Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 8, 1983, 251—256

DIFFEOMORPHIC APPROXIMATION OF QUASICONFORMAL AND QUASISYMMETRIC HOMEOMORPHISMS

MAIRE KIIKKA

Introduction

Piecewise linear approximations for quasiconformal, bilipschitz and quasisymmetric embeddings have been constructed by Carleson [C], Väisälä [V], Kiikka [K] and by Luukkainen and Tukia [LT]. We apply a result of Munkres [M, Theorem 4.1] to smooth these approximations into C^∞-embeddings in the homeomorphic case. By [LV, V. 4.3] every finite K-quasiconformal mapping in dimension 2 is locally the limit of a sequence of regular K-quasiconformal C^ω-mappings.

I wish to thank Pekka Tukia for suggesting this problem and Jouni Luukkainen for a careful reading of the manuscript and for making several corrections.

1. Preliminaries

We will use the same notation and definitions as in [K] and [LT]. If Q is a closed n-cube of R^n with side length $2\lambda_Q$ and center z_Q, define $\alpha_Q(x)=z_Q+\lambda_Q x$. In open sets of R^n we have canonical decompositions $\mathscr{K}$ into cubes as in [K]. Let I^n denote the closed cube $[-1,1]^n$ in R^n and J^n the open cube $]-1,1[^n$. If $Q\in\mathscr{K}$, then $Q=\alpha_Q I^n$. For $Q\in\mathscr{K}$ define

$$P_Q=P_Q(\mathscr{K})=\{\alpha_Q^{-1}Q' \mid Q'\in\mathscr{K},\ Q'\cap Q\neq\emptyset,\ Q'\neq Q\}.$$

Denote by $\mathscr{K}^n$ the set of all canonical decompositions of open sets of R^n. Set

$$\mathscr{P}^n=\{P_Q(\mathscr{K}) \mid Q\in\mathscr{K}\in\mathscr{K}^n\}.$$

Then $\mathscr{P}^n$ is finite.

Let T be a triangulation of an n-dimensional PL manifold X in R^n. If a k-dimensional open simplex σ of T has vertices $a_0,\dots,a_k$, we write $\sigma=(a_0,\dots,a_k)$. If $0\leqq k\leqq n-1$, let $\{\sigma_1,\dots,\sigma_p\}$ be the set of those n-simplices of T which have σ as a face. Let

$$\sigma_i=(a_0,\dots,a_k,a^i_{k+1},\dots,a^i_n),\quad i\in\{1,\dots,p\}.$$

doi:10.5186/aasfm.1983.0814

If $x\in\bar{\sigma}_i$, denote by $\lambda^i_j(x), j\in\{0, \ldots, n\}$, the barycentric coordinates of x in $\bar{\sigma}_i$. If $s\in]0, \infty[$, set

$$V_s(\sigma) = \bigcup_{i=1}^{p} \{x\in\bar{\sigma}_i | \lambda^i_l(x) \geqq s\lambda^i_j(x) \quad \text{for all} \quad l \leqq k,\ j > k\}.$$

Then $V_s(\sigma)$ is a closed neighborhood of σ in X.

For $\sigma\in T$ set $\mathrm{St}(\sigma, T)=\{\tau\in T \mid \sigma \text{ is a face of } \tau\}$.

Denote by T^k the set of k-simplices of T. If $A\subset X$, set $T|A=\{\sigma\in T|\sigma\subset A\}$ and $T^k | A=\{\sigma\in T^k | \sigma\subset A\}$.

2. Results

The following theorem is analogous to [K, Theorem 2.1].

Theorem 1. *Let $n=2$ or $n=3$ and let $K\geqq 1$. Then there exists $\tilde{K}\geqq 1$ with the following property: Let G and G' be domains in R^n, let $f: G\to G'$ be a K-quasiconformal homeomorphism and let $\varepsilon: G\to]0, \infty[$ be continuous. Then there exists a $\tilde{K}$-quasiconformal C^∞-diffeomorphism $\tilde{f}: G\to G'$ such that $|\tilde{f}(x)-f(x)|<\varepsilon(x)$ for every $x\in G$.*

Theorem 2 is analogous to [K, Theorem 3.1] and [LT, Corollary 3.3], and Theorems 3 and 4 to [LT, Corollary 2.21] and [LT, Theorem 2.16], respectively.

Theorem 2. *Let $n=2$ or $n=3$ and let $L\geqq 1$. Then there exists $\tilde{L}\geqq 1$ with the following property: Let G and G' be open sets in R^n, let $f: G\to G'$ be an L-bilipschitz homeomorphism and let $\varepsilon: G\to]0, \infty[$ be continuous. Then there exists an $\tilde{L}$-bilipschitz C^∞-diffeomorphism $\tilde{f}: G\to G'$ such that $|\tilde{f}(x)-f(x)|<\varepsilon(x)$ for every $x\in G$.*

Theorem 3. *Let $n=2$ or $n=3$ and let $\eta: R^1_+\to R^1_+$ be a homeomorphism. Then there exists a homeomorphism $\tilde{\eta}: R^1_+\to R^1_+$ with the following property: Let G and G' be open sets in R^n, let $f: G\to G'$ be an η-quasisymmetric homeomorphism and let $\varepsilon: G\to]0, \infty[$ be continuous. Then there exists an $\tilde{\eta}$-quasisymmetric C^∞-diffeomorphism $\tilde{f}: G\to G'$ such that $|\tilde{f}(x)-f(x)|<\varepsilon(x)$ for every $x\in G$.*

Let ϱ_Q and β_Q be as in [K] and [LT].

Theorem 4. *Let n and η be as in Theorem 3 and let $\varepsilon>0$. Then there exist a homeomorphism $\tilde{\eta}: R^1_+\to R^1_+$ and a finite set $\tilde{D}$ of C^∞-embeddings $\tilde{g}: 2I^n\to R^n$ with the following property: Let G be open in $R^n, f: G\to R^n$ an η-quasisymmetric embedding and let $\mathscr{K}$ be a canonical decomposition of G. Then there exists an*

$\tilde{\eta}$-quasisymmetric C^∞-embedding $\tilde{f}\colon G\to R^n$ such that

(1) $$d(\tilde{f}|Q, f|Q)\leqq \varepsilon\varrho_Q$$

and

(2) $$\beta_Q\tilde{f}\alpha_Q|2I^n\in\tilde{D}$$

for every $Q\in\mathscr{K}$.

3. Proofs

Proof of Theorem 1. Let $n=2$ or $n=3$ and $K\geqq 1$. Let $\delta_M=\delta_{M(n)}>0$ be the constant of [K, p. 12]. The proof of [K, Lemma 2.2] shows that there exists a finite set $D=D(K)$ of PL embeddings of $2I^n$ into R^n with the following property: Let G be a domain in $R^n, f\colon G\to R^n$ a K-QC embedding and $\mathscr{K}\in\mathscr{K}^n$ with $\cup\mathscr{K}=G$. Then there exists a PL embedding $f^*\colon G\to R^n$ such that $\beta_Q f^*\alpha_Q|2I^n\in D$ and $|f^*(x)-f(x)|<\varrho_Q$ if $x\in Q\in\mathscr{K}$. In fact, to see this, consider the maps γ_Q in [K, p. 12].

We choose a triangulation τ of I^n such that $g|\sigma$ is affine and $d(g\sigma)<\delta_M$ if $g\in D$ and $\sigma\in\tau$ and such that $g|\alpha_R\sigma$ is affine if $g\in D$, $R\in P\in\mathscr{P}^n$, $\sigma\in\tau$ and $\alpha_R\sigma\subset 2I^n$. We may assume that $\tau|Q$ is a triangulation of Q and that $\tau|\partial Q$ is a full subcomplex (see [RS, p. 31]) of $\tau|Q$ for each of the 4^n subcubes $Q\in\{z+[0,1/2]^n|\, z\in(1/2)Z^n\}$ of I^n. Furthermore, if $n=3$, we may assume for all 2-dimensional faces S of these cubes Q that $\tau|\partial S$ is a full subcomplex of $\tau|S$.

For each $P\in\mathscr{P}^n$ we next define a subdivision τ_P of τ such that, if G is open in R^n and if $\mathscr{K}$ is a canonical decomposition of G, then

(3) $$T=\{\alpha_Q\sigma|Q\in\mathscr{K},\ \sigma\in\tau_{P_Q}\}$$

is a triangulation of G.

We first suppose that $P\in\mathscr{P}^3$ and set

$$I_P=\cup\{\partial I^3\cap\partial R_1\cap\partial R_2\,|\,R_1, R_2\in P \text{ and } R_1\neq R_2\}.$$

Let γ_1 be the triangulation of ∂I^3 satisfying

$$\gamma_1^0=(\tau^0\cap I_p)\cup\Big(\bigcup_{R\in P}(\alpha_R\tau^0\cap\partial I^3)\Big)$$

and such that $\gamma_1|\partial R\cap\partial I^3$ is a subdivision of $\alpha_R\tau|\partial R\cap\partial I^3$ for every $R\in P$. Let γ_2 be the subdivision of $\tau|\partial I^3$ having

$$\gamma_2^0=(\tau^0\cap\partial I^3)\cup\Big(\bigcup_{R\in P}(\alpha_R\tau^0\cap I_P)\Big).$$

Then $\gamma_1|I_P=\gamma_2|I_P$. Set $\gamma_3=\{\sigma_1\cap\sigma_2\,|\,\sigma_1\in\gamma_1, \sigma_2\in\gamma_2\}$. Then γ_3 is a cell complex on ∂I^3. By replacing every (open) 2-cell $\sigma\in\gamma_3$ which is not a simplex by the simplices obtained by joining the barycenter of σ with the cells of γ_3 in $\partial\sigma$, we get a triangulation γ_4 of ∂I^3. Then γ_4 is a subdivision of $\tau|\partial I^3$.

Let τ_P be the subdivision of τ having $\tau_P^0=\tau^0\cup\gamma_4^0$ and $\tau_P|\partial I^3=\gamma_4$.

To see that $\{\tau_P\,|\,P\in\mathscr{P}^3\}$ is the desired family, suppose that above $P=P_Q$, where $Q\in\mathscr{K}\in\mathscr{K}^n$, and denote $\gamma_i(Q)=\gamma_i$ if $1\leqq i\leqq 4$. Then, for each $Q'\in\mathscr{K}$ with $Q'\cap Q\neq\emptyset$, $Q'\neq Q$, we have $\alpha_Q\gamma_1(Q)\,|\,Q\cap Q'=\alpha_{Q'}\gamma_2(Q')\,|\,Q\cap Q'$, whence $\alpha_Q\gamma_i(Q)\,|\,Q\cap Q'=\alpha_{Q'}\gamma_i(Q')\,|\,Q\cap Q'$ for $i=3$ and thus also for $i=4$. Therefore $\alpha_Q\tau_{P_Q}\,|\,Q\cap Q'=\alpha_{Q'}\tau_{P_{Q'}}\,|\,Q\cap Q'$.

If $P\in\mathscr{P}^2$, the construction of τ_P is similar but easier.

For every $\mathscr{K}\in\mathscr{K}^n$ and $Q\in\mathscr{K}$ we define a triangulation $T_{P_Q}=T(\mathscr{K},Q)$ of $2I^n$ by setting

$$T_{P_Q}=\alpha_Q^{-1}T\,|\,2I^n,$$

where T is defined by (3).

One can show that T_{P_Q} depends only on the set $P_Q\in\mathscr{P}^n$. Instead of this fact we could have used below the easier fact that the set $\{T(\mathscr{K},Q)\,|\,Q\in\mathscr{K}\in\mathscr{K}^n\}$ is finite.

We are going to smooth the PL maps $g\in D$ in some neighborhoods of I^n. The groups Γ^i of Milnor and Thom, cf. [M, Chapter 1], are zero for $i\in\{1,2,3\}$; see [M, Proof of Theorem 6.3].

For $P\in\mathscr{P}^n$ and $0\leqq i\leqq n-1$ set

$$U_P^i=\cup\{V_1(\sigma):\ \sigma\in T_P^i|2J^n\}.$$

Then U_P^i is a closed neighborhood of I^n in $2I^n$.

We first suppose that $n=2$. Let $g\in D$ and $P\in\mathscr{P}^2$. Then $g|\bar\sigma$ is affine if $\sigma\in T_P$. Let $\sigma\in T_P^1\,|\,2J^2$. We apply [M, Theorem 4.1] with $\mathscr{V}=\operatorname{int}V_1(\sigma)$, $\mathscr{U}=\mathscr{V}\setminus\sigma$, $\mathscr{W}=\operatorname{int}V_2(\sigma)$ to obtain a homeomorphism

$$\varphi=\varphi_{g,\sigma}:\ V_1(\sigma)\to gV_1(\sigma)$$

such that

$$\varphi=g\quad\text{in}\quad\sigma\cup(V_1(\sigma)\setminus V_2(\sigma)),$$

$$d(\varphi,g|V_1(\sigma))<d(g\sigma)/2,$$

and such that $\varphi\,|\operatorname{int}V_1(\sigma)$ is a C^1-embedding and φ is smooth on $\operatorname{int}V_1(\sigma)$ near the vertices of σ; see [M, Definition 2.2]. This is possible because $g\,|\,V_1(\sigma)$ is smooth on $\operatorname{int}V_1(\sigma)\setminus\sigma$ near σ and near the vertices of σ (cf. [M, Proof of Theorem 2.8]), and because $\gamma(g\,|\,V_1(\sigma))\in\Gamma^1=0$ (cf. [M, Definition 3.4]).

Define a homeomorphism

$$\tilde g_P^1=\cup\{\varphi_{g,\sigma}:\ \sigma\in T_P^1|2J^2\}:\ U_P^1\to gU_P^1.$$

Let $v\in T_P^0\,|\,2J^2$. Because $\tilde g_P^1\,|\,V_1(v)$ is smooth on $\operatorname{int}V_1(v)\setminus v$ near v and $\gamma(\tilde g_P^1\,|\,V_1(v))\in\Gamma^2=0$, we may apply [M, Theorem 4.1] to get a homeomorphism

$$\varphi=\varphi_{g,v}:\ V_1(v)\to\tilde g_P^1V_1(v)$$

such that

$$\varphi=\tilde g_P^1\quad\text{in}\quad v\cup(V_1(v)\setminus V_2(v)),$$

$$d(\varphi,\tilde g_P^1|V_1(v))<\min\{d(g\sigma^2)|v\in\bar\sigma^2\text{ and }\sigma^2\in T_P^2\}/2,$$

and $\varphi\,|\operatorname{int}V_1(v)$ is a C^1-embedding. We get a homeomorphism

$$\tilde g_P^0=\cup\{\varphi_{g,v}:\ v\in T_P^0|2J^2\}:\ U_P^0\to\tilde g_P^1U_P^0.$$

Then $\tilde{g}_P^0 \mid \mathrm{int}\, U_P^0$ is a C^1-embedding. One may replace above C^1 by C^∞; see [M, Chapter 9].

We construct the maps $\tilde{g}_P^1$ and $\tilde{g}_P^0$ for every $g \in D$ and $P \in \mathscr{P}^2$. It is possible to do this in such a way that the following condition is satisfied:

Let $i \in \{0, 1\}, j \in \{1, 2\}$, $P_j \in \mathscr{P}^2$, $\sigma_j \in T_{P_j}^i \mid 2J^2$ and let $g, h \in D$. If ψ_1 and ψ_2 are affine bijections of R^2, $\psi_1 \operatorname{St}(\sigma_2, T_{P_2}) = \operatorname{St}(\sigma_1, T_{P_1})$ and $h = \psi_2 g \psi_1$ in $V_1(\sigma_2)$, we have

(4) $$\tilde{h}_{P_2}^i = \psi_2 \tilde{g}_{P_1}^i \psi_1$$

in $V_1(\sigma_2)$. Here $V_1(\sigma_2)$ is taken in T_{P_2}.

Let $\tilde{g}_P \colon \mathrm{int}\, U_P^0 \to R^2$ be the C^∞-embedding defined by $\tilde{g}_P^0$.

If $x \in I^2$, we have

$$|\tilde{g}_P(x) - g(x)| \leqq |\tilde{g}_P(x) - \tilde{g}_P^1(x)| + |\tilde{g}_P^1(x) - g(x)| < \delta_M$$

because $d(g\sigma^2) < \delta_M$ for all $\sigma^2 \in T_P^2 \mid I^2$.

If $n=3$, we may proceed in the same way, because $\Gamma^i = 0$ for $i \in \{1, 2, 3\}$.

Let $n=2$ or 3. Let G and G' be domains in R^n, $f \colon G \to G'$ a K-QC homeomorphism, $\varepsilon \colon G \to]0, \infty[$ continuous and $\mathscr{K}$ a canonical $(f, \varepsilon/2)$-decomposition of G. Let $f^* \colon G \to R^n$ be the PL embedding given in the first paragraph of the proof.

We define $\tilde{f} \colon G \to R^n$ setting

(5) $$\tilde{f} | Q = \beta_Q^{-1} \tilde{g}_{P_Q} \alpha_Q^{-1} | Q$$

for $Q \in \mathscr{K}$ whenever $\beta_Q f^* \alpha_Q \mid 2I^n = g \, (\in D)$.

We show that $\tilde{f}$ is well-defined and that

(6) $$\tilde{f} = \beta_Q^{-1} \tilde{g}_{P_Q} \alpha_Q^{-1} \quad \text{in} \quad U_Q = \alpha_Q (\mathrm{int}\, U_{P_Q}^0)$$

for each $Q \in \mathscr{K}$. Let $R \in \mathscr{K}$, $R \neq Q$, $R \cap Q \neq \emptyset$ and $x \in R \cap U_Q$. There is $v \in T^0 \mid R \cap \alpha_Q(2J^n)$ such that $x \in V_1(v)$, where $V_1(v)$ is taken in the triangulation T of G. Let $v_Q = \alpha_Q^{-1} v$, $v_R = \alpha_R^{-1} v$, $g = \beta_Q f^* \alpha_Q \mid 2I^n$ and $h = \beta_R f^* \alpha_R \mid 2I^n$. Then $\operatorname{St}(v_Q, T_{P_Q}) = \alpha_Q^{-1} \alpha_R \operatorname{St}(v_R, T_{P_R})$ and $h = \beta_R \beta_Q^{-1} g \alpha_Q^{-1} \alpha_R$ in $V_1(v_R)$. Because $\psi_1 = \alpha_Q^{-1} \alpha_R$ and $\psi_2 = \beta_R \beta_Q^{-1}$ are affine bijections of R^n, it follows from (4) that

$$\beta_R^{-1} \tilde{h}_{P_R} \alpha_R^{-1}(x) = \beta_R^{-1} (\beta_R \beta_Q^{-1} \tilde{g}_{P_Q} \alpha_Q^{-1} \alpha_R) \alpha_R^{-1}(x) = \beta_Q^{-1} \tilde{g}_{P_Q} \alpha_Q^{-1}(x).$$

Hence $\tilde{f}$ is well-defined and (6) holds.

By the construction, $\tilde{f}$ is a C^∞-embedding. The maps $\tilde{g}_P \mid J^n$ are quasiconformal. Set

$$\tilde{K} = \max \{ K(\tilde{g}_P | J^n) \mid g \in D, \ P \in \mathscr{P}^n \}.$$

Then $\tilde{f}$ is a $\tilde{K}$-QC embedding.

For $Q \in \mathscr{K}$ we have

(7) $$d(\tilde{f}|Q, f|Q) \leqq d(\tilde{f}|Q, f^*|Q) + d(f^*|Q, f|Q) \leqq \delta_M s_Q + \varrho_Q.$$

Hence $|\tilde{f}(x) - f(x)| \leqq 2\varrho_Q < \varepsilon(x)$ if $x \in Q \in \mathscr{K}$. We may assume that

(8) $$\varepsilon(x) \leqq \min \left\{ d(f(x), \partial G'), \ (1 + |f(x)|)^{-1} \right\}$$

for every $x \in G$. Therefore $\tilde{f}G = G'$; see [K, p. 8]. Theorem 1 is proved. □

Proof of Theorem 4. Let D be the set of PL embeddings given by [LT, Theorem 2.16] and let $\delta_M=\delta_{M(n)}>0$ be the constant of [LT, 2.13], both with $q=n$ and with ε replaced by $\varepsilon/2$. Let $D_1=\{g\in D \mid g: 2I^n\to R^n\}$. For each $g\in D_1$ and $P\in\mathscr{P}^n$ we obtain a C^∞-embedding $\tilde{g}_P$: int $U_P^0\to R^n$ with $d(\tilde{g}_P \mid I^n, g \mid I^n)<\delta_M$ in the same way as in proving Theorem 1.

Let G be open in $R^n, f: G\to R^n$ an η-quasisymmetric embedding and $\mathscr{K}\in\mathscr{K}^n$ with $\cup\mathscr{K}=G$. Define $\tilde{f}\mid Q$ by (5) for each $Q\in\mathscr{K}$. Then $\tilde{f}: G\to R^n$ is a C^∞-embedding and (1) holds, because $d(\tilde{f}|Q, f|Q)\leqq\delta_M s_Q+\varepsilon\varrho_Q/2\leqq\varepsilon\varrho_Q$ for every $Q\in\mathscr{K}$; see (7).

One can find $\tilde{D}$, prove (2), and then, since every $\tilde{g}\in\tilde{D}$ is quasisymmetric, construct $\tilde{\eta}$ with $\tilde{f}$ being $\tilde{\eta}$-quasisymmetric as D and η^* were obtained in the proof of [LT, Theorem 2.16]. □

Proof of Theorem 3. Let $\mathscr{K}$ be a canonical (f, ε)-decomposition of G. Apply Theorem 4 with $\varepsilon=1$. It follows from (1) that $|\tilde{f}(x)-f(x)|\leqq\varrho_Q<\varepsilon(x)$ if $x\in Q\in\mathscr{K}$. We may assume (8). Hence $\tilde{f}G=G'$. □

Proof of Theorem 2. Theorem 2 can be proved similarly to Theorem 1; cf. the proof of [K, Theorem 3.1]. Also Theorem 2 follows easily from Theorem 4; cf. the proof of [LT, Theorem 3.2]. □

References

[C] CARLESON, L.: The extension problem for quasiconformal mappings. - Contributions to Analysis, edited by L. V. Ahlfors et al., Academic Press, New York—London, 1974, 39—47.

[K] KIIKKA, M.: Piecewise linear approximation of quasiconformal and Lipschitz homeomorphisms. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 30, 1980, 1—24.

[LV] LEHTO, O., and K. I. VIRTANEN: Quasiconformal mappings in the plane. - Springer-Verlag, Berlin—Heidelberg—New York, 1973.

[LT] LUUKKAINEN, J., and P. TUKIA: Quasisymmetric and Lipschitz approximation of embeddings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 343—367.

[M] MUNKRES, J.: Obstructions to the smoothing of piecewise-differentiable homeomorphisms. - Ann. of Math. (2) 72, 1960, 521—554.

[RS] ROURKE, C. P., and B. J. SANDERSON: Introduction to piecewise-linear topology. - Springer-Verlag, Berlin—Heidelberg—New York, 1972.

[V] VÄISÄLÄ, J.: Piecewise linear approximation of lipeomorphisms. - Ann. Acad. Sci. Fenn Ser. A I Math. 3, 1977, 377—383.

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 7 March 1983