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ON THE INTEGRAL REPRESENTATION OF BISUBHARMONIC FUNCTIONS IN $\boldsymbol{R}^n$

VICTOR ANANDAM

1. Introduction

A summable function ω in $\boldsymbol{R}^n$, $n \geqq 2$, satisfying the condition $\Delta^2\omega \geqq 0$ (Δ is the Laplacian in the sense of distribution) can be identified with the pair (ω, h) satisfying the conditions $\Delta\omega = h$ and $\Delta h \geqq 0$. Then h is an almost subharmonic function; and moreover, remembering the fact that, given any Radon measure $\mu \geqq 0$ in $\boldsymbol{R}^n$, one can construct a subharmonic function u with associated measure μ in the local Riesz representation, ω can be seen to be an almost δ-subharmonic function.

The purpose of this article is to study the properties of such functions (ω, h) with a view to represent them in $\boldsymbol{R}^n$ as integrals.

A subharmonic function of finite order in $\boldsymbol{R}^n$ is the unique sum of a canonical potential and a harmonic function [4]. To begin, we give some properties of subharmonic functions for which the potential part is dominant in determining the growth at infinity. In particular, such functions form a sup-stable convex cone.

Then it is shown that with every δ-subharmonic function ω, one can associate a subharmonic function ω^* and define the order of ω as that of ω^*. This value, of course, is the same as the order of the function $T(r, \omega)$ defined (Privalov) by analogy with meromorphic functions in $\boldsymbol{C}$; that is, if ω is a δ-subharmonic function in $\boldsymbol{R}^n$ (harmonic in a neighbourhood of 0) with associated measure $\mu = \mu^+ - \mu^-$, then

$$T(r, \omega) = M(r, \omega^+) + \alpha_n \int_0^r \frac{\mu^-(B_0^t)}{t^{n-1}}\, dt,$$

where $M(r, \omega^+)$ is the mean of ω^+ on $|x| = r$ and $\alpha_n = \max(1, n-2)$. Then, using the subharmonic function ω^*, we give some integral representation theorems for ω and explain their relation with the Hadamard representation theorem of M. Arsove [2] for δ-subharmonic functions in $\boldsymbol{R}^2$.

Making use of these results, one finally arrives at the integral representation of a bisubharmonic pair (ω, h).

The details are given only in $\boldsymbol{R}^3$.

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2. Preliminaries

Let $S(r)$ be an increasing function in $\boldsymbol{R}^3$. If $S(r)$ is not upper bounded, we define the order of $S(r)$ as

$$\operatorname{ord} S(r) = \limsup_{r\to\infty} \frac{\log S(r)}{\log r};$$

otherwise we take that $\operatorname{ord} S(r)$ is 0.

For a positive measure μ in $\boldsymbol{R}^3$, $\operatorname{ord} \mu(B_0^r)$ is called the order of μ and the smallest integer n (if it exists) such that $\int_1^\infty |y|^{-n-1} d\mu(y)$ is finite is called the genus of μ.

For a signed measure $\mu=\mu^+-\mu^-$, we shall define $\operatorname{ord}\mu$ and $\operatorname{gen}\mu$ as the order and the genus of $|\mu|$. It is then simple to remark that $\operatorname{ord}\mu=\max(\operatorname{ord}\mu^+, \operatorname{ord}\mu^-)$ and $\operatorname{gen}\mu=\max(\operatorname{gen}\mu^+, \operatorname{gen}\mu^-)$.

Define as in [3]

$$B_n'(x, y) = \begin{cases} -|x-y|^{-1} & \text{if} \quad |y| < 1 \\ -|x-y|^{-1}+|y|^{-1}+\sum\limits_{m=1}^{n-1} H_m |x|^m |y|^{-m-1} & \text{if} \quad |y| \geqq 1, \end{cases}$$

where $H_m=P_m(\cos\theta)$, P_m being the Legendre polynomial of degree m and θ the angle between $0x$ and $0y$.

We recall that, given any positive Radon measure μ in $\boldsymbol{R}^3$, we can construct subharmonic functions u in $\boldsymbol{R}^3$ with associated measure μ in a local Riesz representation. If $\mu\geqq 0$ is of genus n, one such function is $\int B_n'(x, y)\,d\mu(y)$, called the *canonical potential* associated with μ [4].

3. Measure-dominant subharmonic functions

Let u be a subharmonic function in $\boldsymbol{R}^3$ with associated measure μ. We say that u is a *normal subharmonic function* if $\operatorname{ord}\mu$ is finite or equivalently if $\operatorname{ord} M(r, u)$ is finite, where $M(r, u)$ is the mean of $u(x)$ on $|x|=r$.

A normal subharmonic function u in $\boldsymbol{R}^3$ has a unique decomposition in the form $u=p+H$, where p is the canonical potential associated with μ and H is a harmonic function in $\boldsymbol{R}^3$.

Theorem 1. *Let u be a subharmonic function in $\boldsymbol{R}^3$ with associated measure μ. If u is normal,* $\operatorname{ord} u=\max(\operatorname{ord} H, -1+\operatorname{ord}\mu)$; *and if u is not normal,* $\operatorname{ord} u$ *is* ∞.

Proof. If u is not normal, $\operatorname{ord} M(r, u)$ is not finite and hence $\operatorname{ord} u = \operatorname{ord} M(r, u^+)=\infty$.

If u is normal, let $u=p+H$ be the canonical decomposition of u. Then, by Corollary 3.6 [4], $\operatorname{ord} u = \max(\operatorname{ord} H, \operatorname{ord} p)$. But, by Theorem 2.4 [4], $\operatorname{ord} p = \max(0, -1+\operatorname{ord}\mu)$.

Hence the theorem is proved.

Note. The condition $\operatorname{ord}\mu < 1$ means that u is a (Newtonian) potential up to an additive harmonic function.

Definition 2. *We say that a subharmonic function u in $\mathbf{R}^3$ with associated measure μ is* measure-dominant *if* $\operatorname{ord} u = \max(0, -1+\operatorname{ord}\mu)$; *this class of functions is denoted by $\mathscr{D}$.*

Remarks. (1) In view of Theorem 2.1 [4], it can be seen that $u \in \mathscr{D}$ if and only if $\operatorname{ord} u = \operatorname{ord} M(r, u)$.

(2) The class $\mathscr{D}$ includes (a) all canonical potentials (Theorem 2.4 [4]), (b) all subharmonic functions whose order is not an integer (Corollary 2.2 [4]), (c) all positive subharmonic functions and (d) all subharmonic functions which are not normal. Moreover, the following results show in particular that $\mathscr{D}$ is a sup-stable convex cone.

For any two subharmonic functions u and v, $\operatorname{ord}(u+v) \leqq \max(\operatorname{ord} u, \operatorname{ord} v)$; but here, in general, we cannot replace the inequality sign by the equality sign. In this context we have the following theorem.

Theorem 3. *Let u be a subharmonic function in $\mathbf{R}^3$ and $v \in \mathscr{D}$. Then* $\operatorname{ord}(u+v) = \max(\operatorname{ord} u, \operatorname{ord} v) = \operatorname{ord}(\sup(u, v))$.

Proof. First we note that for any two subharmonic functions u and v, $\max(\operatorname{ord} u, \operatorname{ord} v) = \operatorname{ord}(\sup(u, v))$.

In the present case, let u be an arbitrary subharmonic function with associated measure μ and $v \in \mathscr{D}$ with associated measure ν.

We know that $\operatorname{ord}(u+v) = \max(\operatorname{ord} u, \operatorname{ord} v)$ in the following two cases: (i) when $\operatorname{ord} u \neq \operatorname{ord} v$ (Theorem 3.1 [4]), and (ii) when $\operatorname{ord} u = \operatorname{ord} v = \lambda$, a non-integer (Theorem 3.4 [4]). Hence the only case that remains to be seen is when $\operatorname{ord} u = \operatorname{ord} v = n$, an integer.

Write $v = p + H$, where p is the canonical potential with associated measure ν and H is harmonic. Then $\operatorname{ord} p = \operatorname{ord} v = n$ (since $v \in \mathscr{D}$) and hence $\operatorname{ord}\nu$ is $n+1$ (Theorem 2.4 [4]).

Consequently $\operatorname{ord}(u+H+p) \geqq n$ (Theorem 2.1 [4]), which implies that $\operatorname{ord}(u+v)$ is n.

Corollary. *Let $u, v \in \mathscr{D}$. Then $u+v \in \mathscr{D}$.*

For
$$\begin{aligned}\operatorname{ord}(u+v) &= \max(\operatorname{ord} u, \operatorname{ord} v)\\ &= \max(\operatorname{ord} M(r, u), \operatorname{ord} M(r, v))\\ &= \operatorname{ord}(M(r, u)+M(r, v))\\ &= \operatorname{ord} M(r, u+v).\end{aligned}$$

Hence $u+v\in\mathscr{D}$.

Proposition 4. *Let* u *be a subharmonic function in* $\boldsymbol{R}^3$ *majorizing some* $v\in\mathscr{D}$. *Then* $u\in\mathscr{D}$.

Proof. Let us suppose that u is a normal subharmonic function with associated measure μ.

Write $u=p+H$, where H is harmonic, and note that $\operatorname{ord} p \geqq \operatorname{ord}(v-H) = \max(\operatorname{ord} v, \operatorname{ord} H)$.

Hence $\operatorname{ord} u = \operatorname{ord} p$; that is, $u\in\mathscr{D}$.

Corollary. *Let* u *be a subharmonic function in* $\boldsymbol{R}^3$ *and* $v\in\mathscr{D}$. *Then* $\operatorname{ord}(u+v)=\operatorname{ord} M(r, \sup(u, v))$.

For, since $\sup(u, v)\in\mathscr{D}$,

$$\begin{aligned}\operatorname{ord} M(r, \sup(u, v)) &= \operatorname{ord}(\sup(u, v))\\ &= \operatorname{ord}(u+v) \quad \text{(Theorem 3).}\end{aligned}$$

4. Normal δ-subharmonic functions

A function ω that is the difference of two subharmonic functions in $\boldsymbol{R}^3$ is called a δ-subharmonic function. We say that ω is a *normal δ-subharmonic function* in $\boldsymbol{R}^3$ if $\operatorname{ord}\mu$ is finite where μ is the measure associated with ω in the local Riesz representation.

4.1. *The order of a δ-subharmonic function.* Let ω be a normal δ-subharmonic function with associated measure μ. Let u and v be the canonical potentials associated with μ^+ and μ^-. Then ω has a unique decomposition of the form $\omega=H+u-v$, where H is harmonic in $\boldsymbol{R}^3$. With this decomposition, we define the order of ω as follows:

Definition 5. *Let* ω *be a δ-subharmonic function in* $\boldsymbol{R}^3$. *When* ω *is normal we define* $\operatorname{ord}\omega=\max(\operatorname{ord} H, -1+\operatorname{ord}\mu)$; *if* ω *is not normal, we take* $\operatorname{ord}\omega$ *as* ∞.

With a normal δ-subharmonic function $\omega=H+u-v$ in $\boldsymbol{R}^3$, we consider the subharmonic function $\omega^*=H+u+v$ in $\boldsymbol{R}^3$, called the *subharmonic function associated* with ω.

Theorem 6. *Let* ω *be a normal δ-subharmonic function in* $\boldsymbol{R}^3$ *with* ω^* *as its associated subharmonic function. Then* $\operatorname{ord}\omega=\operatorname{ord}\omega^*$.

Proof. By Theorem 3 we have

$$\begin{aligned}\operatorname{ord}\omega^* &= \operatorname{ord}(H+u+v)\\ &= \max(\operatorname{ord} H, \operatorname{ord} u, \operatorname{ord} v)\\ &= \max(\operatorname{ord} H, -1+\operatorname{ord}\mu^+, -1+\operatorname{ord}\mu^-)\\ &= \max(\operatorname{ord} H, -1+\operatorname{ord}\mu)\\ &= \operatorname{ord}\omega.\end{aligned}$$

4.2. *Relation with other definitions of order.*

i) *The definition due to Arsove.* Let ω be a δ-subharmonic function with associated measure μ. Write $\omega=u-v$, where u and v are two subharmonic functions with associated measures μ^+ and μ^-. Let $s=\sup(u, v)$.

The function s is unique up to the addition of a harmonic function and hence $M(r, s)$ is determined by ω to within an additive constant. M. Arsove then defines $\operatorname{ord}\omega=\operatorname{ord} M(r, s)$.

Using Corollary to Proposition 4, one shows that this manner of defining the order of ω gives the same value for $\operatorname{ord}\omega$ as in Definition 5 above.

ii) *The relation with meromorphic functions in* $\boldsymbol{C}$. Let ω be a δ-subharmonic function with associated measure μ. We shall take $\omega(0)=0$. Then by analogy with the characteristic function of a meromorphic function in $\boldsymbol{C}$, define (Privalov)

$$T(r, \omega) = M(r, \omega^+)+\int_0^r \frac{\mu^-(B_0^t)}{t^2}\,dt.$$

Now it is immediate that

$$T(r, \omega) = T(r, -\omega) = M(r, \omega^-)+\int_0^r \frac{\mu^+(B_0^t)}{t^2}\,dt$$

and that $\operatorname{ord}\omega=\operatorname{ord} T(r, \omega)$. For, if $\omega=u-v$ is a decomposition as above and if $s=\sup(u, v)$, then

$$T(r, \omega) = M(r, s)-M(r, v)+\int_0^r \frac{\mu^-(B_0^t)}{t^2}\,dt = M(r, s)-v(0).$$

iii) *The natural decomposition of* ω. Let ω be a δ-subharmonic function with associated measure μ. Write $\omega=u-v$, where v is taken as the canonical potential if $\operatorname{ord}\mu^-$ is finite; or if $\operatorname{ord}\mu^-$ is not finite and $\operatorname{ord}\mu^+$ is finite, then u is taken as the canonical potential; or if $\operatorname{ord}\mu^-$ and $\operatorname{ord}\mu^+$ are both infinite, then u and v are taken as subharmonic functions with associated measures μ^+ and μ^-.

Such a decomposition of ω shall be referred to as a *natural decomposition* of ω. Note that when $\operatorname{ord}\mu$ is finite, $u+v$ is the subharmonic function associated with ω.

Let ω be a δ-subharmonic function with a natural decomposition $\omega=u-v$. Then it is easy to prove that $\operatorname{ord}\omega=\operatorname{ord}(u+v)$.

4.3. *Measure-dominant δ-subharmonic functions.* Let h be a subharmonic function in $\boldsymbol{R}^3$. Then $\operatorname{ord} h=\operatorname{ord} h^+$ and the δ-subharmonic function h^- satisfies the condition $\operatorname{ord} h^-\leqq\operatorname{ord} h^+$; when h is harmonic, $\operatorname{ord} h^-=\operatorname{ord} h^+$ of course. Now under what general condition can we say that $\operatorname{ord} h^-=\operatorname{ord} h^+$ and, further, what are the analogous results in the case of a δ-subharmonic function ω?

Theorem 7. *Let* ω *be a* δ*-subharmonic function in* $\boldsymbol{R}^3$. *Then*

$$\operatorname{ord}\omega = \max(\operatorname{ord}\omega^+, \operatorname{ord}\omega^-).$$

Proof. Since $\operatorname{ord}\omega \leqq \max(\operatorname{ord}\omega^+, \operatorname{ord}\omega^-)$, we shall now consider only the case when ω is normal.

Let $\omega = u - v$ be the natural decomposition of ω. Then $\omega^+ = \sup(u, v) - v$ and $\operatorname{ord}\omega^+ \leqq \operatorname{ord}\sup(u, v) = \operatorname{ord}(u+v) = \operatorname{ord}\omega$.

Dealing similarly with ω^-, we obtain $\operatorname{ord}\omega \geqq \max(\operatorname{ord}\omega^+, \operatorname{ord}\omega^-)$. Hence the theorem is proved.

A similar argument proves the following proposition.

Proposition 8. *Let* ω *be a* δ*-subharmonic function with a natural decomposition* $\omega = u - v$. *Then* $\operatorname{ord}\omega = \max(\operatorname{ord}\omega^+, \operatorname{ord} v)$.

Corollary. *Let* ω *be a* δ*-subharmonic function with associated measure* μ. *If* $\operatorname{ord}\mu^- < 1 + \operatorname{ord}\omega$, *then* $\operatorname{ord}\omega = \operatorname{ord}\omega^+$.

Let us say that a δ-subharmonic function ω with associated measure μ is *measure-dominant* if $\operatorname{ord}\omega = (-1 + \operatorname{ord}\mu)^+$. We shall denote this class of δ-subharmonic functions by $\bar{\mathscr{D}}$. Thus when a measure-dominant subharmonic function ω is of finite order, its associated subharmonic function $\omega^* \in \mathscr{D}$. The following lemma, in particular, shows that if a δ-subharmonic function ω majorizes a subharmonic function $s \in \mathscr{D}$, then $\omega \in \bar{\mathscr{D}}$.

Lemma 9. *Let* ω *be a* δ*-subharmonic function with associated measure* μ. *If* ω *majorizes a measure-dominant subharmonic function (in particular if* $\omega \geqq 0$*), then* $\operatorname{ord}\omega = (-1 + \operatorname{ord}\mu^+)^+$.

Proof. If $\operatorname{ord}\mu^+ = \infty$, $\operatorname{ord}\omega = \infty$. Let us suppose then $\operatorname{ord}\mu^+$ is finite.

Note that $\operatorname{ord}\mu^- \leqq \operatorname{ord}\mu^+$ since $\omega \geqq s \in \mathscr{D}$. Hence, in the natural decomposition $\omega = u - v$, v is a canonical potential and hence $v + s \in \mathscr{D}$.

Since $u \geqq v + s$, $u \in \mathscr{D}$ and $\operatorname{ord} u = \operatorname{ord} M(r, u) = (-1 + \operatorname{ord}\mu^+)^+$.

Moreover, $\operatorname{ord}\omega = \operatorname{ord}(u+v) = \operatorname{ord} u$.

Hence the lemma is proved.

Remark. In the above lemma, if $\omega = u - v$ is a natural decomposition of ω, $\operatorname{ord}\omega = \operatorname{ord} u = \operatorname{ord} M(r, u)$.

Theorem 10. *Let* ω *be a* δ*-subharmonic function not in* $\bar{\mathscr{D}}$. *Then* $\operatorname{ord}\omega^- = \operatorname{ord}\omega^+ = \operatorname{ord}\omega$.

Proof. On account of Theorem 7, we shall take for example $\operatorname{ord}\omega = \operatorname{ord}\omega^+$. Since $\omega \notin \bar{\mathscr{D}}$, it is a normal δ-subharmonic function. Let $\omega = u - v$ be the natural decomposition of ω. Then

(i) $\operatorname{ord}\omega = \operatorname{ord} u > \max(\operatorname{ord} M(r, u), \operatorname{ord} M(r, v))$.

Let $\omega^+=u_1-v_1$ and $\omega^-=u_2-v_2$ be the natural decompositions of ω^+ and ω^-. Then

(ii) $\operatorname{ord}\omega=\operatorname{ord}\omega^+=\operatorname{ord}M(r, u_1)$.

Note also that $\operatorname{ord}M(r, v_1)\leqq\operatorname{ord}M(r, v)$ since $u_1-v_1=\omega^+=\sup(u, v)-v$; and since $\omega^-=\omega^+-\omega$, by the property of natural decompositions, there exists a subharmonic function s in $\boldsymbol{R}^3$ such that

(iii)
$$u_2+s = u_1+v$$
and

(iv)
$$v_2+s = u+v_1.$$
From (iv) we obtain by using (i)
$$\begin{aligned}\operatorname{ord}M(r, s)&\leqq\max(\operatorname{ord}M(r, u), \operatorname{ord}M(r, v_1))\\ &\leqq\max(\operatorname{ord}M(r, u), \operatorname{ord}M(r, v))\\ &<\operatorname{ord}u.\end{aligned}$$
From (iii) it follows by using (ii)
$$\begin{aligned}\max(\operatorname{ord}M(r, u_2), \operatorname{ord}M(r, s))&=\max(\operatorname{ord}M(r, u_1), \operatorname{ord}M(r, v))\\ &=\operatorname{ord}u.\end{aligned}$$

Consequently, $\operatorname{ord}M(r, u_2)=\operatorname{ord}u$; but $\operatorname{ord}\omega^-=\operatorname{ord}M(r, u_2)$ by Remark above and hence $\operatorname{ord}\omega^-=\operatorname{ord}\omega^+=\operatorname{ord}\omega$.

4.4. *Integral representation.* Let ω be a normal δ-subharmonic function with associated measure μ. Let ω^* be the subharmonic function associated with ω. Then we can prove the following two theorems, similar to Theorem 3.2 and Theorem 3.3 in [3].

Theorem 11. *Let ω be a normal δ-subharmonic function in $\boldsymbol{R}^3$, with ω^* as its associated subharmonic function. Then the following statements are equivalent:*

i) $$\int_R^\infty r^{-n-1}\,d|\mu|(B_0^r) \quad \textit{is finite.}$$

ii) $$\int_R^\infty r^{-n-2}|\mu|(B_0^r)\,dr \quad \textit{is finite.}$$

iii) $$\int_R^\infty r^{-n-1}M(r, \omega^*)\,dr \quad \textit{is finite.}$$

iv) $$\omega(x) = \int B_n'(x, y)\,d\mu(y)+\text{a harmonic function.}$$

Remark. The implication i)$\Rightarrow$iv) in essentially the Hadamard representation theorem for δ-subharmonic functions proved by M. Arsove in [2]. The converse as in the proof of Theorem 3.1 in [3], is a little more involved.

Theorem 12. *Let ω be a normal δ-subharmonic function in $\boldsymbol{R}^3$, with ω^**

as its associated subharmonic function. If $\int_R^\infty r^{-n-1}M(r, \omega^{*+})dr$ *is finite, then* $\omega(x)$ *is of the form*

$$\omega(x) = \int B_n'(x, y)\, d\mu(y) + h(x),$$

where $h(x)$ *is a harmonic polynomial of degree* $<n$.

5. Canonical bipotentials

We shall consider in this section a δ-subharmonic function ω in $\boldsymbol{R}^3$ as a pair (ω, μ), where μ is the measure associated with ω in a local Riesz representation; that is, $d\mu(x) = (1/4\pi)\, \Delta\omega dx$. Of particular interest is the case when μ is given by a density function which is subharmonic in $\boldsymbol{R}^3$.

5.1. *Bisubharmonic pair.*

Definition 13. *Let* (ω, h) *be a pair of functions defined in* $\boldsymbol{R}^3$, *where* ω *is a* δ*-subharmonic function satisfying the condition* $\Delta\omega = h$. *Then we say that*

i) (ω, h) *is a* bisubharmonic pair *if* h *is subharmonic; and*

ii) (ω, h) *is a* biharmonic pair *if* h *is harmonic.*

Remark. If (ω, h) is a bisubharmonic pair, then $\Delta^2\omega \geqq 0$. On the other hand, if ω is a locally summable function such that $\Delta^2\omega \geqq 0$, let h be the subharmonic function such that $h = \Delta\omega$ a.e. and let ω_0 be a δ-subharmonic function such that $\Delta\omega_0 = h$. Then we can write $\omega = \omega_0 + H$ a.e, where H is a harmonic function in $\boldsymbol{R}^3$. Consequently, when $\Delta^2\omega \geqq 0$, $(\omega, \Delta\omega)$ coincides a.e. with the bisubharmonic pair $(\omega_0 + H, h)$.

Definition 14. *Let* (ω, h) *be a bisubharmonic pair in* $\boldsymbol{R}^3$. *Then* $\operatorname{ord}\omega$ *is taken as* the order of the bisubharmonic pair (ω, h).

Theorem 15. *Let* (ω, h) *be a bisubharmonic pair in* $\boldsymbol{R}^3$. *If* μ *is the measure associated with* ω, *then* $\operatorname{ord}\mu^+ = \operatorname{ord}\mu = 3 + \operatorname{ord} h$.

Proof. Since h is subharmonic, $M(r, h^+) - M(r, h^-) = M(r, h)$ is an increasing function of r, and hence

$M(r, h^+) \leqq M(r, |h|) \leqq 2M(r, h^+) +$ a constant, which implies that

$$\text{(i)} \qquad \operatorname{ord} h = \limsup \frac{\log M(r, h^+)}{\log r} = \limsup \frac{\log M(r, |h|)}{\log r}.$$

Now

$$\text{(ii)} \qquad |\mu|(B_0^r) = \frac{1}{4\pi} \int_{B_0^r} |h(x)|\, dx = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \int_0^r |h(t, \theta, \varphi)|\, t^2 \sin\theta\, dt\, d\theta\, d\varphi$$

$$= \int_0^r M(t, |h|)\, t^2\, dt.$$

Suppose $M(R, h^+) \geqq R^\alpha$ for some R; then from (ii)

$$|\mu|(B_0^{2R}) \geqq \int_0^{2R} M(t, h^+) t^2\, dt \geqq \int_R^{2R} M(t, h^+) t^2\, dt$$

$$\geqq \frac{7}{3} M(R, h^+) R^3 \geqq \frac{7}{3} R^{3+\alpha},$$

This implies that

(iii) $$\operatorname{ord} \mu = \operatorname{ord} |\mu|(B_0^r) \geqq 3 + \operatorname{ord} h.$$

In particular, if $\operatorname{ord} h = \infty$, then $\operatorname{ord} \mu = \infty$.

Let us suppose that $\operatorname{ord} h = \lambda < \infty$.

Then from (i), $M(t, |h|) \leqq t^{\lambda+\varepsilon}$ if $t \geqq R$ and hence, using (ii),

$$|\mu|(B_0^r) \leqq \int_0^R M(t, |h|) t^2\, dt + \int_R^r t^{\lambda+\varepsilon} t^2\, dt \leqq A r^{\lambda+3+\varepsilon} + \text{a constant},$$

which implies that

(iv) $$\operatorname{ord} \mu \leqq \lambda + 3 = 3 + \operatorname{ord} h.$$

From (iii) and (iv) we get $\operatorname{ord} \mu = 3 + \operatorname{ord} h$.

A similar argument dealing only with h^+ instead of $|h|$ shows that $\operatorname{ord} \mu^+ = 3 + \operatorname{ord} h$.

Hence the theorem is proved.

Corollary. *Let* (ω, h) *be a biharmonic pair in* $\mathbf{R}^3$. *If* $\operatorname{ord}(\omega, h)$ *is finite, then it is an integer.*

For, by hypothesis, the order of the δ-subharmonic function ω is finite. Then if $\omega = H + u - v$ is the natural decomposition of ω, $\operatorname{ord} \omega = \max(\operatorname{ord} H, -1 + \operatorname{ord} \mu)$.

Now the fact that $\operatorname{ord} H$ is finite implies that it is an integer; and the fact that $\operatorname{ord} \mu$ is finite implies that $\operatorname{ord} h$ is finite, and hence an integer. So is $\operatorname{ord} \mu = 3 + \operatorname{ord} h$.

Consequently, $\operatorname{ord} \omega$ is an integer.

5.2. *Decomposition of a bisubharmonic pair.*

Definition 16. *If* μ *is a signed measure of genus* n, $\int B_n'(x, y)\, d\mu(y)$ *is called the* canonical δ-potential *associated with* μ. *A bisubharmonic pair* (ω, h) *is called a* canonical bipotential pair *if* h *is a canonical potential and* ω *is a canonical δ-potential.*

Theorem 17. *Every bisubharmonic pair of finite order is the unique sum of a canonical bipotential pair and a biharmonic pair.*

Proof. Let (ω, h) be a bisubharmonic pair of finite order. This implies (Theorem 15) that h is of finite order. Write $h = p + H$, where p is a canonical potential and H is a harmonic function.

Let ω_0 be the unique canonical δ-potential such that $\Delta\omega_0 = p$. Then $(\omega, h) = (\omega_0, p) + (\omega - \omega_0, H)$, where (ω_0, p) is a canonical bipotential pair and $(\omega - \omega_0, H)$ is a biharmonic pair.

5.3. *Integral representation.* In view of Theorem 11, we have the following *integral representation theorem for a bisubharmonic pair* (ω, h). Note that here ω is a normal δ-subharmonic function if and only if $\operatorname{ord} h$ is finite.

Theorem 18. *Let* (ω, h) *be a bisubharmonic pair,* $h^+ \not\equiv 0$. *Then the following statements are equivalent:*

(i)
$$\int_{|y| \geqq R} \frac{|h(y)|}{|y|^{n+1}}\, dy \quad \text{is finite.}$$

(ii)
$$\int_R^\infty \frac{M(r, |h|)}{r^{n-1}}\, dr \quad \text{finite.}$$

(iii)
$$\int_R^\infty \frac{M(r, h^+)}{r^{n-1}}\, dr \quad \text{is fiinite.}$$

(iv)
$$\omega(x) = (1/4\pi) \int B_n'(x, y) h(y)\, dy + a \quad \text{harmonic function.}$$

Corollary. *Let* (ω, h) *be a bisubharmonic pair. If* $\operatorname{ord} h$ *is a non-integer* λ *and if* n *is the greatest integer* $< \lambda$, *then*

$$\omega(x) = (1/4\pi) \int B_{n+3}'(x, y) h(y)\, dy + H(x),$$

where $H(x)$ *is a harmonic function. Moreover, if* $\omega(x)$ *majorizes a measure-dominant subharmonic function (in particular if* $\omega(x) \geqq 0$*), then* $H(x)$ *is a harmonic polynomial of degree* $\leqq n+2$.

5.4. *Some extensions.* The results in this section can be easily modified for the class (ω, h) where ω and h are locally summable functions such that $\Delta\omega = h$ and $\Delta h \geqq 0$.

These in turn can be generalized to the class (ω, h) of locally summable functions where $\Delta\omega = h$ and $\Delta h = \varphi$, φ being a locally summable function (or, more generally, $\Delta\omega \geqq h$ and $\Delta h \geqq \mu$, where μ is a signed measure).

Of special interest is the case where φ satisfies the condition $\int_{B_0^r} |\varphi(x)|\, dx \leqq r^\lambda$ for some λ and $r \geqq R$. In this case we can find a pair (ω, h) that is the difference of two canonical bipotentials satisfying the equation $\Delta^2\omega = \varphi$ a.e.

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Université Mohammed V
Département de Mathématiques
Faculté des Sciences
Rabat

Morocco

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