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ON THE SPHERICAL DERIVATIVE OF A MEROMORPHIC FUNCTION WITH A NEVANLINNA DEFICIENT VALUE

SAKARI TOPPILA

1. Introduction and results

I thank Professor D. Shea for suggesting this subject to me.

Let f be a meromorphic function in the complex plane. We write

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}$$

and

$$\mu(r, f) = \sup\{\varrho(f(z))\colon |z| = r\}.$$

We shall use the usual notations of the Nevanlinna theory. The following result is proved in [11].

Theorem A. *Let f be a transcendental meromorphic function in the plane such that $\delta(\infty, f) > 0$. Then*

$$\limsup_{\substack{z\to\infty \\ z\in E(f)}} \frac{|z|\varrho(f(z))}{T(|z|, f)} \geqq A_0(1+t)\delta(\infty, f),$$

where $A_0 > 0$ is an absolute constant, t is the order of f and

$$E(f) = \{z\colon |f(z)| = 1\}.$$

In the other direction, we shall prove the following result.

Theorem 1. *For any d, $0 < d \leqq 1$, and t, $0 < t < \infty$, there exists a meromorphic function f of order t such that $\delta(\infty, f) = d$ and that*

$$\limsup_{r\to\infty} \frac{r\mu(r, f)}{T(r, f)} \leqq 60(1+t)\delta(\infty, f). \tag{1.1}$$

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2. Some lemmas

LEMMA 1. *Let k be a positive integer,*

$$g(z) = \frac{1}{1-z^{8k}},$$

and

$$g_p(z) = g(2^{-p/k}z) \quad for \quad p = 1, \ldots, k,$$

$$f_k(z) = \sum_{p=1}^{k} (-1)^p g_p(z).$$

Then $n(r, \infty, f_k) = 8k^2$ for $r \geqq 2$,

(2.1) $$\varrho(f_k(z)) < 72k$$

for all z in the finite complex plane C, and if $|z| \geqq 4$, then

(2.2) $$|f_k(z)| \leqq |2/z|^{6k}.$$

Proof. It follows immediately from the definition of f_k that the number of the poles of f_k is $8k^2$ and that all the poles lie on $|z| \leqq 2$.

Let $|z| \geqq 4$. We get

$$|f_k(z)| \leqq \sum_{p=1}^{k} (|2^{-p/k}z|^{8k}-1)^{-1} \leqq k(|z/2|^{8k}-1)^{-1}$$

$$\leqq 2k|2/z|^{6k}(2/4)^{2k} \leqq |2/z|^{6k},$$

which proves (2.2).

Let s, $1 \leqq s \leqq k$, be an integer. If

$$|z| \geqq 2^{(s+1/2)/k}$$

and $p \leqq s$, we get

(2.3) $$|g_p(z)| \leqq ((2^{(s+1/2)/k}/2^{p/k})^{8k}-1)^{-1} = (16(2^8)^{s-p}-1)^{-1} \leqq (15(2^8)^{s-p})^{-1}.$$

If

$$|z| \leqq 2^{(s-1/2)/k}$$

and $p \geqq s$, we get

(2.4) $$|g_p(z)-1| = |(z^{-1}2^{p/k})^{8k}-1|^{-1} \leqq ((2^8)^{p-s+1/2}-1)^{-1} \leqq (15(2^8)^{p-s})^{-1}.$$

Since

(2.5) $$g_p'(z) = 8kz^{-1}g_p(z)(g_p(z)-1),$$

we get in both cases the estimate

(2.6) $$|g_p'(z)| \leqq 8k|z|^{-1}(1+1/15)(15(2^8)^{|s-p|})^{-1} = \frac{128k}{225|z|}(2^8)^{-|s-p|}.$$

From (2.6) we deduce that if $|z| = 2^{1/(2k)}$ or $|z| \geqq 2^{(k+1/2)/k}$, then

$$|f_k'(z)| \leqq \frac{128k}{225} \sum_{t=0}^{\infty} (2^8)^{-t} \leqq k,$$

and using the maximum principle and the fact that

$$\varrho(f_k(z)) \leqq |f_k'(z)|,$$

we conclude that

$$\varrho(f_k(z)) \leqq k \tag{2.7}$$

if $|z| \leqq 2^{1/(2k)}$ or $|z| \geqq 2^{(k+1/2)/k}$.

Let s, $1 \leqq s \leqq k$, be fixed and let

$$2^{(s-1/2)/k} \leqq |z| \leqq 2^{(s+1/2)/k}.$$

We write $f_k(z) = g_s(z) + h(z)$. It follows from (2.3) and (2.4) that

$$|h(z)| \leqq \sum_{p=1}^{s-1} |g_p(z)| + \left| \sum_{p=s+1}^{k} (-1)^p \right| \tag{2.8}$$

$$+ \sum_{p=s+1}^{k} |1 - g_p(z)| \leqq 1 + (2/15) \sum_{q=0}^{\infty} (2^8)^{-q} \leqq 8/7,$$

and from (2.6) we get

$$|h'(z)| \leqq \frac{256k}{225} \sum_{q=0}^{\infty} (2^8)^{-q} \leqq \frac{115k}{100}. \tag{2.9}$$

From (2.5) and (2.9) we deduce that if $|g_s(z)| \leqq 5/2$, then

$$\varrho(f_k(z)) \leqq |g_s'(z)| + |h'(z)| \leqq 8k(5/2)(1+5/2) + \frac{115k}{100} < 72k, \tag{2.10}$$

and if $|g_s(z)| > 5/2$, it follows from (2.5), (2.8) and (2.9) that

$$\varrho(f_k(z)) \leqq |h'(z)| + \frac{|g_s'(z)|}{1 + (|g_s(z)| - |h(z)|)^2} \tag{2.11}$$

$$\leqq \frac{115k}{100} + \frac{|g_s'(z)|}{|g_s(z)/2|^2} \leqq \frac{115k}{100} + 32k \left| \frac{g_s(z)-1}{g_s(z)} \right| \leqq \frac{115k}{100} + 32k(1+2/5) < 72k.$$

Combining (2.10), (2.11) and (2.7), we get (2.1). Lemma 1 is proved.

Lemma 2. *Let* $k \geqq 9$ *be an integer,* $k \leqq \log A \leqq k+1$, *and*

$$f(z) = \prod_{n=1}^{\infty} (1 - z/A^n)^{k^n}.$$

Then

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{N(r, 0, f)} \leqq 12, \tag{2.12}$$

the order of f *is* $(\log A)^{-1} \log k$,

$$|z|^{-2} |f(z)| \to \infty \tag{2.13}$$

as $z\to\infty$ *outside the union of the discs* $|z-A^n|\leqq A^n/2$,

(2.14) $$|z|^2|f(z)|\to 0$$

as $z\to\infty$ *through the union of the discs* $|z-A^n|\leqq A^n/9$, *and*

(2.15) $$N(A^n, 0, f)=(1+o(1))k^{n+1}(k-1)^{-2}\log A$$

as $n\to\infty$.

Proof. We have

(2.16) $$N(A^n, 0, f)=\sum_{p=1}^{n-1} k^p\log(A^n/A^p)$$

$$=\sum_{p=1}^{n-1} k^p n\log A-\log A\sum_{p=1}^{n-1} pk^p=\left(\frac{n(k^n-k)}{k-1}-\frac{k(1+(n-1)k^n-nk^{n-1})}{(k-1)^2}\right)\log A$$

$$=(1+o(1))k^{n+1}(k-1)^{-2}\log A \quad (n\to\infty),$$

which proves (2.15).

For $n\geqq 2$, we write

$$g_n(z)=\prod_{p=1}^{n-1}(1-z/A^p)^{k^p}\prod_{p=n+1}^{\infty}(1-z/A^p)^{k^p}.$$

We have

(2.17) $$\log|g_n(A^n)|\leqq\log\prod_{p=1}^{n-1}(A^n/A^p)^{k^p}=N(A^n, 0, f)$$

and for $\pi/2\leqq\varphi\leqq 3\pi/2$

(2.18) $$\log|g_n(A^n e^{i\varphi})|\geqq\log\prod_{p=1}^{n-1}(A^n/A^p)^{k^p}=N(A^n, 0, f).$$

Let $A^n/2-1\leqq|z|\leqq 2A^n$. We get

(2.19) $$\left|\frac{g_n'(z)}{g_n(z)}\right|=\Big|\sum_{p\neq n} k^p/(z-A^p)\Big|\leqq(|z|-A^{n-1})^{-1}\sum_{p=1}^{n-1} k^p+2\sum_{p=n+1}^{\infty}(k/A)^p$$

$$\leqq\frac{(A^n/2)k^{n-1}}{(A^n/2-A^{n-1}-1)|z|(1-1/k)}+\frac{2(k/A)^{n+1}}{1-k/A}\leqq\frac{k^{n-1}}{(1-3/A)|z|(1-1/k)}+\frac{4k^{n+1}}{|z|A(1-k/A)}$$

$$\leqq\frac{k^{n-1}}{|z|}\left(\frac{1}{(1-3/A)(1-1/k)}+\frac{4k^2}{A-k}\right)\leqq\frac{7k^{n-1}}{6|z|}.$$

Let $|z-A^n|\leqq A^n/9$. It follows from (2.17) and (2.19) that

(2.20) $$\log|g_n(z)|=\log|g_n(z)/g_n(A^n)|+\log|g_n(A^n)|$$

$$\leqq N(A^n, 0, f)+\left|\int_z^{A^n}(g_n'(w)/g_n(w))\,dw\right|$$

$$\leqq N(A^n, 0, f)+(A^n/9)(7k^{n-1})(6(1-1/9)A^n)^{-1}\leqq N(A^n, 0, f)+(7/48)k^{n-1},$$

and we deduce from (2.16) and (2.20) that

$$(2.21)\qquad \log|z^2f(z)| \leqq 3\log A^n - k^n\log 9 + N(A^n, 0, f) + (7/48)k^{n-1}$$
$$\leqq -k^{n-1}\big((\log A-1)\log 9 - k^2(k-1)^{-2}\log A\big) + \big(7/48+o(1)\big)k^{n-1}$$
$$\leqq -k^{n-1}\big(6+o(1)\big) \quad (n\to\infty).$$

This proves (2.14).

Let $A^n/2 \leqq |z| \leqq 2A^n$, $|z-A^n| \geqq A^n/2$ and $\operatorname{Im} z \geqq 0$. Integrating along the positive imaginary axis and the circle $|w|=|z|$, we get from (2.18) and (2.19)

$$(2.22)\qquad \log|g_n(z)| \geqq \log|g_n(iA^n)| - \Big|\int_{A^n}^{z} \big(g_n'(w)/g_n(w)\big)\,dw\Big|$$
$$\geqq N(A^n, 0, f) - (7k^{n-1}/6)\Big(\Big|\int_{A^n}^{|z|} r^{-1}\,dr\Big| + \pi/2\Big)$$
$$\geqq N(A^n, 0, f) - (7k^{n-1}/6)(\log 2+\pi/2) \geqq N(A^n, 0, f) - 2.643k^{n-1}.$$

This implies together with (2.16) that

$$(2.23)\qquad \log|z^{-2}f(z)| \geqq N(A^n, 0, f) - 2.643k^{n-1} - k^n\log 2 - 3\log A^n$$
$$\geqq k^{n-1}\big(\log A - k\log 2 - 2.643 + o(1)\big) \geqq k^{n-1}\big((1-\log 2)\log A - 2.643 + o(1)\big)$$
$$\geqq \big(1/9+o(1)\big)k^{n-1} \quad (n\to\infty).$$

Since $|f(\bar z)| = |f(z)|$, we deduce that (2.23) holds for all z satisfying the conditions $A^n/2 \leqq |z| \leqq 2A^n$ and $|z-A^n| \geqq A^n/2$. Using the minimum principle, we get (2.13) from (2.23).

It follows from (2.16) that

$$\lim_{n\to\infty} \frac{\log N(A^n, 0, f)}{\log A^n} = \frac{\log k}{\log A},$$

which shows that the order of f is at least $(\log A)^{-1}\log k$.

Let $2A^{n-1} \leqq r \leqq 2A^n$. It follows from (2.13) that

$$m(2A^n, 0, f) = o(1) \quad (n\to\infty),$$

and we deduce from the first main theorem of the Nevanlinna theory and (1.16) that

$$\frac{\log T(r, f)}{\log r} \leqq \frac{(1+o(1))\log N(2A^n, 0, f)}{(n-1)\log A}$$
$$\leqq \frac{(1+o(1))\log N(A^{n+1}, 0, f)}{(n-1)\log A} \leqq (\log A)^{-1}\log k + o(1) \quad (n\to\infty),$$

which shows that the order of f is at most $(\log A)^{-1}\log k$. We have shown that the order of f is $(\log A)^{-1}\log k$.

Let $h(z)=f(z)$ if $|z-A^n|\leqq A^n/9$ and $h(z)=1/f(z)$ if z lies outside the union of the discs $|z-A^n|<A^n/2$. It follows from (2.13) and (2.14) that

$$\varrho(f(z)) \leqq |h'(z)| = \left|(2\pi i)^{-1} \int_{|z-w|=1} \frac{h(w)}{(w-z)^2}\,dw\right| = O(|z^{-2}|) \tag{2.24}$$

as $z\to\infty$ outside the union of the annuli

$$D_n = \{z\colon A^n/9-1 < |z-A^n| < 1+A^n/2\}.$$

Let $z\in D_n$. It follows from (2.19) that

$$|z|\,\varrho(f(z)) \leqq |z|\,|f'(z)/f(z)| \leqq |z|\,|g_n'(z)/g_n(z)|+|z|\,k^n|z-A^n|^{-1}$$
$$\leqq 7k^{n-1}/6+k^n(A^n+A^n/9)(A^n/9-1)^{-1} \leqq k^{n-1}(7/6+10k+o(1)) \quad (n\to\infty).$$

This implies together with (2.24) that

$$r\mu(r,f) \leqq k^{n-1}(7/6+10k+o(1)) \quad (n\to\infty) \tag{2.25}$$

for $A^n/2-1<r<2A^n$. For these values of r we get from (2.16)

$$N(r,0,f) \geqq N(A^n/2-1,0,f) \geqq N(A^n,0,f)-\frac{k^{n-1}}{1-1/k}\log\frac{A^n}{A^n/2-1}$$
$$\geqq k^{n-1}(\log A-(9/8)\log 2+o(1)) \quad (n\to\infty),$$

which together with (2.25) and the fact that $9\leqq k\leqq\log A$ implies that

$$\frac{r\mu(r,f)}{N(r,0,f)} \leqq \frac{7/6+10\log A}{\log A-(9/8)\log 2}+o(1)$$
$$\leqq \frac{7/6+90}{9-(9/8)\log 2}+o(1) \leqq 12+o(1) \quad (n\to\infty).$$

This together with (2.24) proves (2.12). Lemma 2 is proved.

Lemma 3. *Let $0<d\leqq 1$ and $0<\lambda\leqq(\log 9)/9$ be given. There exists a meromorphic function g of order λ such that $\delta(\infty,g)=d$ and that*

$$\limsup_{r\to\infty}\frac{r\mu(r,g)}{T(r,g)} \leqq 12\,d. \tag{2.26}$$

Proof. We choose a positive integer $k\geqq 9$ such that

$$\frac{\log(k+1)}{k+1} \leqq \lambda \leqq \frac{\log k}{k} \tag{2.27}$$

and choose $A>0$ such that $(\log A)^{-1}\log k=\lambda$. It follows from (2.27) that

$$\log A = \frac{\log k}{\lambda} \geqq k$$

and that

$$\log A = \frac{\log k}{\lambda} \leqq \frac{(k+1)\log k}{\log(k+1)} < k+1.$$

We choose $f(z)$ as in Lemma 2 corresponding to these values of k and A.

If $d=1$, we set $g=f$, and deduce from Lemma 2 that g is an entire function satisfying the assertions of Lemma 3.

Let us suppose that $0<d<1$. We set

$$b = 1/d-1, \tag{2.28}$$

and we denote by $[x]$ the integral part of a positive real number x. We set

$$s_p = 1+[(bk^p/8)^{1/2}], \quad h_p(z) = f_{s_p}(8pA^{-p}(z-A^p)), \tag{2.29}$$

where f_{s_p} is as in Lemma 1, and

$$h(z) = \sum_{p=1}^{\infty} h_p(z).$$

It follows from Lemma 1 that

$$\varrho(h_p(z)) = 8pA^{-p}\varrho(f_{s_p}(8pA^p(z-A^p)) \leqq 576ps_pA^{-p} \tag{2.30}$$

for any $z\in C$ and that

$$|h_p(z)| \leqq \left|\frac{A^p}{4p(z-A^p)}\right|^{6s_p} \tag{2.31}$$

if $|z-A^p|\geqq A^p/(2p)$.

Let $n\geqq 9$, $A^n/2\leqq|z|\leqq A^{n+1}/2$ and $|z-A^n|\geqq A^n/(2n)$. It follows from (2.29) and (2.31) that

$$\begin{aligned}
&|z|^2(|h_n(z)|+|h(z)-h_n(z)|)\\
&\leqq A^{2n+2}\left(2^{-6s_n}+\sum_{p=1}^{n-1}(A^p/A^n)^{6s_p}+\sum_{p=n+1}^{\infty}\exp(-2s_p)\right)\\
&\leqq A^{2n+2}\Big(\sum_{1\leqq p\leqq n/2}(A^{-n/2})^6+\sum_{p>n/2}\exp(-2s_p)\Big)\\
&\leqq (1+o(1))A^{2n+2}\Big(nA^{-3n}+\sum_{p>n/2}\exp(-k^{p/3})\Big)\\
&\leqq (1+o(1))A^{2n+2}(nA^{-3n}+\exp(-k^{n/6})) = o(1) \quad (n\to\infty).
\end{aligned}$$

This implies that

$$|z^2h(z)| \to 0 \tag{2.32}$$

as $z\to\infty$ outside the union of the discs $|z-A^n|<A^n/(2n)$, and, together with the

maximum principle, that

$$(2.33)\qquad |z^2||h(z)-h_n(z)| \leqq o(1) \quad (n \to \infty)$$

in $|z-A^n| \leqq 3A^n/(4n)$.

We set $g(z)=f(z)+h(z)$. Let $n>9$ and $|z-A^n| \leqq 3A^n/(4n)$. We write

$$g(z) = h_n(z)+H_n(z).$$

It follows from (2.33) and Lemma 2 that

$$(2.34)\qquad |z^2 H_n(z)| \leqq o(1) \quad (n \to \infty),$$

which implies that

$$(2.35)\qquad |H_n'(z)| = \left|(2\pi i)^{-1} \int_{|w-z|=1} H_n(w)(w-z)^{-2}\,dw\right| \leqq o(|z^{-2}|) \quad (n \to \infty)$$

in $|z-A^n| \leqq 5A^n/(8n)$. Since

$$\varrho(g(z)) \leqq \frac{|h_n'(z)|}{1+|h_n(z)+H_n(z)|^2} + |H_n'(z)|,$$

we get from (2.29), (2.30), (2.34), (2.35) and Lemma 2

$$(2.36)\qquad |z|\varrho(g(z)) \leqq (576+o(1))ns_n A^{-n}|z| \leqq (576+o(1))ns_n = o(s_n^2) = o(k^n)$$
$$= o(N(|z|, 0, f)) \quad (n \to \infty)$$

in $|z-A^n| \leqq 5A^n/(8n)$.

As in the connection of (2.35), we deduce from (2.32) that

$$(2.37)\qquad |h'(z)| = o(|z|^{-2})$$

as $z \to \infty$ outside the union of the discs $|z-A^n| < 5A^n/(8n)$. Since

$$\varrho(g(z)) \leqq \frac{|f'(z)|}{1+|f(z)+h(z)|^2} + |h'(z)|,$$

it follows from Lemma 2, (2.32) and (2.37) that

$$|z|\varrho(g(z)) \leqq (12+o(1))N(|z|, 0, f)$$

as $z \to \infty$ outside the union of the discs $|z-A^n| < 5A^n/(8n)$. This together with (2.36) implies that

$$(2.38)\qquad r\mu(r, g) \leqq (12+o(1))N(r, 0, f) \quad (r \to \infty).$$

It follows from (2.29) that

$$(2.39)\qquad n(r, \infty, g) = (b+o(1))n(r, 0, f)$$

as $r \to \infty$ outside the union of the intervals $[A^n(1-1/(2n)), A^n(1+1/(2n))]$. Let

$A^n \leqq r \leqq A^{n+1}$. It follows from (2.39) that

$$|N(r, \infty, g)-bN(r, 0, f)| = \left|\int_0^r (n(t, \infty, g)-bn(t, 0, f))t^{-1}dt\right|$$

$$\leqq o(N(r, 0, f))+\sum_{p=1}^{n+1}\int_{A^p(1-1/(2p))}^{A^p(1+1/(2p))}|n(t, \infty, g)-bn(t, 0, f)|t^{-1}dt$$

$$\leqq o(N(r, 0, f))+O\left(\sum_{p=1}^{n+1} k^p \log\frac{1+1/(2p)}{1-1/(2p)}\right)$$

$$\leqq o(N(r, 0, f))+O\left(\sum_{p=1}^{n+1} k^p/p\right) \leqq o(N(r, 0, f))+o(k^{n+1}) \quad (n \to \infty),$$

which together with Lemma 2 implies that

(2.40) $$N(r, \infty, g) = (b+o(1))N(r, 0, f) \quad (r \to \infty).$$

Let $A^p(1-1/p) \leqq r \leqq A^p(1+1/p)$. From the first main theorem and (2.32) it follows that

$$m(r, \infty, h) = T(r, h)-N(r, \infty, h) \leqq T(A^p(1+1/p), h)-N(A^p(1-1/p), \infty, h)$$
$$\leqq N(A^p(1+1/p), h)-N(A^p(1-1/p), h)+o(1),$$

and since $N(t, h)=N(t, g)$ for all $t>0$, we deduce from (2.40) and Lemma 2 that

$$m(r, h) \leqq (b+o(1))N(A^p(1+1/p), 0, f)-bN(A^p(1-1/p), 0, f)$$
$$\leqq O\left(k^p \log\frac{1+1/p}{1-1/p}\right)+o(N(A^p, 0, f)) \leqq o(T(r, g)) \quad (p \to \infty).$$

This together with (2.32) implies that

(2.41) $$m(r, \infty, h) = o(T(r, g)) \quad (r \to \infty).$$

Since

$$m(r, g) \leqq m(r, f)+m(r, h)+\log 2$$

and

$$m(r, f) \leqq m(r, g)+m(r, h)+\log 2,$$

we deduce from (2.41) that

(2.42) $$m(r, g) = (1+o(1))m(r, f) = (1+o(1))T(r, f) \quad (r \to \infty).$$

From (2.40) and (2.42) it follows that the functions f and g have the same order, so it follows from Lemma 2 that the order of g is $(\log A)^{-1}\log k=\lambda$.

From (2.40), (2.42) and Lemma 2 we get

$$\frac{m(4A^n, \infty, g)}{T(4A^n, g)} = \frac{T(4A^n, f)}{T(4A^n, f)+bN(4A^n, 0, f)}+o(1)$$
$$= (1+b)^{-1}+o(1) = d+o(1) \quad (n \to \infty),$$

which implies that $\delta(\infty, g) \leqq d$. From (2.40) and (2.42) we deduce that

$$\frac{m(r, \infty, g)}{T(r, g)} = \frac{T(r, f)}{T(r, f)+bN(r, 0, f)}+o(1) \geqq \frac{T(r, f)}{T(r, f)+bT(r, f)}+o(1)$$
$$= d+o(1) \quad (r \to \infty),$$

which implies that $\delta(\infty, g) \geqq d$, and we get $\delta(\infty, g)=d$.

From (2.38), (2.40) and (2.42) it follows that

$$\frac{r\mu(r, g)}{T(r, g)} \leqq \frac{12N(r, 0, f)}{T(r, f)+bN(r, 0, f)}+o(1) \leqq \frac{12N(r, 0, f)}{N(r, 0, f)+bN(r, 0, f)}+o(1)$$
$$= 12\,d+o(1) \quad (r \to \infty).$$

This completes the proof of Lemma 3.

3. Proof of Theorem 1

Let d and t be as in Theorem 1. If $0<t \leqq (\log 9)/9$, then the function g of Lemma 3 satisfies the assertions of Theorem 1.

Let us suppose that $t>(\log 9)/9$. We choose a positive integer k such that

$$k-1 < \frac{9t}{\log 9} \leqq k, \tag{3.1}$$

$\lambda=t/k$, and $f(z)=g(z^k)$, where g is the function of Lemma 3 corresponding to these values of d and λ.

Since

$$m(r, f) = m(r^k, g) \tag{3.2}$$

and

$$N(r, f) = N(r^k, g) \tag{3.3}$$

for all $r>0$, we deduce from Lemma 3 that $\delta(\infty, f)=\delta(\infty, g)=d$ and that the order of f is $k\lambda=t$.

Since

$$|z|\varrho(f(z)) = k|z|^k\varrho(g(z^k))$$

for all $z \in C$, we get from (3.2), (3.3) and Lemma 3

$$\frac{r\mu(r, f)}{T(r, f)} = \frac{kr^k\mu(r^k, g)}{T(r^k, g)} \leqq 12k\,d+o(1) \quad (r \to \infty). \tag{3.4}$$

Since $1/5<(\log 9)/9<1$, we get from (3.1)

$$k \leqq 5((k\log 9)/9+1-(\log 9)/9) = 5(1+(k-1)(\log 9)/9) < 5(1+t),$$

which together with (3.4) proves (1.1). Theorem 1 is proved.

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University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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