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SIMPLIFIED PROOFS OF SOME BASIC THEOREMS FOR QUASIREGULAR MAPPINGS

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1. Introduction

In what follows f will always denote a non-constant n-dimensional quasiregular mapping of a domain $G \subset \boldsymbol{R}^n$ into $\boldsymbol{R}^n$. We recall that the branch set B_f is the set of those points in G at which f is not locally homeomorphic, and that $N(y, f, A)$ is the number of all points in the set $f^{-1}(y) \cap A$. Our notation and terminology is adopted from [1].

The purpose of this paper is to present new simplified proofs for the following well-known theorems in the theory of quasiregular mappings.

1.1. Theorem. *The condition* (N) *is satisfied, i.e., if* $A \subset G$ *and* $m(A)=0$, *then* $m(fA)=0$. *Moreover* $m(fB_f)=0$.

1.2. Theorem. *The transformation formula*

$$\int_E (h \circ f)\, J_f \, dm = \int_{\boldsymbol{R}^n} h(y)\, N(y, f, E)\, dm(y)$$

holds whenever $h\colon \boldsymbol{R}^n \to [0, \infty]$ *and* $E \subset G$ *are measurable.*

1.3. Theorem. *For a.e.* $x \in G$, $J_f(x) \neq 0$. *Consequently* $m(B_f)=0$.

Rešetnjak's original proof for the condition (N) does not make use of the fact that f is discrete and open. In the present proof these properties of f play an essential role. It should be noted that 1.1 is not needed in proving the discreteness and openness of f (see [4]).

Theorem 1.2 is a direct consequence of the proof of Theorem 1.1. Earlier the transformation formula was obtained by the use of a general theorem [3, p. 364] the proof of which requires a heavy machinery of algebraic topology.

The original proof [1, 8.2] of Theorem 1.3 is based on the K_I-capacity inequality. Our proof instead is, based on the use of the K_0-path family inequality and Poleckii's lemma.

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2. The proofs of Theorems 1.1 and 1.2

Because f is continuous and a.e. differentiable, it is a basic fact of real analysis that *if f is injective,* then

$$m(fE) \geqq \int_E J_f \, dm \tag{2.1}$$

for every Borel set E in G. In fact, the equality holds in (2.1). This is a consequence of the following result, which is obtained by a C^1-approximation.

2.2. Proposition. *For every Borel set E in G*

$$m(fE) \leqq \int_E J_f \, dm.$$

Proof. We first show that n-intervals in G can be approximated by n-intervals whose boundaries f maps into null-sets. To do this, fix a closed n-interval Q in G and let ε be positive. Let Q' be a closed n-interval in G so that $Q \subset \operatorname{int} Q'$ and $m(Q' \setminus Q) < \varepsilon$. If $m(f\partial Q_0) > 0$ for every n-interval Q_0, $Q \subset Q_0 \subset Q'$, then there is a positive number p and a sequence of n-intervals Q_i, $Q \subset Q_i \subset Q'$, with disjoint boundaries such that $m(f\partial Q_i) \geqq p$ for every i. But this is impossible, since

$$\sum_i m(f\partial Q_i) = \int_{R^n} \sum_i \chi_{f\partial Q_i} \, dm \leqq N(f, Q') m(fQ') < \infty,$$

where

$$N(f, Q') = \sup \{N(y, f, Q') \mid y \in R^n\}.$$

Hence, $m(f\partial Q_0) = 0$ for some n-interval $Q_0 \supset Q$ with $m(Q_0 \setminus Q) < \varepsilon$.

Let ε_1 be positive. It follows from the definition of the Lebesgue measure and from the approximation result mentioned above, that since J_f is locally integrable (f is ACL^n), there exists a sequence of closed n-intervals $Q_i \subset G$ with $m(f\partial Q_i) = 0$, such that $E \subset \bigcup_i Q_i$ and

$$\sum_i \int_{Q_i} J_f \, dm \leqq \int_E J_f \, dm + \varepsilon_1.$$

On the other hand, $m(fE) \leqq \sum_i m(fQ_i)$, so that it remains to show that the proposition holds for any closed n-interval Q in G satisfying $m(f\partial Q) = 0$. By [5; 27.7] there are C^1-mappings $f_1, f_2, \ldots$, which converge c-uniformly to f and whose Jacobians J_{f_i} converge to J_f in L^1_{loc}. Set $\chi = \chi_{fQ}$ and $\chi_j = \chi_{f_j Q}$. In order to show that $\chi_j \to \chi$ a.e., we first pick a point y in $fQ \setminus f\partial Q$ and note that the local topological degree μ satisfies $\mu(y, f_j, \operatorname{int} Q) = \mu(y, f, \operatorname{int} Q) > 0$ for $j \geqq j_0$, since the convergence is c-uniform and f is sense-preserving. Hence $y \in f_j Q$ if $j \geqq j_0$, and $\chi_j(y) \to \chi(y)$. Outside fQ the convergence $\chi_j \to \chi$ is obvious, so that $\chi_j \to \chi$

a.e. in $\boldsymbol{R}^n$. To complete the proof we apply Fatou's lemma, and get

$$m(fQ) = \int_{R^n} \chi\, dm \leqq \varliminf_{j\to\infty} \int_{R^n} \chi_j\, dm = \varliminf_{j\to\infty} m(f_j Q) \leqq \varliminf_{j\to\infty} \int_Q |J_{f_j}|\, dm = \int_Q J_f\, dm,$$

where the latter inequality comes from elementary calculus.

Theorem 1.1 is an immediate consequence of 2.2; for the last statement, recall that $J_f=0$ a.e. in B_f. Since f satisfies the condition (N), it is obvious that (2.1) and 2.2 hold in fact for any measurable set E in G.

To prove the transformation formula, we first consider the case that $h=\sum_{j=1}^{\infty} a_j \chi_{B_j} (\geqq 0)$ is a simple Borel function. Since $m(fB_f)=0$ and $J_f=0$ a.e. in B_f, we may assume that E does not meet B_f. Let $E_1, E_2, \ldots$ be a measurable partition of the set E, such that each E_k is contained in a domain on which f is injective. Then

$$\int_E (h\circ f) J_f\, dm = \sum_{j,k} a_j \int_{E_k \cap f^{-1}B_j} J_f\, dm = \sum_{j,k} a_j m(fE_k \cap B_j)$$

$$= \int_{R^n} \sum_j a_j \chi_{B_j} \sum_k \chi_{fE_k}\, dm = \int_{R^n} hN(\cdot, f, E)\, dm.$$

Finally, if $h\geqq 0$ is measurable, then there is an increasing sequence (h_i) of simple Borel functions, which converge to h a.e.. It follows from (2.1) that also $h_i\circ f\to h\circ f$ a.e. outside the set $\{x\colon J_f(x)=0\}$, and hence Theorem 1.2 follows by the monotonic convergence theorem.

3. Proof of Theorem 1.3

From 1.2 it follows easily (see [1; 3.2]) that

$$M(\Gamma) \leqq K_0(f) N(f, A) M(f\Gamma) \tag{3.1}$$

if Γ is a path family in a Borel set $A\subset G$, and $N(f, A)<\infty$. This path family inequality and Poleckii's lemma 3.2 will be needed in the proof of 1.3.

3.2. Lemma [2]. *If Γ_0 is the family of all closed paths in G on which f is not absolutely precontinuous, then $M(f\Gamma_0)=0$.*

Recall that f is called absolutely precontinuous on γ if $f\circ\gamma$ is rectifiable and if the reparametrization γ^* of γ with

$$f\circ\gamma^* = (f\circ\gamma)^0$$

is absolutely continuous. Here α^0 denotes the parametrization of α by means of path length.

Proof of 1.3. Suppose that $J_f=0$ in a set of positive measure. This set then contains a Borel set B of positive measure such that $B\subset Q$, where Q is a closed n-interval in G, and that f is differentiable and $f'(x)=0$ for every $x\in B$. Let

Γ_B be the family of all closed intervals γ in Q parallel to $e_1=(1, 0, \ldots, 0)$ with $\int_\gamma \chi_B\, ds>0$. Fubini's theorem implies that $M(\Gamma_B)>0$. By 3 1

$$0 < M(\Gamma_B)/K_0(f)N(f, Q) \leqq M(f\Gamma_B),$$

so that according to Poleckii's lemma there is a path $\gamma\in\Gamma_B$ such that γ^* is absolutely continuous. Thus

$$0 < \int_\gamma \chi_B\, ds = \int_0^{l(f\circ\gamma)} (\chi_B\circ\gamma^*)\,|\gamma^{*\prime}|\, dm_1 = \int_{\gamma^{*-1}B} |\gamma^{*\prime}|\, dm_1,$$

and consequently $m_1(\gamma^{*-1}B)>0$. On the other hand, for m_1-a.e. $t\in\gamma^{*-1}B$,

$$1 = |(f\circ\gamma)^{0\prime}(t)| = |(f\circ\gamma^*)'(t)| = |f'(\gamma^*(t))\gamma^{*\prime}(t)| = 0,$$

which is clearly absurd. Therefore $J_f\neq0$ a.e. Since $J_f=0$ a.e. in B_f, it follows that $m(B_f)=0$.

3.3. Remark. In [2] Poleckii uses 3.2 to prove his celebrated K_I-path family inequality. In his proof he needs the result 1.3, whose original proof requires the use of the K_I-capacity inequality. This latter inequality is quite hard to prove, and, on the other hand, is a special case of the K_I-path family inequality. It is therefore important to have a proof for 1.3 which does not make use of the K_I-capacity inequality.

S. Rickman has pointed out that it would also be possible to modify the proof of the K_I-path family inequality in such way that 1.3 is not needed in the proof.

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