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ON MAXIMIZING $a_4+\mu a_3$ FOR REAL BOUNDED UNIVALENT FUNCTIONS

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1. Introduction

Denote by $S(b)$ the class of bounded univalent functions f defined in the unit disc U: $|z|<1$:

$$S(b)=\{f|f(z)=b(z+a_2z^2+\ldots),\ |f(z)|<1,\quad 0<b<1\}.$$

Let $S_R(b)$ be the subclass of this with all the coefficients a_ν real. In [4] the combination $a_4+\mu a_2$, $\mu\in\mathbf{R}$, was maximized in $S_R(b)$ for such values of μ that produced algebraic extremal functions f. The present paper deals with the same problem for $a_4+\mu a_3$. The formulae, however, are much more involved than those in [4].

It appears that also in the present case all algebraic extremal domains can be determined. In [5] K. Zyskowska considered the combination $a_h+\mu a_k$, h even and k odd, in $S_R(b)$ and proved the existence of such an interval $0<b<b_\mu$ that the left radial-slit mapping maximizes the combination. In the present paper the exact value of $\sup b_\mu$ is determined for $h=4$, $k=3$.

2. The use of the Power inequality

Start from the Power inequality true for a_2, a_3, a_4 in $S(b)$ and $S_R(b)$:

$$a_4\leqq\frac{2}{3}(1-b^3)-\frac{b}{2}a_2^2-\frac{13}{12}a_2^3+2a_2a_3-2\lambda\left(a_3-\frac{3}{4}a_2^2+ba_2\right)+\lambda^2[2(1-b)-a_2],\quad \lambda\in R.$$

This implies for the combination:

$$\begin{aligned}&a_4+\mu a_3\leqq(\mu+2a_2-2\lambda)a_3\\ (1)\qquad &+\frac{2}{3}(1-b^3)-\frac{b}{2}a_2^2-\frac{13}{12}a_2^3+2\lambda\left(\frac{3}{4}a_2^2-ba_2\right)+\lambda^2[2(1-b)-a_2].\end{aligned}$$

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We can eliminate the effect of a_3 directly by choosing

$$\lambda = a_2+\frac{\mu}{2}; \tag{2}$$

$$a_4+\mu a_3 \leqq \frac{2}{3}(1-b^3)+\frac{1}{2}(1-b)\mu^2+\mu\left(2-3b-\frac{\mu}{4}\right)a_2+\left(2-\frac{9}{2}b-\frac{\mu}{4}\right)a_2^2-\frac{7}{12}a_2^3 = F_1.$$

The same result follows also if we transform (1) in the optimized form and choose λ such that the right side is minimized:

$$\begin{aligned} &\lambda = \frac{a_3-\frac{3}{4}a_2^2+ba_2}{2(1-b)-a_2}; \\ &a_4 \leqq \frac{2}{3}(1-b^3)-\frac{1}{2}ba_2^2-\frac{13}{12}a_2^3+2a_2a_3-\frac{\left(a_3-\frac{3}{4}a_2^2+ba_2\right)^2}{2(1-b)-a_2}. \end{aligned} \tag{3}$$

Add here μa_3 on both sides and write the right side in the form where a_3 is included in a perfect square. This allows an estimation, free of a_3:

$$\begin{aligned} a_4+\mu a_3 &\leqq \frac{2}{3}(1-b^3)-\frac{1}{2}ba_2^2-\frac{13}{12}a_2^3-\left(\frac{\mu}{2}+a_2\right)\left[-\frac{a_2^2}{2}+(4b-2)a_2-\frac{\mu}{2}(2(1-b)-a_2)\right] \\ &\quad -\frac{1}{2(1-b)-a_2}\left[a_3+ba_2-\frac{3}{4}a_2^2-\left(\frac{\mu}{2}+a_2\right)(2(1-b)-a_2)\right]^2 \\ &\leqq \frac{2}{3}(1-b^3)-\frac{1}{2}ba_2^2-\frac{13}{12}a_2^3-\left(\frac{\mu}{2}+a_2\right)\left[-\frac{a_2^2}{2}+(4b-2)a_2-\frac{\mu}{2}(2(1-b)-a_2)\right] \\ &= \frac{2}{3}(1-b^3)+\frac{1}{2}(1-b)\mu^2+\left(2-3b-\frac{\mu}{4}\right)a_2+\left(2-\frac{9}{2}b-\frac{\mu}{4}\right)a_2^2-\frac{7}{12}a_2^3 = F_1. \end{aligned}$$

The equality condition defines the parabola in the a_2a_3-plane:

$$1^\circ: \qquad a_3 = -\frac{a_2^2}{4}+(2-3b)a_2+\frac{\mu}{2}[2(1-b)-a_2].$$

The corner point $[2(1-b),\ 3-8b+5b^2]$ of the coefficient body lies on this curve.

When substituting in the optimizing expression of λ the value of a_3 from 1° we obtain, again, the choice (2) by aid of which a_3 was previously eliminated.

The equality function f of (2) is determined by Löwner's $\varkappa(u)=e^{-i\vartheta(u)}$, $b\leqq u\leqq 1$. In $S_R(b), f$ is obtained already by aid of $\cos\vartheta$ from (2) in [4]. The existence of this, again, is equivalent to the existence of the number σ. This, in turn, is connected with λ and a_2 by (3) in [4] and the first formula of (4) in [4]. When substituting $\lambda=a_2+\mu/2$ we obtain the following necessary and sufficient conditions for the existence of σ and thus for that of $\cos\vartheta$:

$$8\sigma+(6a_2-2+3\mu)\sigma^{-1/2}-(9a_2+6b+3\mu)=0,$$

$$\frac{1}{3}-\frac{4}{3}\sigma^{3/2}\leqq a_2+\frac{\mu}{2}\leqq\frac{1}{3}+\frac{8}{3}\sigma^{3/2}, \tag{4}$$

$$b\leqq\sigma\leqq 1.$$

3. The use of the Jokinen inequality

Next, consider the inequality of Jokinen for a_2, a_3, a_4 in $S_R(b)$ (cf. (10) in [4] or directly (6.36) in [1]):

$$a_4-2a_2a_3+a_2^3-b^2a_2+2\lambda(a_3-a_2^2+1-b^2)\leqq\frac{2}{3}(1+\lambda)^3,\quad -1\leqq\lambda\leqq 0. \tag{5}$$

This yields

$$a_4+\mu a_3\leqq(2a_2-2\lambda+\mu)a_3-a_2^3+b^2a_2-2\lambda(1-b^2-a_2^2)+\frac{2}{3}(1+\lambda)^3.$$

Again, the effect of a_3 is eliminated by the choice (2):

$$\lambda=a_2+\frac{\mu}{2}\in[-1,0], \tag{6}$$

giving

$$a_4+\mu a_3\leqq -a_2^3+b^2a_2-(2a_2+\mu)(1-b^2-a_2^2)+\frac{2}{3}\left(1+a_2+\frac{\mu}{2}\right)^3.$$

The meaning of the choice (6) will be fully understood if we start from the optimized form of (5) (cf. (15) in [4]):

$$a_4\leqq a_2^3+(3b^2-2)a_2+2(a_2+1)x_0^2-\frac{4}{3}x_0^3,$$

$$0\leqq x_0=\lambda+1=(a_3-a_2^2+1-b^2)^{1/2}\leqq 1,$$

yielding

$$a_4+\mu a_3\leqq a_2^3+(3b^2-2)a_2+2(a_2+1)x_0^2-\frac{4}{3}x_0^3+\mu a_3;$$

$$a_3=a_2^2-1+b^2+x_0^2;$$

$$a_4+\mu a_3\leqq a_2^3+(3b^2-2)a_2+\mu(a_2^2-1+b^2)+[2(a_2+1)+\mu]x_0^2-\frac{4}{3}x_0^3 \tag{7}$$
$$=\mathcal{M}(a_3;\,a_2).$$

Figure 1 illustrates the maximizing choice of a_3 or x_0:

$$\frac{d}{dx_0}=2[2(a_2+1)+\mu]x_0-4x_0^2=0\Rightarrow\begin{cases}x_0=a_2+1+\dfrac{\mu}{2}\in[0,1];\ 2^\circ,\\ x_0=0;\ 3^\circ.\end{cases}$$

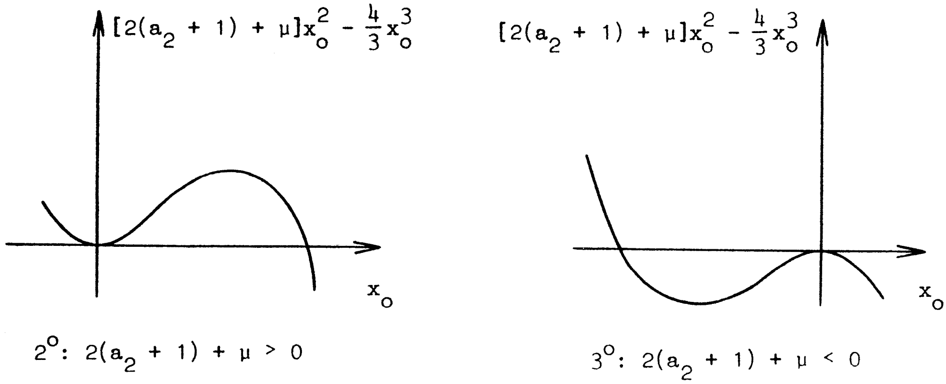

Figure 1.

Thus, we obtain $\max \mathcal{M}(a_3; a_2)$ by choosing a_3 according to the following cases:

$$(8) \qquad a_4+\mu a_3 \leqq \begin{cases} a_2^3+(3b^2-2)a_2+\mu(a_2^2-1+b^2)+\dfrac{2}{3}\left(a_2+1+\dfrac{\mu}{2}\right)^3 = F_2; \ 2^\circ, \\ a_2^3+(3b^2-2)a_2+\mu(a_2^2-1+b^2) = F_3; \ 3^\circ. \end{cases}$$

The upper bound F_2 is sharp for

$$0 \leqq x_0 = \lambda+1 = (a_3-a_2^2+1-b^2)^{1/2} = a_2+1+\frac{\mu}{2} \leqq 1; \ \lambda = a_2+\frac{\mu}{2},$$

i.e.,

$$a_3 = 2a_2^2+(2+\mu)a_2+\left(b+\frac{\mu}{2}\right)^2,$$

2°:

$$-1-\frac{\mu}{2} \leqq a_2 \leqq -\frac{\mu}{2}.$$

The sharpness of F_3 requires that

$$x_0 = \lambda+1 = (a_3-a_2^2+1-b^2)^{1/2} = 0$$

i.e.,

3°:

$$a_3 = a_2^2-1+b^2.$$

In the case 2° the equality is really reached provided that the equality function $\cos\vartheta$, determined by the formulae (11) in [4], exists. It is necessary and sufficient for this that there exist the numbers σ_1 and σ_2 defined by (12) in [4] for $\lambda = a_2+\mu/2$.

This yields the necessary and sufficient existence conditions

(9)
$$\sigma_2=\left(\frac{1-3a_2-\dfrac{3}{2}\mu}{4}\right)^{2/3},$$
$$\sigma_1=\sigma_2+\frac{3a_2+2b+\mu}{4},$$
$$b\leqq\sigma_1\leqq\sigma_2\leqq 1.$$

The equality in the case 3° is reached on the lower boundary arc of the coefficient body (a_2, a_3).

4. The upper bound for $a_4+\mu a_3$ in terms of $a_2\in[-2(1-b), 2(1-b)]$

The curves 1° and 2° have a joint tangent in their common point where

(10)
$$a_2=-\frac{2}{3}b-\frac{\mu}{3}.$$

A similar situation is valid in the common point of 2° and 3° where

(11)
$$a_2=-1-\frac{\mu}{2}.$$

The order of the numbers F_ν is changed at the points (10) and (11). We see this by rewriting

$$F_2=\frac{5}{3}a_2^3+2(1+\mu)a_2^2+\left(3b^2+2\mu+\frac{\mu^2}{2}\right)a_2+\mu b^2+\frac{2}{3}+\frac{\mu^2}{2}+\frac{\mu^3}{12},$$

which yields

$$F_2-F_1=\frac{9}{4}\left(a_2+\frac{2}{3}b+\frac{\mu}{3}\right)^3.$$

Thus

$$F_2\leqq F_1\quad\text{for}\quad a_2\leqq-\frac{2}{3}b-\frac{\mu}{3}.$$

For these values of a_2 the upper bound F_2 is thus better than F_1 obtained from the Power inequality.

F_2 preserves its maximal meaning so far as $a_2+1+\mu/2\geqq 0$, i.e., in the interval

$$-1-\frac{\mu}{2}\leqq a_2\leqq-\frac{2}{3}b-\frac{\mu}{3}.$$

If $a_2+1+\mu/2\leqq 0$, the number F_3 assumes the role of the maximum. This is seen if we compare the numbers F_2 and F_3 for which

$$F_2-F_3=\frac{2}{3}\left(a_2+1+\frac{\mu}{2}\right)^3.$$

Thus

$$F_3 \geqq F_2 \quad \text{for} \quad a_2+1+\frac{\mu}{2}\leqq 0, \quad \text{i.e.,} \quad a_2\leqq -1-\frac{\mu}{2}.$$

Altogether, we are led to the following estimation of $a_4+\mu a_3$ in terms of a_2:

$$a_4+\mu a_3 \leqq \tag{12}$$

$$\leqq \begin{cases} F_1 = F_2-\frac{9}{4}\left(a_2+\frac{2}{3}b+\frac{\mu}{3}\right)^3, & -\frac{2}{3}b-\frac{\mu}{3}\leqq a_2\leqq 2(1-b);\ \mu\geqq 4b-6, \\ F_2 = F_3+\frac{2}{3}\left(a_2+1+\frac{\mu}{2}\right)^3, & -1-\frac{\mu}{2}\leqq a_2\leqq -\frac{2}{3}b-\frac{\mu}{3};\ \mu\geqq 4b-6, \\ F_3 = a_2^3+(3b^2-2)a_2+\mu(a_2^2-1+b^2), & -2(1-b)\leqq a_2\leqq -1-\frac{\mu}{2};\ \mu\leqq 2-4b. \end{cases}$$

Because the limits of a_2 do change with μ we see that the use of (12) must be regulated according to the fixed value of μ in question. Moreover, the order of the points

$$a_2=-\frac{\mu}{2} \quad \text{and} \quad a_2=-\frac{2}{3}b-\frac{\mu}{3}$$

is to be taken into consideration and, finally, the existence conditions (4) and (9) governing the existence of the possible algebraic extremal function. Thus, the most reliable procedure for performing the rather elementary but numerous comparisons involved is to leave them to the unerroneus computer memory. In addition, one must further take into account the comparisons of the local maxima achieved at the end points of the validity intervals and at the vanishing points of the derivatives on them.

5. The local maxima

Consider the functions F_ν in (12) separately.

1) $-\frac{2}{3}b-\frac{\mu}{3}\leqq a_2\leqq 2(1-b)$.

$$F_1'=-\frac{7}{4}\left[a_2^2-\frac{4}{7}\left(4-9b-\frac{\mu}{2}\right)a_2-\frac{4}{7}(2-3b)\mu+\frac{\mu^2}{7}\right]=0. \tag{13}$$

The vanishing points of F_1' are

$$\alpha_1,\alpha_2=\frac{2}{7}\left(4-9b-\frac{\mu}{2}\right)\pm\left(\frac{4}{49}\left(4-9b-\frac{\mu}{2}\right)^2+\frac{4}{7}(2-3b)\mu-\frac{\mu^2}{7}\right)^{1/2}. \tag{14}$$

α_1 is the local maximum point of F_1. The locally maximizing f exists provided that the conditions (4) are satisfied for $a_2=\alpha_1$.

2) $-1-\frac{\mu}{2} \leqq a_2 \leqq -\frac{2}{3}b-\frac{\mu}{3}$.

(15) $$F_2' = 5a_2^2+4(1+\mu)a_2+3b^2+2\mu+\frac{\mu^2}{2} = 0.$$

The equation (15) holds at the points

(16) $$\beta_1, \beta_2 = -\frac{2}{5}(1+\mu)\pm\left(\frac{4}{25}(1+\mu)^2-\frac{6b^2+4\mu+\mu^2}{10}\right)^{1/2}$$

β_2 is the local maximum point of F_2. The existence conditions for the corresponding f are those of (9) for $a_2=\beta_2$.

3) $-2(1-b) \leqq a_2 \leqq -1-\frac{\mu}{2}$.

(17) $$F_3' = 3a_2^2+2\mu a_2+3b^2-2 = 0.$$

This holds at

(18) $$\gamma_1, \gamma_2 = -\frac{\mu}{3}\pm\left(\frac{\mu^2}{9}+\frac{2-3b^2}{3}\right)^{1/2}.$$

γ_2 is the local maximum point. When existing on the interval 3) it gives the corresponding f which is of the type 2:2 and belongs to the lower boundary of the coefficient body (a_2, a_3).

6. The types of the extremal domains and their ranges

The types of the extremal domains maximizing the functional $a_4+\mu a_3$ are denoted as follows:

A = the left radial-slit mapping,

B = 2:2 belonging to the lower boundary arc of the coefficient body (a_2, a_3),

C = 2:3,

D = 1:3 or 3:3,

E = elliptic type.

The types A—D are those determined by the conditions (2)—(3) and (11)—(12) in [4]. The elliptic type is not characterized in terms of Löwner's $\varkappa$ and there is no sharp inequality available for it. On the other hand, we know that the types A—E represent all the solutions of the problem $\max(a_4+\mu a_3)$ and that the algebraic cases A—D are completely governed by (12). Thus we are sure that in the cases

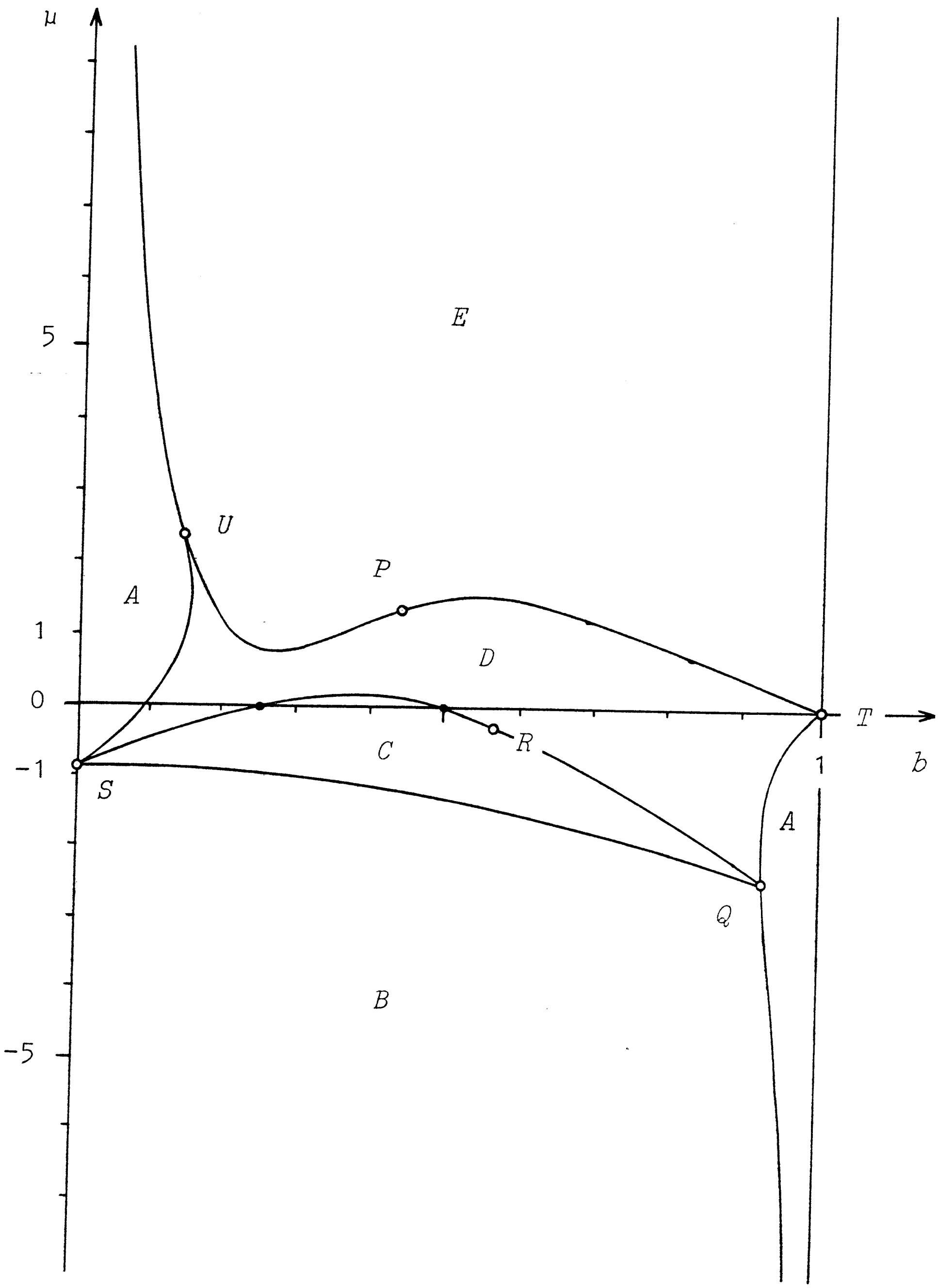

Figure 2.

where (12) fails to yield a sharp result, the corresponding pair (b, μ) is connected with E.

In Figure 2 there are the ranges of the different types in the $b\mu$-plane obtained from comparisons mentioned above. The boundary curves of these ranges will next be studied more closely.

1) $A \cap D$.

Consider those values of b for which

$$2(1-b) \leqq \alpha_1$$

i.e.,

$$\frac{6+4b+\mu}{7} \leqq \left(\frac{4}{49}\left(4-9b-\frac{\mu}{2}\right)^2+\frac{4}{7}(2-3b)\mu-\frac{\mu^2}{7}\right)^{1/2}. \tag{19}$$

This means that the local maximum given by A for $a_4+\mu a_3$ exceeds that connected with D.

If $\mu \geqq -6-4b$, we can square (19) in the form

$$\mu^2-4(1-2b)\mu+4(1-b)(11b-1) \leqq 0 \tag{20}$$

yielding either

$$b \leqq b_2 = \frac{6+\mu}{11}-\frac{(100+4\mu+15\mu^2)^{1/2}}{22} = \frac{1-\mu-\dfrac{\mu^2}{4}}{6+\mu+\dfrac{1}{2}(100+4\mu+15\mu^2)^{1/2}} \tag{21}$$

or

$$b \geqq b_1 = \frac{6+\mu}{11}+\frac{(100+4\mu+15\mu^2)^{1/2}}{22}. \tag{22}$$

The number $b_2 \geqq 0$ provided that

$$-0.828{\cdot}427 = 2-2\sqrt{2} \leqq \mu \leqq 2+\sqrt{2} = 4.828{\cdot}427.$$

The number $b_1 \leqq 1$ for

$$-4 \leqq \mu \leqq 0.$$

The conditions (21) and (22) determine two regions inside of which the type A is the maximizing case. However, according to Figure 2 the regions where A remains the extremal function are somewhat larger. The extensions will be studied later on.

2) $D \cap E$ so that $\sigma = b$; PT.

The type D can vanish so that the number σ decreases below the limit b. At the endpoint $\sigma = b$ we obtain from (4) and (13) for $a_2 = \alpha_1$:

$$\begin{aligned} &\sigma = b \Rightarrow \alpha_1 = \frac{3(b^{1/2}-1)\mu+2(1-b^{3/2})}{6-9b^{1/2}} = h\mu+k, \\ &-4F_1' = 7\alpha_1^2-4\left(4-9b-\frac{\mu}{2}\right)\alpha_1-4(2-3b)\mu+\mu^2 = 0. \end{aligned} \tag{23}$$

Substituting $\alpha_1 = h\mu + k$ in the latter condition (23) we obtain for μ an equation of the type $\mu^2 + 2A\mu + B = 0$. From this $\mu = \mu(b)$ can be solved:

$$\mu_1, \mu_2 = -A \pm (A^2 - B)^{1/2} = \mu(b);$$

$$A = \frac{7hk + (18b-8)h + k + 6b - 4}{7h^2 + 2h + 1},$$

$$B = \frac{7k^2 + 4(9b-4)k}{7h^2+2h+1}; \tag{24}$$

$$h = \frac{b^{1/2}-1}{2-3b^{1/2}},$$

$$k = \frac{2(b^{3/2}-1)}{9b^{1/2}-6}.$$

The boundary arc $D \cap E$ obtained from this is the arc PT in Figure 2. We shall return to the point P later on.

3) $D \cap E$ so that σ disappears through a double root; UP.

The type D may cease to exist also so that σ disappears from $\mathbf{R}$ through a double root of the equation (4). Thus, in the limit case

$$F(\sigma) = 8\sigma + (6\alpha_1 - 2 + 3\mu)\sigma^{-1/2} - (9\alpha_1 + 6b + 3\mu) = 0,$$

$$F'(\sigma) = 0 \Rightarrow \sigma = \left(\frac{6\alpha_1 - 2 + 3\mu}{16}\right)^{2/3}, \tag{25}$$

$$-4F_1' = 7\alpha_1^2 - 4\left(4 - 9b - \frac{\mu}{2}\right)\alpha_1 - 4(2-3b)\mu + \mu^2 = 0.$$

The solution of (25) can be parametrized in σ. We have first from (25):

$$\mu = 16\sigma^{3/2} - 16\sigma + 2 + 4b,$$

$$\alpha_1 = -\frac{16}{3}\sigma^{3/2} + 8\sigma - \frac{2}{3} - 2b. \tag{26}$$

This implies

$$\mu = -3\alpha_1 + 8\sigma - 2b,$$

which, together with the last condition (25), yields,

$$5\alpha_1^2 + (4 + 4b - 16\sigma)\alpha_1 - 10b^2 + 32b\sigma + 32\sigma^2 + 8b - 32\sigma = 0,$$

$$\alpha_1 = -\frac{16}{3}\sigma^{3/2} + 8\sigma - \frac{2}{3} - 2b.$$

This allows eliminating of α_1 and hence we obtain

(27)
$$b^2+8(4\sigma-s)b+\frac{5}{2}s^2-8\sigma s+2s+16\sigma^2-16\sigma=0;$$
$$s=-\frac{16}{3}\sigma^{3/2}+8\sigma-\frac{2}{3}.$$

This, finally, yields the parametrized solution:

(28)
$$b=4(s-4\sigma)\pm\left([4(s-4\sigma)]^2-\frac{5}{2}s^2-16\sigma^2+8s\sigma-2s+16\sigma\right)^{1/2}=b(\sigma),$$
$$\mu=4b+16\sigma^{3/2}-16\sigma+2=4b-3s+8\sigma=\mu(\sigma),$$
$$s=-\frac{16}{3}\sigma^{3/2}+8\sigma-\frac{2}{3},$$
$$b\leqq\sigma\leqq 1.$$

This gives at the endpoint $\sigma=1$:

(29) U: $b=-8+\sqrt{66}=0.124{\cdot}038{\cdot}405,\quad \mu=-30+4\sqrt{66}=2.496{\cdot}153{\cdot}620.$

At the other endpoint $\sigma=b$, according to (26),

(30)
$$\mu=16b^{3/2}-12b+2,$$
$$\alpha_1=-\frac{16}{3}b^{3/2}+6b-\frac{2}{3}.$$

By using these numbers in (25) or by substituting $\sigma=1$ in (27), we obtain for b the condition

(31)
$$\frac{640}{9}b^3-128b^{5/2}+81b^2+\frac{64}{9}b^{3/2}-16b-\frac{2}{9}=0.$$

Observe that the point (b,μ) in question satisfies also the conditions (23), because the value of μ in (30) gives for α_1 in (23) the special value in (30). This means that the arcs obtained from 2) and 3) meet each other at the same point

$$P:\quad b=0.428{\cdot}576{\cdot}811,\quad \mu=1.346{\cdot}216{\cdot}580.$$

Unfortunately, (31) does not allow a simple algebraic presentation for P.

4) $D\cap A$ so that $\sigma=1$.

For completeness observe that the type D can finally cease to exist so that the number σ increases above the limit 1. At the endpoint $\sigma=1$ we obtain from (4) and (13)

$$8+(6a_2-2+3\mu)-(9a_2+6b+3\mu)=0,$$
$$7\alpha_1^2-4(4-9b)\alpha_1+2\alpha_1\mu-4(2-3b)\mu+\mu^2=0;$$
$$a_2=\alpha_1=2(1-b),$$
$$7\alpha_1^2-4(4-9b)\alpha_1+2\alpha_1\mu-4(2-3b)\mu+\mu^2.$$

The elimination of α_1 yields the equality case of (20), i.e., we are at the equality cases of (21) and (22), as was to be expected.

5) $C \cap D$ so that there exist two extremal domains; SR.

The type C can be transformed into the type D so that the maxima F_1 and F_2 of (12) assume equal values for different values of arguments:

$$F_1(\alpha_1) = F_2(\beta_2); \quad \alpha_1 \neq \beta_2. \tag{32}$$

This leads to the arc SR, according to Figure 2, with

$$S = (0, 2-\sqrt{2}), \quad R = \left(\frac{20-\sqrt{22}}{27}, \frac{38-10\sqrt{22}}{27}\right).$$

The coordinates of S follow from 1), those of R will be determined in the following Section 6).

6) $C \cap D$ so that there exists one extremal domain; RQ.

The type C can be transformed into the type D so that the maxima F_1 and F_2 of (12) are equal for the same value of the argument $\alpha_1 = \beta_2 = a_2$. From the first formula (12) and from (15) it follows for the boundary points:

$$F_2 - F_1 = \frac{9}{4}\left(a_2 + \frac{2}{3}b + \frac{\mu}{3}\right)^3 = 0,$$

$$\beta_2 = a_2 \colon F_2' = 5a_2^2 + 4(1+\mu)a_2 + 3b^2 + 2\mu + \frac{\mu^2}{2} = 0;$$

$$\beta_2 = a_2 = -\frac{2}{3}b - \frac{\mu}{3},$$

$$5\left(\frac{2}{3}b + \frac{\mu}{3}\right)^2 - \frac{4}{3}(1+\mu)(2b+\mu) + 3b^2 + 2\mu + \frac{\mu^2}{2} = 0.$$

Thus the boundary arc $C \cap D$ with only one extremal domain satisfies the equation

$$47b^2 - 4b\mu - \frac{5}{2}\mu^2 - 24b + 6\mu = 0, \tag{33}$$

from which the connection $\mu = \mu(b)$ follows:

$$\mu = \frac{6-4b}{5} - \frac{6(1-8b+13.5b^2)^{1/2}}{5}. \tag{34}$$

The common point R of the curves in 5) and 6) is now determined by the application of the procedure for finding the corresponding point in [4]. This means that the numbers $\alpha_1 = \beta_2$ must be double zeros of the equations $F_1' = F_2' = 0$. According to (14) and (16) this implies

$$\alpha_1 = \frac{2}{7}\left(4-9b-\frac{\mu}{2}\right),$$

$$\beta_2 = -\frac{2}{5}(1+\mu),$$

$$\frac{2}{7}\left(4-9b-\frac{\mu}{2}\right) = -\frac{2}{5}(1+\mu);$$

$$\mu = 10b-6,$$

(35)
$$\alpha_1 = \beta_2 = -4b+2.$$

This is in agreement with the requirement $\alpha_1=\beta_2=-2b/3-\mu/3$ in 6), implying $F_2=F_1$.

From (14) and (16) we obtain further the discriminant conditions:

$$\frac{4}{49}\left(4-9b-\frac{\mu}{2}\right)^2+\frac{4(2-3b)\mu-\mu^2}{7} = 0,$$

$$\frac{4}{25}(1+\mu)^2-\frac{3b^2+2\mu+\frac{\mu^2}{2}}{5} = 0;$$

$$162b^2-3\mu^2-8b\mu-144b+20\mu+32 = 0,$$

(36)
$$30b^2 = 8-4\mu+3\mu^2.$$

The first condition (35) and the second condition (36) now yield

$$27b^2-40b+14 = 0;$$

(37) R: $b = \dfrac{20-\sqrt{22}}{27} = 0.567\cdot021\cdot639,\quad \mu = \dfrac{38-10\sqrt{22}}{27} = -0.329\cdot783\cdot615.$

The validity of the first condition (36) can be checked directly. Similarly, the condition $F_1(\alpha_1)-F_2(\beta_2)=0$ of 6) holds at R.

7) $B\cap C$ with one extremal domain.

The types B and C give the same maxima for the same value of a_2 provided that, according to (12),

$$F_2(a_2)-F_3(a_2) = \frac{2}{3}\left(a_2+1+\frac{\mu}{2}\right)^3 = 0.$$

Thus in this case

$$\beta_2 = \gamma_2 = a_2 = -1-\frac{\mu}{2}.$$

Thus the conditions $F_2'(\beta_2)=F_3'(\gamma_2)=0$, according to (15) and (17), assume the form

$$3b^2 = -1-\mu+\frac{\mu^2}{2}, \tag{38}$$

which is the equation of the boundary curve in question.

The boundary curves 7) and 4) satisfy the conditions (38) and (cf. (20))

$$11b^2-(2\mu+12)b+1+\mu-\frac{\mu^2}{4}=0$$

yielding

$$S:\ b=0,\quad \mu=2-2\sqrt{2}; \tag{39}$$

$$Q:\ b=8-5\sqrt{2},\quad \mu=26-20\sqrt{2}. \tag{40}$$

Direct calculation shows that the boundary curve (33) of 6) is satisfied at Q. Similarly, the boundary curve of 5) holds in S.

8) $B\cap A$.

The type B shrinks into the left radial-slit mapping A provided that $\gamma_2=2(1-b)$:

$$\gamma_2:\ F_3'=3a_2^2+2\mu a_2+3b^2-2=0,$$

$$a_2=\gamma_2=2(1-b);$$

$$\mu=\frac{-10+24b-15b^2}{4(1-b)}. \tag{41}$$

As can be directly checked, this holds at the point Q found above.

7. Completing the range of the type A. The Zyskowska-boundary.

1) The use of the upper boundary of the coefficient body.

We want to extend the validity of the maximizing type A for small values of b. First, find a curve in the $b\mu$-plane, above which the type A cannot exist.

Take the right upper part of the coefficient body (a_2, a_3), where ([2])

$$\frac{f(z,u)}{(1-f(z,u))^2}=\frac{u}{\sigma}\frac{f(z,\sigma)}{(1-f(z,\sigma))^2};\quad u\in[b,\sigma],$$

$$u\left(f(z,u)-f(z,u)^{-1}\right)=z-z^{-1}-2\sigma\ln\frac{f(z,u)}{z};\quad u\in[\sigma,1].$$

From this we obtain for the coefficients a_ν ($\nu=2, 3, 4$) of $f(z)=f(z, b)$:

$$a_2 = a_2(\sigma)+\sigma A_2,$$
$$a_3 = a_3(\sigma)+2\sigma A_2 a_2(\sigma)+\sigma^2 A_3,$$
$$a_4 = a_4(\sigma)+\sigma A_2(2a_3(\sigma)+a_2(\sigma)^2)+3\sigma^2 A_3 a_2(\sigma)+\sigma^3 A_4;$$
$$a_2(\sigma) = -2\sigma \ln \sigma,$$
$$a_3(\sigma) = a_2(\sigma)^2-2\sigma a_2(\sigma)+1-\sigma^2,$$
$$a_4(\sigma) = \sigma a_2(\sigma)^2-\sigma^2 a_2(\sigma)-2\sigma a_3(\sigma)+2a_2(\sigma)a_3(\sigma)-a_3(\sigma)^3;$$
$$A_2 = 2\left(1-\frac{b}{\sigma}\right),$$
$$A_3 = 3-8\frac{b}{\sigma}+5\left(\frac{b}{\sigma}\right)^2,$$
$$A_4 = 4-20\frac{b}{\sigma}+30\left(\frac{b}{\sigma}\right)^2-14\left(\frac{b}{\sigma}\right)^3.$$

Put

(42) $$\sigma = 1-d, \quad d>0$$

and let $d\to 0$. For the combination $a_4+\mu a_3$ this yields the development

(43) $$a_4+\mu a_2$$
$$= 4-20b+30b^2-14b^3+\mu(3-8b+5b^2)+(-2+8b-15b^2+4b\mu)_0 d^2+O(d^3),$$

$$(\ )_0>0 \quad \text{for} \quad \mu>\frac{1}{2b}-2+\frac{15}{4}b=\mu_0(b).$$

For b fixed choose $\mu>\mu_0(b)$. For d small enough the left radial-slit mapping (having $d=0$) cannot maximize the expression (43). Hence the extremal domain of the type A cannot exist above the curve

(44) $$\mu=\mu_0(b)=\frac{1}{2b}-2+\frac{15}{4}b.$$

2) The use of the upper boundary of the algebraic part of the coefficient body. We refer to the curve 1′ of Figure 38 in [3], where the coefficients a_2 and a_3 of the boundary function of the coefficient body (a_2, a_3, a_4) are (cf. p. 151 in [3])

$$a_2 = -\frac{2}{3}-2b+8\sigma-\frac{16}{3}\sigma^{3/2},$$

$$a_3 = a_2^2+\frac{7}{9}+b^2-\frac{48}{3}\sigma^2+\frac{64}{9}\sigma^{3/2}+\frac{64}{9}\sigma^3; \quad b\leqq\sigma\leqq 1.$$

The equality case of the optimized Power inequality gives a_4 and thus $a_4+\mu a_3$:

$$(45)\qquad a_4+\mu a_3=\frac{2}{3}(1-b^3)-\frac{1}{2}ba_2^2-\frac{13}{12}a_2^3+\mu a_3+2a_2a_3-\frac{\left(a_2-\frac{3}{4}a_2^2+ba_2\right)^2}{2(1-b)-a_2}.$$

The substitution (42) applied in this formula yields now

$$(46)\qquad a+\mu a_3=4-20b+30b^2-14b^3+\mu(3-8b+15b^2)+2(\ )_0 d^2+O(d^3).$$

This confirms the above conclusion and strongly suggests that the curve (44) forms a sharp boundary for the extremal type A.

3) The upper bound given by the Power inequality.
The maximizing point (a_2, a_3) lies on the curve

$$1^\circ \cup 2^\circ \cup 3^\circ.$$

If μ is big enough, this lies outside the coefficient body (a_2, a_3) in the a_2a_3-plane. The right side of the Power inequality (cf. (45)) is maximized on the upper boundary arc of the coefficient body. We try to find those values of μ for which the maximum is attained for functions of type A. *Necessary* for this is that the unsharp upper bound decreases when moved along the upper boundary arc to the left from the point $a_2=2(1-b)$.

In Section 6. 1) we have the coefficients a_2 and a_3 on the upper boundary arc. In the vicinity of the point $a_2=2(1-b)$ we may, again, use the number $\sigma=1-d$, $d>0$. The Power inequality now yields

$$(47)\qquad a_4+\mu a_3\leqq 4-20b+30b^2-14b^3+\mu(3-8b+5b^2)+(\ )_0 d^2+O(d^3).$$

The necessary condition for the decreasing of this upper bound for small values of d is thus

$$(\ )_0=-2+8b-15b^2+4b\mu\leqq 0;$$

$$(48)\qquad \mu\leqq\frac{1}{2b}-2+\frac{15}{4}b.$$

It appears that this condition is also sufficient to guarantee that the upper bound obtained from Power inequality is maximized at $a_2=2(1-b)$. Hence the curve (44) forms an extended boundary for the extremal domain A.

The curves (44) and (20) have a common tangent at a point where

$$11b^2-(2\mu+12)b+1+\mu-\frac{\mu^2}{4}=0,$$

$$15b^2-(4\mu+8)b+2=0.$$

This point appears to be U of (29).

From 6. 1) we obtain the parametric presentation of the right part of the upper boundary of the coefficient body. Thus we have for $a_4+\mu a_3$

$$a_2 = 2(\sigma \ln \sigma - b),$$

$$a_3 = a_2^2 - 2\sigma a_2 + 2(\sigma - b)^2 + 1 - b^2, \quad b \leqq \sigma \leqq 1;$$

(49)
$$a_4+\mu a_3 \leqq \frac{2}{3}(1-b^3) - \frac{1}{2} b a_2^2 - \frac{13}{12} a_2^3 + 2a_2 a_3 + \mu a_3 - \frac{\left(a_3 - \frac{3}{4} a_2^2 + b a_2\right)^2}{2(1-b) - a_2}.$$

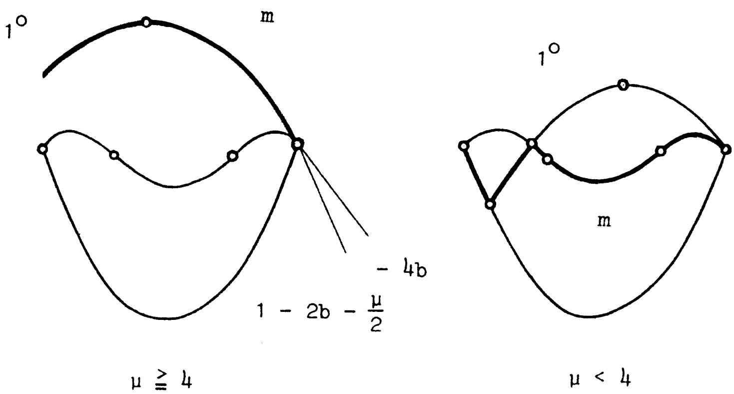

Figure 3.

Compare the tangents of the curves meeting at $a_2=2(1-b)$, $a_3=3-8b+5b^2$.

$$\frac{da_3}{da_2} = \frac{da_3}{d\sigma} : \frac{da_2}{d\sigma} = -4\sigma \ln \sigma - 4b;$$

$$\left(\frac{da_3}{da_2}\right)_{\sigma=1} = -4b.$$

For the curve 1°

$$\left(\frac{da_3}{da_2}\right)_{a_2=2(1-b)} = 1 - 2b - \frac{\mu}{2}.$$

The necessary condition for 1° to lie in the complement of the coefficient body is

(50)
$$1 - 2b - \frac{\mu}{2} \leqq -4b;$$

$$\mu \geqq 2 + 4b.$$

Require now that the point $a_2 = -2(1-b)$, $a_3=3-8b+5b^2$ of the coefficient body is below the corresponding point of 1°:

$$a_3(-2(1-b)) = -5 + 12b - 7b^2 + 2(1-b)\mu \geqq 3 - 8b + 5b^2;$$

(51)
$$\mu \geqq 4 - 6b.$$

We obtain the same condition by requiring that the vertex of the parabola 1° has

$$a_2 = 4-6b-\mu \leqq 0.$$

Now

$$4-6b > 2+4b \quad \text{for} \quad b < 0.2.$$

Thus for the relevant values of b (51) implies (50). By requiring $\mu \geqq 4$ we are thus sure that 1° lies above the coefficient body. For

$$2.496 \cdot 153 \cdot 620 = -30+4\sqrt{66} \leqq \mu < 4$$

the curve 1° meets the lower boundary arc of the coefficient body (Figure 3).

The maximum of the upper bound in (49) is now determined on the arc m (Figure 3). It is achieved at $a_2=2(1-b)$ if

$$\mu \geqq -30+4\sqrt{66},$$

(52)

$$0 \leqq b \leqq b_0 = \frac{4+2\mu-(-14+16\mu+4\mu^2)^{1/2}}{15}.$$

For

$$-0.828 \cdot 427 = 2-2\sqrt{2} \leqq \mu \leqq 2+2\sqrt{2} = 4.828 \cdot 427,$$

(53)

$$0 \leqq b \leqq b_2 = \frac{6+\mu}{11} - \frac{(100+4\mu+15\mu^2)^{1/2}}{22}$$

the maximum is determined already from (12) and the extremal function is of the type A. Thus, the validity of A is extended up to the value b_0. (52) and (53) determine together the boundary of the region of A, the Zyskowska-boundary, for $a_4+\mu a_3$.

References

[1] JOKINEN, O.: On the use of Löwner identities for bounded univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 41, 1982, 1—52.

[2] KORTRAM, R., and O. TAMMI: On the first coefficient regions of bounded univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 592, 1—27.

[3] TAMMI, O.: Extremum problems for bounded univalent functions. - Lecture Notes in Mathematics 913, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

[4] TAMMI, O.: On maximizing certain fourth-order functionals of bounded univalent functions. - To appear in Ann. Univ. Mariae Curie—Skłodowska.

[5] ZYSKOWSKA, K.: On general estimations of coefficients of bounded symmetric univalent functions. - To appear.

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