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CONVERGENCE OF INFINITE EXPONENTIALS

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1. Introduction and results

Suppose that a is a complex number and set $b=e^a$, $T(z)=e^{az}$. Define the sequence

$$w_n = T^n(1) = T \circ T \circ \quad \circ T(1), \quad n = 1, 2, \dots \tag{1}$$

where T^n denotes the n-th iterate of the map T. If the sequence converges its limit may be regarded as defining the infinite exponential

$$b^{b^{b^{\cdot^{\cdot^{\cdot}}}}}$$

The long history of investigations of the convergence of (1) goes back at least to Euler and is described with an extensive bibliography by R. A. Knoebel [6].

If w_n converges with limit λ we have $T(\lambda)=e^{a\lambda}=\lambda$, so that $\lambda\neq 0$ and we may put $\lambda=e^t$, giving $\exp(ae^t)=e^t$, and among the possible choices of t we take the one which gives $ae^t=t$. We also have $T'(\lambda)=ae^{a\lambda}=a\lambda=t$.

If $w_n=\lambda$ for some n_0, and so for all $n\geqq n_0$, we call the convergence terminating. This happens if and only if one of the equations $T^{n+1}(1)-T^n(1)=0, n=0,1,2,\dots,$ holds, that is $e^a-1=0$, $\exp(ae^a)-e^a=0, \dots,$ each equation expressing the vanishing of an entire function of a. Thus terminating convergence occurs for at most a countable set of values a.

For non-terminating convergence the $w_n(=T(w_{n-1}))$ approach $\lambda=e^t$ but $w_n\neq\lambda$, while locally near λ the map $T(w)$ behaves like $\lambda+t(w-\lambda)+o(|w-\lambda|)$. Thus convergence can occur only if $|t|\leqq 1$, that is for a which belong to the set

$$K_c = \{a;\ a = te^{-t} \text{ for some } |t| \leqq 1\}. \tag{2}$$

This was observed by A. Carlsson [2]. It remains an open problem to find whether it is not only necessary but indeed sufficient that a belongs to K_c for (1) to converge.

Positive results include the assertion that (1) converges if $a\in K_c$ and a is real

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(Euler, see e.g. [6]), or if $a \in K_s$ (Shell [9]) or $a \in K_T$ (Thron [11]) where

$$K_s = \{a;\ a = te^{-t} \quad \text{for} \quad |t| \leqq \log 2\},$$
$$K_T = \{a;\ |a| \leqq e^{-1}\}.$$

The set $K_c \cap \mathbf{R}$ is the segment $-e \leqq a \leqq 1/e$.

By applying results of the Fatou—Julia theory of iteration [3, 4, 5] one can settle most cases.

Theorem 1. *If* $a = te^{-t}$, $|t| < 1$ *or* t *a root of unity, then the sequence* (1) *converges to* e^t.

For almost all t *such that* $|t| = 1$, *the sequence diverges.*

Thron considered also the composition of functions $T_i(z) = e^{a_i z}$ with differing values a_i. He proved

Theorem A [11]. *If* $a_i \in K_T$, $i = 1, 2, \ldots$, *then the sequence*

$$w_n = T_1 \circ T_2 \circ \ldots \circ T_n(1), \quad T_i(z) = e^{a_i z}, \tag{3}$$

converges to a limit u *such that* $|\log u| \leqq 1$.

This may be regarded as expressing stability of infinite exponentiation with respect to changes of the exponents a_i within the region K_T. A result of this type remains true for other regions.

Theorem 2. *If* $a \in \mathring{K}_c$, *so that* $a = te^{-t}$ *for some* t *with* $|t| < 1$, *then for any neighbourhood* N *of* e^t *there is a corresponding neighbourhood* U *of* a *such that for any sequence* a_i *of points in* U *the sequence* (3) *converges to a limit in* N.

Returning to the case of equal a_i we can show

Theorem 3. *For each* $n = 1, 2, \ldots$ *there is a countable set of values* a *such that* $w_n = T^n(1)$ *in* (1) *satisfies* $w_n = w_{n+k}$, $k \geqq 1$, *while* w_n *is different from* w_i *for* $i < n$. *One may find such that values* a *with arbitrarily large real part.*

A value of a in Theorem 3 leads to a sequence (1) with terminating convergence and there is a countable set of such values a which lie outside K_c. However a's of this type obviously fail to have the stability property of Theorem 2.

2. Lemmas from iteration theory

If f is an entire or rational function the n-th iterate f^n (where $f^1 = f$, $f^{n+1} = f \circ f^n$, $n = 1, 2, \ldots$) is a function of the same type. Iteration of rational functions was studied extensively by Fatou [3] and Julia [5] and the analogous theory for transcendental entire functions more briefly by Fatou [4].

Suppose that f is a non-linear entire function. In fact in our applications $f(z)$ will always have the form $f(z)=e^{az}$, a constant. Denote by $\mathscr{F}(f)$ the set of points of the complex plane in whose neighbourhood the sequence f^n, $n\geqq 1$, fails to be a normal family. The complement of $\mathscr{F}$ will be denoted by $\mathscr{C}(f)$.

A fixed point α of f is a solution of $f(\alpha)=\alpha$ and $f'(\alpha)$ is called the multiplier of α. If $|f'(\alpha)|<1$ the fixed point is called attractive and $f^n(z)\to\alpha$ as $n\to\infty$, uniformly in a neighbourhood of α, so that $\alpha\in\mathscr{C}(f)$. If $|f'(\alpha)|>1$ then α is called repulsive and clearly $\alpha\in\mathscr{F}(f)$. If $f'(\alpha)=1$ then $\alpha\in\mathscr{F}(f)$ since the expansion near α of f gives

$$f(z) = \alpha+(z-\alpha)+a_{m+1}(z-\alpha)^{m+1}+\dots, \quad a_{m+1}\neq 0, \quad m\geqq 1$$

$$f^n(z) = \alpha+(z-\alpha)+na_{m+1}(z-\alpha)^{m+1}+\dots.$$

If $\alpha\in\mathscr{C}(f)$ then for any limit function φ of a subsequence f^{n_κ} in the component of $\mathscr{C}(f)$ which contains α we have $\varphi(\alpha)=\alpha$ so that φ is analytic and $\varphi^{(m+1)}(\alpha)$ is the limit of the $(m+1)$st derivative of the f^{n_κ} at α, which leads to a contradiction.

We state some general properties of $\mathscr{F}$ and $\mathscr{C}$.

I. *$\mathscr{C}(f)$ is open. $\mathscr{F}(f)$ is perfect and non-empty* [4].

In fact $\mathscr{C}$ is open by definition so $\mathscr{F}$ is at least closed. For $f(z)=e^{az}$ all large solutions of $e^{az}=z$ are repulsive fixed points so that in this case $\mathscr{F}$ is clearly non-empty.

II. *$\mathscr{C}(f)$ and $\mathscr{F}(f)$ are completely invariant under f in the sense that if $z\in\mathscr{C}$ then $f(z)\in\mathscr{C}$, and if further $f(w)=z$ then $w\in\mathscr{C}$* [4].

III. *For any integer $p>1$, $\mathscr{F}(f)=\mathscr{F}(f^p)$*; [4].

IV. *If in a component D of $\mathscr{C}(f)$ the sequence f^n converges to a finite limit function then D is simply-connected.*

(This follows from applying the maximum principle to f^n-f^m on any closed curve which lies in D.)

V. *If α is an attractive fixed point of f then the component of $\mathscr{C}(f)$ which contains α is simply connected and contains a singular point of f^{-1}*; [3, 4].

If D is the component in question then $f^n\to\alpha$ in D, which is simply connected by IV. If D contains no singularity of f^{-1} then continuing the branch for which $f^{-1}(\alpha)=\alpha$ yields a function $g(=f^{-1})$ which is analytic and univalent in D and by II maps D into D with $g(\alpha)=\alpha$. If h is the conformal map of the unit disc Δ to D such that $h(0)=\alpha$ the application of Schwarz's Lemma to $h^{-1}\circ g\circ h=k$ shows that $|k'(0)|\leqq 1$, which yields $|f'(\alpha)|=1/|g'(\alpha)|\geqq 1$, a contradiction.

VI. *If* α *is a fixed point of* f *such that* $f'(\alpha)$ *is a root of unity, then* $\alpha \in \mathcal{F}(f)$ *but* α *lies on the boundary of one or more components* D *of* $\mathcal{C}(f)$ *in which* $f^n \to \alpha$ *as* $n \to \infty$, *and at least one such* D *contains a singularity of* f^{-1}. (Proved in [3] for rational f.)

If α is a fixed point of f such that $f'(\alpha)=\lambda$ is a primitive p-th root of unity, then $f^p(\alpha)=\alpha$, $(f^p)'(\alpha)=\lambda^p=1$ so that $\alpha \in \mathcal{F}(f^p)=\mathcal{F}(f)$.

Let us simplify the notation by putting $\alpha=0$. As shown in [1, Theorem 2] the expansion of $F=f^p$ about 0 has the form

(4) $$F(z) = z+a_{m+1}z^{m+1}+\dots, \quad a_{m+i} \neq 0,$$

where $m=kp$ for some positive integer k. It sufficies to study the iteration of F near 0 (since $\mathcal{F}(F)=\mathcal{F}(f)$) and this has been worked out e.g. in [3] and in somewhat greater detail in [1].

Near $z=0$ (see e.g. [1, Lemma 4]) the set $\mathcal{C}(F)$ contains a star of m equally spaced domains G_j, $1 \leqq j \leqq m$, where each G_j is bounded by a simple closed curve which lies in the region $\alpha_j < \arg z < \beta_j$ and approaches $z=0$ in the directions $\arg z = \alpha_j$, β_j, where

(5) $$\alpha_j = -\gamma+\pi/(3m)-(2j-1)\pi/m,$$
$$\beta_j = -\gamma-\pi/(3m)-(2j-3)\pi/m,$$
$$\gamma = (\pi+\arg a_{m+1})/m.$$

Thus $\beta_j-\alpha_j=(4\pi)/3m$. Moreover we have $F(G_j)\subset G_j$ and $F^n(z)\to 0$ uniformly as $n\to\infty$ for $z\in G_j$.

Now f and the branches of f^{-1} which vanish at zero permute the components of $\mathcal{C}$ of which the G_j form part. If f^{-1} has no singularity in any of these components then f^{-1} (and so $F^{-1}=f^{-p}$) is univalent in each such component D and F^{-1} maps D into itself. Here F^{-1} is understood to be the analytic continuation of

(6) $$F^{-1}(z) = z-a_{m+1}z^{m+1}+\dots$$

throughout D. The iterates $(F^{-1})^n$ are normal in the components D and in particular in the star $\cup G_j$.

By applying the theory described above to the local iteration of F^{-1} (6) rather than F (4) near 0 we see that there is a star of domains G_j', $1 \leqq j \leqq m$, of the same form as G_j, but rotated through an angle π/m (by (5)) such that $F^{-1}(G_j')\subset G_j'$ and the iterates $(F^{-1})^n$ converge uniformly to 0 in $\cup G'$. Together the G_j and G_j' form a region H which includes a punctured neighbourhood $0<|z|\leqq \varrho$ of 0. In H the sequence $(F^{-1})^n$ is normal, analytic and converges uniformly to 0. Hence $(F^{-1})^n \to 0$ uniformly in the whole neighbourhood $|z|<\varrho$.

This is impossible by the same argument which was used to show that a fixed point of multiplier 1 is a member of $\mathcal{F}$. Thus one of the components D_j of $\mathcal{C}$

which contains a G_j will also contain a singularity of f^{-1}. In the components D_j we have $F^n = f^{pn} \to 0$ since this holds in each G_j. The function f permutes the D_j cyclically and we have $f^k(z) \to 0$ as $k \to \infty$ for z in each D_j.

The behaviour of iterates near a fixed point whose multiplier has the form $\lambda = e^{2\pi i\theta}$, θ irrational, may be approached via the centrum problem (assume the fixed point is at the origin): Find if possible a local change of variable

$$z = \varphi(t) = t + b_2 t^2 + \dots, t \text{ near } 0, \tag{7}$$

which reduces the transformation

$$z_1 = f(z) = \lambda z + a_2 z^2 + \dots \tag{8}$$

to the rotation $t_1 = \lambda t$.

Such a φ must satisfy $\varphi(\lambda t) = f(\varphi(t))$ and the coefficients b_n of φ are uniquely determined by recursions which involve division by $\lambda^n - \lambda$. Thus we have a small divisor problem in which convergence of the series for φ depends on how well λ^{n-1} approximates 1 (or θ is approximated by rationals). Siegel [10] has proved the following result, which has been further refined by Rüssmann [8].

VII. *There is a subset E of the unit circumference which has Lebesgue measure 2π and is such that for any f which is analytic near 0 and has the form* (8) *with $\lambda \in E$, the corresponding series* (7) *for φ has positive radius of convergence.*

It follows that if f is entire then the fixed point 0 belongs to $\mathscr{C}(f)$ and that there is a neighbourhood N of 0 such that for any non-zero z_0 in N the images $f^n(z_0)$, $n = 1, 2, \dots$, are dense in a simple closed curve which lies in N and has positive distance from 0.

3. Proof of Theorem 1

Suppose that $a = te^{-t}$, with $|t| \leqq 1$, so that e^t is a fixed point of $T(z) = e^{az}$ with multiplier t. Since te^{-t} is univalent in $|t| \leqq 1$ there is only one such t for a given a and e^t is the only possible limit for w_n in (1).

Suppose first that $|t| < 1$. By property V e^t belongs to a component D of $\mathscr{C}(T)$ which contains the only singular point of T^{-1}, namely the origin. But $T(D) \subset D$ by II so that $1 \in D$ and thus $w_n = T^n(1)$ converges to e^t. A similar argument applies if t is a root of unity, except that V is replaced by VI.

If $|t| = 1$ and t belongs to the subset E of VII, then the fixed point e^t belongs to $\mathscr{C}(T)$ and there is a neighbourhood N of e^t such that for any z_0 such that $z_0 \neq e^t$, $z_0 \in N$ the images $T^n(z_0)$ remain in N for $n = 1, 2, \dots$ but fail to converge to e^t. Thus the only way in which $w_n = T^n(1)$ can converge to e^t is for w_n to be equal to e^t for $n \geqq n_0$. Since this happens for at most a countable set of values of a (and hence of t), removing such a countable set of values from E leaves a set of measure 2π on $|t| = 1$ for which (1) diverges.

4. Terminating convergence

Consider first the case when $w_1=w_2=w_3=\ldots$, that is when $e^a=\exp(ae^a)$, so that $ae^a=a+2n\pi i$ for some integer n. Theorem 3 asserts that this last equation has solutions of arbitrarily large real part. This is easy enough to prove directly but it is convenient to quote the

Lemma 1 [Littlewood [7]). *Suppose that* $f(z)=a_0+a_1z+\ldots$ *is analytic in* D: $|z|<1$ *and that* u_m *is a sequence such that for some constant* $K>1$ *we have*

$$|u_m| \leqq |u_{m+1}| \leqq K|u_m|, \quad 1 \leqq m < \infty, \tag{9}$$

$$u_m \to \infty \quad as \quad m \to \infty. \tag{10}$$

Then if $f(z)\neq u_m$, $m=1, 2, \ldots$, *in* D *we have*

$$\operatorname*{Max}_{|z|=r} |f(z)| \leqq C_1(1-r)^{-C_2}, \quad 0 < r < 1,$$

where C_1 *depends on* u_1, a_0 *and* C_2 *depends only on* K.

From this follows

Lemma 2. *If* f *is an entire function and* u_m *is a sequence which satisfies* (9) *and* (10), *then if* f *omits the values* u_m, $m=1, 2, \ldots$, *in a half-plane* H *it follows that* f *has at most polynomial growth as* $z\to\infty$ *in* H.

To prove Lemma 2 it suffices to consider H as $\operatorname{Im} z>0$. Then $z=\varphi(t)=i(1+t)/(1-t)$, $t=(z-i)/(z+i)$ maps D: $|t|<1$ onto H. Applying Lemma 1 to $f(\varphi(t))$ shows that

$$|f(\varphi(t))| \leqq C_1(1-|t|)^{-C_2}, \quad |t| < 1.$$

If $z=re^{i\theta}$ we have

$$1-|t|^2 = 4r\sin\theta/(r^2+2r\sin\theta+1)$$

whence

$$1-|t| > 2r\sin\theta/(r^2+2r\sin\theta+1)$$

and

$$|f(z)| < C_1\{(r+1)^2/2r\sin\theta\}^{C_2} < K(r/\sin\theta)^{C_2},$$

if $r>1$ $\sin\theta>0$.

Applying Lemma 2 to $f(z)=ze^z-z$, and the sequence $u_m=2m\pi i$ in a half-plane $\operatorname{Re} x>A$ proves the claim made at the beginning of § 4.

To complete the proof of Theorem 3 we need to find a such that $w_{n-1}\neq w_n=w_{n+1}$ (for given $n>1$). We have $w_n=w_{n+1}$ if $T_n(1)=\exp(aT_{n-1}(1))=T_{n+1}(1)=\exp(aT_n(1))$, that is if $aT_{n-1}(1)=aT_n(1)+2k\pi i$ for some integer $k\neq 0$ ($k=0$ is equivalent to $w_{n-1}=w_n$). We have only to note that $aT_n(1)-aT_{n-1}(1)$ is an entire function of a which has very large growth on the positive real axis. The application of Lemma 2 to this function and to the sequence $u_m=2m\pi i$ shows that there are solutions a of our problem in any region $\operatorname{Re} a>A$.

5. Proof of Theorem 2

Suppose that $a=te^{-t}$ where $|t|<1$. Given a neighbourhood N of e^t and ϱ such that $|t|<\varrho<1$ choose a disc $\Delta=\{z: |z-e^t|<d, d>0\}$ such that $\bar{\Delta}\subset N$ and also $|T_a'(z)|<\varrho$ in Δ, $T_a(\Delta)\subset\Delta'=\{z: |z-e^t|<\varrho d\}$, where $T_a(z)=e^{az}$ (and $T_a'(e^t)=t$).

Now Δ belongs to the component D of $\mathscr{C}(e^{az})$ which contains e^t and in which $T_a^n\to e^t$. Thus there is a positive integer p such that $T_a^p(1)\in\Delta'$. By continuity there is a neighbourhood U of a such that

(i) $$U\subset K_c,$$

(ii) for any $a_1, \ldots, a_p$ in U we have

$$T_{a_1}\circ T_{a_2}\circ\ldots\circ T_{a_p}(1)\in\Delta,$$

(iii) for any b in U we have $T_b(\Delta)\subset\Delta$,

(iv) for all b in U we have $|T_b'(z)|<\lambda=1/2\,(1+\varrho)<1$ for all z in Δ.

Suppose that a_i is any sequence of points in U and set $T_i=T_{a_i}$. For any n we have $T_{n+1}\circ\ldots\circ T_{n+p}(1)\in\Delta$ by (ii), and $w_{n+p}=T_1\circ\ldots\circ T_n\circ T_{n+1}\circ\ldots\circ T_{n+p}(1)\in T_1\circ\ldots\circ T_n(\Delta)$ which by (iv) has diameter at most $2\lambda^n d$. If $n>k$ by (iii) both w_{n+p}, w_{k+p} are in $T_1\circ\ldots\circ T_k(\Delta)$ so that $|w_{n+p}-w_{k+p}|<2\lambda^k d$. Thus w_m is a Cauchy sequence which converges to a limit inside $\bar{\Delta}\subset N$. The proof is complete.

6. Periodic sequences of exponents

Suppose that for some natural number k and for all n we have $a_{n+k}=a_n$. As in Theorems A and 2 set $T_i(z)=e^{a_i z}$.

Theorem 4. *If the sequence of exponents is periodic with period k and if*

$$a_n = t_n \exp(-t_{n+1}), \quad t_{n+k} = t_n, \quad n = 1, 2, \ldots,$$

where either $|t_1 t_2 \ldots t_k|<1$ or $t_1 t_2 \ldots t_k$ is a root of unity, and $w_n=T_1\circ T_2\circ\ldots\circ T_n(1)$, then for at least one p with $0\leq p\leq k$ the sequence w_{mk+j} converges to e^{t_1} as $m\to\infty$.

In the case $k=1$ this reduces to Theorem 1. For $k>1$ it has some similarity to Theorem 2.

Put $\varphi_i=T_i\circ T_{i+1}\circ\ldots\circ T_{i+k-1}$. Then if t_i are as in Theorem 4 and $\lambda_i=e^{t_i}$ we have $T_n(\lambda_{n+1})=\lambda_n$ so that λ_i is a fixed point of φ_i and further $\varphi_i'(\lambda_i)=t_1\ldots t_n$. Thus under the assumptions of the theorem λ_i belongs to a domain D in which the iterates $\varphi_i^N\to\lambda_i$ as $N\to\infty$. D contains at least one of the singularities of φ_i^{-1}, that is one of the k values

$$0, \quad T_i(0) = 1, \ldots, T_i \circ T_{i+1} \circ \ldots \circ T_{i+k-2}(0).$$

Thus for such a value β we have $\varphi_i^m(\beta) \to 0$ as $m \to \infty$, that is $T_i \circ T_{i+1} \circ \ldots \circ T_{i+mk-1+p}(0) \to \lambda_i$ as $m \to \infty$ for some $0 \leqq p < k$. Choosing $i=1$ gives the result claimed.

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