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FOURIER—STIELTJES COEFFICIENTS AND CONTINUATION OF FUNCTIONS

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1. To introduce our subject we recall two classical problems of continuation for certain functions of a complex variable.

(CA) Here f is continuous in R^2 and analytic outside a closed set E; E is *removable* for this problem if it is always true that f is entire.

(QC) In this problem f is a homeomorphism of the extended plane, which is K-quasiconformal outside E; E is *removable* if f is always quasiconformal in the extended plane.

For both problems there is a best-possible theorem. E is removable if it is of σ-finite length (Besicovitch [2]; Gehring [5]). In each problem, a product set $F\times[0, 1]$ is non-removable precisely when F is uncountable (Carleson [3], Gehring [5]). Our purpose is to find nonremovable sets contained in $F\times[0, 1]$, not of the product type nor even approximately so. To describe these sets we denote by Γ a compact set in R^2 meeting each line $x=x_0$ at most once, so that Γ is the *graph* of a real function whose domain is a compact set in R; since Γ is closed, that function is continuous.

Theorem. (a) *In each compact set* $E_1\times E_2$, *where* E_1 *is uncountable and* E_2 *has positive linear measure, there is a graph* Γ *non-removable for* (CA).

(b) *In each set* $E_1\times[0, 1]$, *where* E_1 *is uncountable, there is a graph* Γ, *non-removable for* (QC).

The reason for requiring an interval on the y-axis, and not merely a set of positive measure in (b), can be seen from [1, p. 128]. Perhaps the correct class of sets E^2 could be found. When E_2 is an interval, the non-removability of Γ can be improved in two directions.

2. Both proofs are based on a theorem of Wiener (1924) about the Fourier—Stieltjes transforms of measures on R; a streamlined version of this theorem is presented in [6; p. 42]. We adopt the symbol $e(t)\equiv e^{2\pi it}$ and the notation $\hat{\mu}(u)\equiv\int e(-ut)\cdot\mu(dt)$. Wiener's theorem is then the relation $\lim_N (2N+1)^{-1}\sum_{-N}^{N} \hat{\mu}(k)=\mu(Z)$, and in fact is an easy consequence of dominated convergence. When μ is *continuous,* i.e. has no jumps, and $\lambda=\mu*\tilde{\mu}$ is defined by $\hat{\lambda}=|\hat{\mu}|^2$, this becomes $\sum_{-N}^{N}|\hat{\mu}(k)|^2=o(N)$. This means that there is a set N_1 of positive integers, of asymptotic density 1, such

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that $\hat{\mu}(k)\to 0$ as $k\to\infty$ in N_1. The Lebesgue space $L^1(d\mu)$ is separable so there is a set N_2, again of asymptotic density 1, such that $e(-kt)\to 0$ weak* in $L^\infty(d\mu)$ as $k\to\infty$ in N_2, that is $\int e(-kt)f(t)\mu(dt)\to 0$ for each f in $L^1(d\mu)$. One more use of the same device yields the following variant of Wiener's theorem for continuous measures μ in R:

(W) *There is a sequence* $1\leqq q_1\leqq q_2\leqq\ldots\leqq q_\nu\leqq\ldots$ *such that* $e(-pq_\nu t)\to 0$ *weak* in* $L^\infty(d\mu)$ *as* $\nu\to\infty$, *for each* $p=\pm 1, \pm 2, \pm 3, \ldots$. (Plainly we could add the condition $q_\nu=\nu+o(\nu)$, but asymptotic density 1 is used merely to find the sequence and has no further use.)

Following [3] and [5], we fix a continuous probability measure μ in E_1.

3. *Proof of* (a). Let now E_2 be a compact set of positive linear measure on the y axis; there exists a function $\varphi(z)$, analytic off E_2, such that $\varphi=z^{-1}+\ldots$ near ∞ and $|\varphi|\leqq C_1$ on $R^2\setminus E_2$. (The constant $C_1=4/m(E_2)$ was found by Pommerenke [8] and is always the minimum value; see also [4; pp. 28—30]. The value of C_1 has no significance in the sequel.)

By Fatou's theorem for the half-plane, φ admits one-sided limits a.e. on E_2; using Cauchy's formula and taking limits we obtain $\varphi(\zeta)=\int g(y)\,(iy-\zeta)^{-1}dy$, for all $\zeta\notin E_2$, where $g\in L^\infty$, $g=0$ off E_2.

Let now $H\in L^\infty(R)\cap C^1(R)$ and $\psi(\zeta)=\int g(y)H(y)(iy-\zeta)^{-1}dy$, $\zeta\notin E_2$. Then with $\zeta=\xi+i\eta$, we can write

$$\psi(\zeta)-H(\eta)\varphi(\zeta)=\int[H(\eta)-H(\zeta)]\,g(y)(iy-\zeta)^{-1}\,dy.$$

The last formula shows plainly that ψ can be estimated by means of $\|\varphi\|_\infty$, $\|H\|_\infty$, $\|H'\|_\infty$, and the measure of E_2.

4. *Proof of* (a), *completed.* Suppose now that the function g, the measure μ, the sequence (q_ν), and an integer $p\neq 0$ are held fixed. We form the sequence of functions

$$\psi_\nu(\zeta)=\int e(-pq_\nu x)\int e(py)g(y)(x+iy-\zeta)^{-1}\,dy\,\mu(dx).$$

For fixed $x\in E_1$, the inner integral is defined whenever $\xi\neq x$, and is $O(p)$, or indeed $O(1+\log|p|)$, hence $\psi_\nu(\zeta)$ is defined for all ζ, and is continuous in R^2, by the continuity of μ and dominated convergence. We claim that $\psi_\nu\to 0$ *uniformly* as $\nu\to+\infty$. To verify this claim we consider the integrals $\int e(py)g(y)(x+iy-\zeta)^{-1}dy$ as elements of $L^1(d\mu)$, parametrized by a complex number ζ. Dominated convergence shows that this collection of functions is *norm-compact* in $L^1(d\mu)$; since $e(-pq_\nu x)\to 0$ weak* in $L^\infty(d\mu)$ the uniform convergence follows.

It is now a simple matter to complete the proof. We suppose that E_2 has diameter $<1/2$, as we can without loss of generality. Beginning with

$$f_0(\zeta)=\int\varphi(\zeta-x)\mu(dx)=\iint g(y)(x+iy-\zeta)^{-1}\,dy\,\mu(dx)$$

we replace f_0 by an integral

$$f_1(\zeta)=\iint H_1(y-q_\nu x)g(y)(x+iy-\zeta)^{-1}\,dy\,\mu(dx)$$

where $H_1\in C^2(R)$, H_1 has period 1 and mean 1, and $H_1(t)=0$ outside the set $|t|\leqq 1/4$ (modulo 1). Hence

$$H_1(y-q_\nu x)=1+\sum{}'a_p e(py-pq_\nu x)$$

where $\sum|pa_p|<+\infty$. Thus

$$f_1(\zeta)-f_0(\zeta)=\sum{}'a_p\int e(-pq_\nu x)\int e(py)g(y)(x+iy-\zeta)^{-1}\,dy\,\mu(dx).$$

The previous analysis establishes that f_1 is continuous in the plane, and that $f_1\to f_0$ uniformly as $\nu\to\infty$. Moreover f_1 is analytic off the set defined by $x\in E_1$, $y\in E_2$ and the relation $|y-q_\nu x|\leqq 1/4$ (modulo 1). Since E_2 has diamter $<1/2$, each line $x=x_0$ meets the set of singularities in a set of diameter $<1/2$. We choose $\nu=\nu_1$ so that $|f_1-f_0|<1/2$, say, and then construct f_2 by inserting a factor $H_2(y-q_{\nu_2}x)$, etc. $H_2(t)=0$ outside the set $|t|<1/8$ (modulo 1), etc. The limit f is then continuous in R^2 and analytic off the support of each function $H_k(y-q_{\nu_k}x)$ and off $E_1\times E_2$. Hence the (closed) set of singularities is already a graph $\Gamma\subseteqq E_1\times E_2$. To ensure that f is not entire we have only to control the Taylor expansion of $f_1, f_2, \ldots$ at ∞. Now $f_0(\zeta)=\zeta^{-1}+\ldots$, and the functions f_k are analytic outside a fixed compact set; hence the first coefficient at ∞ can be controlled simply by writing it as an integral around a large circle. (If f were entire, it would be constant.)

As mentioned before, a better result is possible when E_2 is an interval: all the derivatives $f', f'', \ldots$ are uniformly continuous off Γ. To see this we choose $g(y)$ to be smooth, as well as $H_1, H_2, \ldots$, and estimate the partial derivatives $\partial/\partial y$, $\partial^2/\partial y^2$, ... at each step, using Leibniz' formula. We observe that if E_1 and E_2 have no interior, if f is analytic off $E_1\times E_2$, and f' is uniformly continuous there, then f is entire. (Thus the improvement just mentioned is not possible if E_2 has no interior; if E_1 has measure 0 and E_2 has no interior, then f' cannot even remain bounded unless f is entire.)

5. *Proof of* (b). Let $g(y)=\sin^2 2\pi y$, $0\leqq y\leqq 1/2$ and $g(y)=0$ otherwise. We construct a sequence of real-valued functions

$$u(x,y)=\int_{-\infty}^{x} g(y)\mu(dy)\equiv g(y)\mu(-\infty,x)$$

$$u_k(x,y)=\int_{-\infty}^{x} A_1(t,y)\ldots A_k(t,y)g(y)\mu(dt),$$

where each $A_k\geqq 0$ and $A_k\in C^2(R)$. Then of course $u_k(x,y)$ is continuous and increasing for each fixed y, $\partial u_k(x,y)/\partial y$ exists everywhere as a classical derivative and $\partial u_k/\partial x=0$ away from the support of the measure $A_1(x,y)\ldots A_k(x,y)\mu(dx)$. Also, $A_k(x,y)=H_k(y-q_\nu x)$ where $\nu=\nu_k$ is chosen as follows. Abbreviating $G_k=A_1\ldots A_k$

we have

$$u_{k+1}(x, y) = \int_{-\infty}^{x} H_{k+1}(y-q_v t) G_k(t, y) g(y) \mu(dt),$$

$$\partial(u_{k+1}-u_k)/\partial y = \int_{-\infty}^{x} H'_{k+1}(y-q_v t) G_k(t, y) g(y) \mu(dt)$$

$$+\int_{-\infty}^{x} [H_{k+1}(y-q_v t)-1](\partial[G_k(t, y) g(y)]/\partial y) \mu(dt).$$

Since G_k and $g(y)$ are at least C^1, the following estimation of the first term here will also be valid for the second one. We expand H'_{k+1} in a Fourier series; observing that the constant term $(p=0)$ is now absent. Hence everything is reduced to estimation of the integrals

$$2\pi ip \int_{-\infty}^{x} G_k(t, y) e(py-pq_v t) g(y) \mu(dt),$$

which are clearly $O(p)$. To proceed as in (a), we need a norm-compact subset in $L^1(d\mu)$; we define it as the set of all functions $G_k(t, y) I(t \leqq x)$ with $x+iy$ in R^2, I means characteristic function. With these adaptations, we obtain the uniform convergence of u_{k+1} and $\partial u_{k+1}/\partial y$. A further property of u_k is necessary and easily obtained: $u_k(+\infty, y)-u_k(-\infty, y) \geqq c > 0$ when $1/8 \leqq y \leqq 3/8$, with c independent of k; obviously u_0 has this property. We carry out an infinite sequence of approximations, observing that $|\partial u_0/\partial y| \leqq 2\pi < 7$, obtaining a real function u such that $|\partial u/\partial y| < 7$ everywhere and $\partial u/\partial x = 0$ away from Γ, for a certain graph $\Gamma \subseteqq E_1 \times [0, 1]$. Now $f(x, y) = u(x, y) + x + iy$ is a homeomorphism of R^2 onto itself with $f(\infty) = \infty$, f is of class C^1 off Γ and f is K-quasiconformal off E. We verify the latter point following a suggestion of the referee: off Γ we have $|f_x|^2 + |f_y|^2 = 1 + u_y^2 + 1 \leqq 51$ while the determinant $J = u_x + 1 = 1$. However f cannot be quasiconformal in the plane, because $f(\Gamma)$ has area at least $c/4$, as seen from the properties of u on horizontal lines.

By Lemma 3.1 of [8, p. 200], the curve Γ is not removable even when $K=1$, i.e. when the homeomorphism is complex analytic off Γ; this means that it is in general not even K-quasiconformal in the plane for any $K < +\infty$.

(In a subsequent note on exceptional sets we present a method for constructing conformal mappings that avoids the Beltrami equation but uses more algebra.)

6. The sets $E_1 \times E_2$ are not quite the most general that can be handled by Carleson's method and through which a curve Γ can be passed.

Suppose that $E \subset R^2$ is compact and the set $\{x \in R: m(E(x)) > 0\}$ is uncountable. (Here $E(x)$ is the section of E through x.) Each set $\{m(E(x)) \geqq c\}$ is closed, so that for some $c > 0$ that set carries a continuous probability measure μ. Let $h_n(x, y)$ be continuous in R^2, $0 \leqq h_n \leqq 1$, and $\lim h_n = \chi_E$ everywhere. Supposing that $\int\int |h_n(x, y) - \chi_E(x, y)| \mu(dx) dy < n^{-2}$, we can find a set $B \subseteqq R$ of positive μ-measure, such that $\int |h_n(x, y) - \chi_E(x, y)| dy \to 0$ uniformly for $x \in B$ as $n \to +\infty$. To each section $E(x)$, $x \in B$, there is a function φ, analytic off $E(x)$ and bounded by some c', while $\varphi(\zeta) = \zeta^{-1} + \dots$ near ∞. This can be obtained by an explicit construction, showing that φ depends continuously on x in an appropriate topology. This is a perfect substitute for the compactness used before and so our proof for (CA) can be effected.

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