemma which we mention without proof:

Lemma 3. *Let W be any nondegenerate prediagonal space and \bar{W} its μ -diagonal hull, $\mu \geq 0$ a fixed ordinal. Let $U \subset W$ be a μ -diagonal nondegenerate subspace which is maximal in the sense that $\dim W|U$ is minimal. If W admits a totally isotropic supplements in \bar{W} then $\dim \bar{W}|W = \dim W|U$.*

If \bar{W} is anisotropic in Lemma 3 we may very well have $\dim \bar{W}|W < \dim W|U$; see [7, p. 110, Satz 5].

III.7. Topological characterization of the isometry classes of certain prediagonal spaces. From an isometry $(E, \Phi) \cong (E', \Phi')$ between two arbitrary sesquilinear spaces we obtain of course a homeomorphism $(E, \tau_0(\Phi)) \cong (E', \tau_0(\Phi'))$. When do we have the converse?

$$(25) \quad (E, \tau_0(\Phi)) \cong (E', \tau_0(\Phi')) \Rightarrow (E, \Phi) \cong (E', \Phi').$$

That there are interesting instances of (25) is demonstrated by Theorem 8 and

Theorem 10. *If (V, Φ) and (V', Φ') are alternate spaces of the shape described in (16) then the implication (25) is valid. In particular, for the nondegenerate, alternate,*

prediagonal spaces of dimension \aleph_n with $n < \omega_0$ isometry classes and τ_0 -homeomorphy classes coincide.

Proof. If $(V, \tau_\mu(\Phi)) \cong (V', \tau_\mu(\Phi'))$ holds for a particular value of μ , then it holds for all larger values as well. Thus, if it holds for $\mu=0$ we obtain equality of all indices, $\dim \sigma_\mu V / \sigma_{\mu+1} V = \dim \sigma_\mu V' / \sigma_{\mu+1} V'$. Hence the assertion.

In [3, p. 37, IV. 2] an example of a homeomorphism $(E, \tau_0(\Phi)) \cong (E', \tau_0(\Phi'))$ is given with one space (E, Φ) prediagonal and not the other. A newer result by Băni [6] tells that, in contrast to τ_0 , the topology σ_0 "decides" on prediagonality: if under a homeomorphism $(E, \sigma_0(\Phi)) \cong (E', \sigma_0(\Phi'))$ one space is prediagonal then so is the other, whence

Theorem 11. *Let (V, Φ) be a nondegenerate alternate prediagonal space such that the chain $\mathcal{V}(V)$ is compatible with the form $\tilde{\Phi}$ on the diagonal hull \tilde{V} . Let (V', Φ') be any alternate space (over the same base field) such that $(V, \sigma_0(\Phi)) \cong (V', \sigma_0(\Phi'))$. If $\dim V < \aleph_{\omega_1}$ then there is an isometry $(V, \Phi) \cong (V', \Phi')$.*

If we do not assume compatibility (which means passing to dimensions $\cong \aleph_{\omega_0}$) then there are results which point to quite different directions. We give just one illustration:

(26) There are prediagonal spaces $(E, \Phi), (E', \Phi')$ of dimension \aleph_{ω_0} such that all closures $\tau_\iota E, \tau_\iota E'$ ($0 \leq \iota < \omega_0$) are isometric yet (E, Φ) and (E', Φ') are not isometric.

Problem 7. What classes of spaces (other than that described in Theorem 10) satisfy the implication (25)?

Remark 11. Theorem 10 is particularly satisfying in that there is no reference to the diagonal hull of the spaces to be classified. We are aware of the fact that behind our treatment of prediagonal spaces V (II.5) there lurks the possibility to treat these spaces from within, i.e. without plunging them into their hulls. One should first try to compute the invariants $\dim \sigma_\iota V / \sigma_{\iota+1} V$ inside V by expressing them as generalized defects in the spirit of [7, Chapter III].

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Received 22 June 1982