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ON f''/f' AND INJECTIVITY

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Duren, Shapiro and Shields [DSS] seem to have been the first to observe that there is a constant $A>0$ such that for a function f analytic in the unit disk D, $\sup\{(1-|z|^2)|f''(z)/f'(z)|: z\in D\}\leqq A$ implies that f is univalent in D. Their proof with $A=2(\sqrt{5}-2)$ was based on a univalence criterion involving the Schwarzian derivative due to Nehari [N]. Using the Löwner differential equation, Becker [B] subsequently showed that the same theorem holds with $A=1$. By means of an elementary argument Martio and Sarvas [MS] established the following analogous fact for analytic functions in a uniform domain U: there exists a constant A, depending on two parameters which roughly limit the shape of U, such that $\sup\{\operatorname{dist}(z,\partial U)|f''(z)/f'(z)|: z\in U\}<A$ implies that f is univalent in U, where $\operatorname{dist}(z,\partial U)$ is the distance from z to ∂U. The purpose of this note is to show how the argument of Martio and Sarvas may be used to obtain a similar injectivity criterion for mappings of a uniform domain in a normed linear space.

First we set down our notation and terminology. X and Y will always be real normed linear spaces and U will always be a domain in X. The conjugate space of Y is denoted by Y^*. Norms of elements, linear functionals and linear transformations are all denoted by $|\cdot|$. For $f: U\to Y$ we let $\Delta_1 f(x,h)=|f(x+h)-f(x)|/|h|$ and $\Delta_2 f(x,h)=|f(x+h)+f(x-h)-2f(x)|/|h|^2$. We define $D^+f(x)$ and $D^-f(x)$ to be, respectively, the upper and lower limits of $\Delta_1 f(x,h)$ as $h\to 0$, and we denote by $D_L f(x)$ the upper limit of $\Delta_2 f(x,h)/\Delta_1 f(x,h)$ as $h\to 0$. Furthermore, for $x\in U$ and $a\in X$ we denote by $f'(x,a)$ the derivative of f in the direction a; that is, the limit in the norm topology of Y of $(f(x+ha)-f(x))/h$ as $h\to 0$. Obviously, if $f'(x,a)$ exists, then $f'(x,ta)=tf'(x,a)$ and $D^-f(x)\leqq|f'(x,a)|/|a|\leqq\leqq D^+f(x)$. The mapping f is said to be differentiable at x if there exists a bounded linear transformation $T: X\to Y$ for which $|f(x+h)-f(x)-T(h)|/|h|\to 0$ as $h\to 0$. This linear transformation is denoted by $f'(x)$.

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If X and Y are both the complex plane considered as normed spaces with the usual Euclidean norm and f is an analytic function with $f'(z)\neq 0$, then clearly $D_L f(z)=|f''(z)/f'(z)|$. We generalize Martio's and Sarvas' injectivity criterion to cover the class of mappings we now define by using $D_L f$ as a substitute for $|f''(z)/f'(z)|$.

Definition 1. If $K\geqq 1$, then $Q(U, Y, K)$ denotes the class of locally Lipschitz mappings $f: U\to Y$ such that $D^+f(x)\leqq KD^-f(x)$ for all $x\in U$ and for each $x\in U$ there is a $\delta>0$ such that $f(x+h)\neq f(x)$ for $0<|h|<\delta$.

For example, if X and Y are both the complex plane and f is an analytic function on U with $f'(z)\neq 0$, then $f\in Q(U, Y, 1)$. Similarly, if X and Y are n-dimensional Euclidean spaces and $f: U\to Y$ is continuously differentiable and locally K-quasiconformal, then $f\in Q(U, Y, K)$. We point out, however, that a mapping in $Q(U, Y, K)$ need be neither locally one-to-one nor open.

Aside from changes in the meaning of the parameters the following definition is due to Martio [M].

Definition 2. An open subset U of X is an (a, b)-*uniform* domain if any two points x and y of U may be joined by a curve $C\subset U$ such that C has finite length $L\leqq a|x-y|$, and if $\varphi: [0, L]\to U$ is the arc length parametrization of C, then $\operatorname{dist}(\varphi(t), \partial U)\geqq b\min\{t, L-t\}$ for all $t\in[0, L]$.

The result we establish is the following

Theorem. *Let X and Y be real normed linear spaces and let $U\subset X$ be an (a, b)-uniform domain. If $f\in Q(U, Y, K)$ and*

$$\sup_{x\in U}\operatorname{dist}(x, \partial U)\, D_L f(x) < \frac{8b}{3(1+5K)\left(2+Ka+(4Ka+K^2a^2)^{1/2}\right)},$$

then f is injective.

Henceforth we shall assume that the image space Y is complete. This, of course, constitutes no loss of generality in the theorem since Y can always be embedded in its completion. This assumption is necessary since we shall be differentiating functions with values in Y, which in general requires the completeness of Y. In several places in what follows we use some elementary properties of integrals of continuous functions with values in Banach spaces; for a complete treatment of integration in this context the reader is referred to [HP, p. 62—67].

The following six lemmas lead up to the proof of the theorem; the first of them is the Fundamental Lemma of [J, p. 82].

Lemma 1. *Let $I=[a, b]$ be a real interval and let $f: I\to Y$ satisfy $D^+f(t)\leqq M$ for all $t\in I$. Then $|f(b)-f(a)|\leqq M(b-a)$.*

Lemma 2. *Let $I=[a, b]$ be a real interval and let $f: I \to Y$ be continuous. Let r be a continuous real valued function on I. If $D^+f(t) \leqq r(t)|f(t)|$ on I, then*

$$(1) \qquad |f(x)-f(a)| \leqq |f(a)| \Big(\int_a^x r(t)\,dt \Big) \exp \Big(\int_a^x r(t)\,dt \Big)$$

holds for all $x \in I$.

Proof. Let $g(t)=f(t)-f(a)$. Then $D^+g(t)=D^+f(t) \leqq r(t)|f(t)| \leqq r(t)|g(t)|+r(t)|f(a)|$. Let $a=s_0<s_1<\ldots<s_n=t$ and let $\delta=\max\{s_i-s_{i-1}: 1 \leqq i \leqq n\}$. Let R_i and G_i denote the maxima of r and $|g|$ on $[s_{i-1}, s_i]$, respectively. By Lemma 1 we have $|g(s_i)-g(s_{i-1})| \leqq (R_iG_i+R_i|f(a)|)(s_i-s_{i-1})$. Summing from 1 to n and letting $\delta \to 0$ we have

$$|g(t)| \leqq \int_a^t r(s)|g(s)|\,ds + |f(a)| \int_a^t r(s)\,ds.$$

Applying the Gronwall inequality (see [W, p. 14]) to this we conclude that $|g(x)|$ is indeed bounded above by the expression on the right hand side of (1).

Lemma 3. *Let $I=(a, b)$ be a real interval and let $f: I \to Y$ be locally Lipschitz continuous. Let r be a continuous real valued function on I. If $D_L f(x) \leqq r(x)$ on I, then f is continuously differentiable and $D^+f'(x) \leqq r(x)|f'(x)|$ for $x \in I$.*

Proof. Let $\varphi \in Y^*$ with $|\varphi|=1$. Let $g=\varphi \circ f$. Let $J=(x-\varepsilon, x+\varepsilon)$ and assume that f has Lipschitz constant M on J and that $r(t) \leqq R$ on J. Then for $t \in J$ we have $\limsup_{h \to 0} \Delta_2 g(t, h) \leqq RM$. Consequently the functions $g^{\pm}=MRt^2/2 \pm g$ satisfy $\liminf_{h \to 0} (g^{\pm}(t+h)+g^{\pm}(t-h)-2g^{\pm}(t))/h^2 \geqq 0$ for $t \in J$. This implies (see [N, p. 39]) that g^+ and g^- are convex. Thus for $x-\varepsilon<s_1<s_2 \leqq t_1<t_2<x+\varepsilon$ we have

$$\frac{g^{\pm}(t_2)-g^{\pm}(t_1)}{t_2-t_1} \geqq \frac{g^{\pm}(s_2)-g^{\pm}(s_1)}{s_2-s_1},$$

from which we conclude that

$$(2) \qquad \left| \frac{g(t_2)-g(t_1)}{t_2-t_1} - \frac{g(s_2)-g(s_1)}{s_2-s_1} \right| \leqq MR \left(\frac{t_2+t_1}{2} - \frac{s_2+s_1}{2} \right)$$

for such values of s_1, s_2, t_1, t_2. Since φ is any element of Y^* with norm 1, (2) holds with g replaced by f. This clearly means that f is differentiable. What is more, it means that for $t_1, t_2 \in J$ we have $|f'(t_2)-f'(t_1)| \leqq MR|t_2-t_1|$, from which it follows that f' is continuous. Since we may use the value $M=\sup\{|f'(t)|: t \in J\}$ and since R may be taken to be the corresponding supremum of r, we conclude that $D^+f'(x) \leqq r(x)|f'(x)|$.

Lemma 4. *Let $f: U \to Y$ be locally Lipschitz continuous and satisfy $D_L f(x) \leqq R$ on U. Then $f'(x, a)$ exists for all $x \in U$ and $a \in X$. Furthermore, if U contains* *the closed segment joining* x *and* $x+ta$, *then*

$$|f'(x+ta, a)-f'(x, a)| \leqq |f'(x, a)|\,|a|Rte^{|a|Rt} \tag{3}$$

and

$$|f(x+ta)-f(x)-tf'(x, a)| \leqq |f'(x, a)|\varrho(|a|R, t), \tag{4}$$

where $\varrho(q, t)=(qte^{qt}-e^{qt}+1)/q$.

Proof: Since the function g given by $g(s)=f(x+sa)$ satisfies $D_L g(s) \leqq |a|R$ in a neighborhood of $[0, t]$, we may apply Lemma 3 to conclude that $D^+g'(s) \leqq |a|R|g'(s)|$. Applying Lemma 2 and the fact that $f'(x+sa, a)=g'(s)$, we obtain (3). Integration from 0 to t then gives (4).

Lemma 5. *Let* $f \in Q(U, Y, K)$ *satisfy* $D_L f(x) \leqq R$ *on* U. *If* $|a|=|b|=1$ *and* dist $(x, \partial U)>2t$, *then*

$$|f'(x+ta, b)-f'(x, b)| \leqq (1+Rt)e^{Rt}|f'(x, a)|(1+5K)\varrho(3R/2, t)/t.$$

Proof: Let $y=x+ta/2$ and $c=a/2-b$. Since $x=y-ta/2$ and $x+ta=y+ta/2$, Lemma 4 implies that $|f(x)-f(y)+tf'(y, a)/2|$ and $|f(x+ta)-f(y)-tf'(y, a)/2|$ are bounded above by $|f'(y, a)|\varrho(R, t/2)$. Similarly, since $x+tb=y-tc$ and $x+ta-tb=y+tc$, we have that $|f(x+tb)-f(y)+tf'(y, c)|$ and $|f(x+ta-tb)-f(y)-tf'(y, c)|$ are bounded above by $|f'(y, c)|\varrho(|c|R, t)$. Also by Lemma 4 we see that $|f(x+tb)-f(x)-tf'(x, b)|$ and $|f(x+ta)-f(x+ta-tb)-tf'(x+ta, b)|$ are bounded above by $|f'(x, b)|\varrho(R, t)$ and $|f'(x+ta, b)|\varrho(R, t)$, respectively. Together these six bounds imply that

$$\begin{aligned} &|f'(x+ta, b)-f'(x, b)| \\ &\leqq (2|f'(y, a)|\varrho(R, t/2)+2|f'(y, c)|\varrho(|c|R,t)+ \\ &\quad |f'(x, b)|\varrho(R, t)+|f'(x+ta, b)|\varrho(R, t))/t. \end{aligned} \tag{5}$$

But since $f \in Q(U, Y, K)$, we have

$$|f'(y, c)|/|c| \leqq K|f'(y, a)|/|a| \leqq K|f'(x, a)|(1+Rt/2)e^{Rt/2}$$

and

$$|f'(x+ta, b)| \leqq K|f'(x+ta, a)| \leqq K|f'(x, a)|\,(1+Rt)e^{Rt},$$

by Lemma 4. Finally, $|f'(x, b)| \leqq K|f'(x, a)|$. Since $|c| \leqq 3/2$ and $\varrho(q, t)$ is increasing in q and $\varrho(q, st)=s\varrho(sq, t)$, (5) yields the desired conclusion. (The condition dist $(x, \partial U)>2t$ was tacitly used in the various applications of Lemma 4.)

Lemma 6. *Let* $f \in Q(U, Y, K)$ *satisfy* $D_L f(x) \leqq r(x)$ *on* U, *where* r *is continuous on* U. *Then* f *is continuously differentiable on* U *and*

$$D^+f'(x) \leqq \frac{3(1+5K)r(x)}{4}|f'(x)|.$$

Proof. Let $x \in U$ and let $a, b \in X$. Let X' be the subspace of X spanned by a and b. Let $U'=\{u \in X' : x+u \in U\}$. Let $\varphi \in Y^*$ and let g denote the real valued function on U' defined by $g(u)=\varphi(f(x+u))$. Since f is locally Lipschitz continuous, g is also, so that g is differentiable a.e. on U'. If g is differentiable at u and $c\in X'$, then $g'(u)(c)=\varphi(f'(x+u, c))$. By Lemma 5, $f'(x+u, c)\to f'(x, c)$ as $u\to 0$. Since $g'(u)$ exists a.e. on U' and is linear wherever it exists, we have that $\varphi(f'(x, c))$ is linear in c. Since $\varphi\in Y^*$ is arbitrary, $f'(x, ta+sb)=tf'(x, a)+sf'(x, b)$ for all $s, t\in\mathbf{R}$. Lemma 4 now implies that f is differentiable at x and that $f'(x)(a)=f'(x, a)$. Finally, Lemma 5 implies that f' is continuous and that $D^+f'(x)$ is bounded above by

$$\left(\lim_{t\to 0}\,(1+5K)\,\varrho\left(\frac{3r(x)}{2},\,t\right)\Big/t^2\right)\sup_{|a|=1}|f'(x, a)| = \frac{3(1+5K)r(x)}{4}\,|f'(x)|.$$

With these lemmas we now prove the theorem by extending Martio's and Sarvas' argument to this more general context. Let f be as in the statement and let $b'<b$ be such that

$$A = \sup_{x\in U}\operatorname{dist}(x, \partial U)\, D_L f(x) < \frac{8b'}{3(1+5K)\left(2+Ka+(4Ka+K^2a^2)^{1/2}\right)}. \tag{6}$$

Let x and y be any two distinct points of U. Since U is (a, b)-uniform, there is a piecewise linear curve C joining x and y in U such that if $\varphi\colon[-M, M]\to U$ is the arc length parametrization of C with $\varphi(-M)=x$ and $\varphi(M)=y$, then $2M\leqq a|y-x|$ and $\operatorname{dist}(\varphi(t), \partial U)\geqq b'(M-|t|)$ for $t\in[-M, M]$. Let g denote $f'\circ\varphi$. Lemma 6 implies that g is continuous and satisfies

$$D^+g(t)\leqq \frac{3A(1+5K)}{4b'(M-|t|)}\,|g(t)|,$$

so that by applying Lemma 2 to g as a mapping of $[-M, M]$ into the Banach space of all bounded linear transformations of X into Y we have

$$|g(s)-g(0)|\leqq |g(0)|\,B\left(\frac{M}{M-|s|}\right)^B\log\left(\frac{M}{M-|s|}\right), \tag{7}$$

where for convenience we have set $B=3A(1+5K)/(4b')$. Now,

$$\begin{aligned} f(y)-f(x) &= \int_{-M}^{M} g(s)(\varphi'(s))\,ds = \int_{-M}^{M} g(0)(\varphi'(s))\,ds+E \\ &= g(0)(y-x)+E, \end{aligned}$$

where

$$|E|\leqq \int_{-M}^{M}|g(s)-g(0)|\,|\varphi'(s)|\,ds\leqq 2MB|g(0)|/(1-B)^2$$

by (7). Since $f\in Q(U, Y, K)$, $|g(0)(y-x)|\geqq|g(0)|\,|y-x|/K$. Thus, $|f(y)-f(x)|\geqq |g(0)|\,|y-x|(1/K-aB(1-B)^{-2})$. If $g(0)=0$, then by (7) f' is identically 0 on the curve C. This means that f is constant on C, which cannot be by the last condition of Definition 1. Thus $|g(0)|>0$. Hence by (6) we have that $f(y)\neq f(x)$, so that f is indeed injective.

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