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# ON THE LENGTH OF ASYMPTOTIC PATHS OF MEROMORPHIC FUNCTIONS OF ORDER ZERO

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## 1. Introduction

We use the usual notation of the Nevanlinna theory. We shall consider the following problem of Erdös (Clunie and Hayman [2, Problem 2.41]): Suppose that $f$ is an entire function of finite order and $\Gamma$ is a locally rectifiable path on which $f(z)\to\infty$. Let $l(r,\Gamma)$ be the length of $\Gamma$ in $|z|\leqq r$. Find a path for which $l(r,\Gamma)$ grows as slowly as possible and estimate $l(r,\Gamma)$ in terms of $M(r,f)$. If $f$ has zero order, or, more generally, finite order, can a path $\Gamma$ be found for which $l(r,\Gamma)=O(r)$ as $r\to\infty$?

First we shall consider the case when we may choose a ray for $\Gamma$. We denote by $U(a,r)$ the open disc $|z-a|<r$. Following Hayman [4], we say that the union of the open discs $U(a_n, r_n)$, $n=1,2,\ldots$, is an $\varepsilon$-set if $a_n\to\infty$ as $n\to\infty$ and the series $\sum r_n/|a_n|$ converges. We note that if $f(z)\to a$ as $z\to\infty$ outside an $\varepsilon$-set, then $a$ is a radial asymptotic value of $f$; in fact, $f(re^{i\theta})\to a$ as $r\to\infty$ for almost all $\theta$. Hayman [4] has shown that if $f$ is an entire function satisfying $T(r,f)=O((\log r)^2)$, then

$$\log|f(z)| = (1+o(1))T(|z|,f)$$

as $z\to\infty$ outside an $\varepsilon$-set. Anderson [1] proved that if $f$ is a meromorphic function such that $\delta(\infty,f)>0$ and

$$T(2r,f)-T(r,f) = o(T(r,f))(\log\log r)^{-1},$$

then

$$\liminf\frac{\log|f(z)|}{T(|z|,f)} \geqq \delta(\infty,f)$$

as $z\to\infty$ outside an $\varepsilon$-set. We shall prove the following theorem.

Theorem 1. *Suppose that $f$ is meromorphic in the finite complex plane $C$ and that $\delta(\infty,f)=\delta>0$. If there exists $m$, $0<m<\log 2$, such that*

$$T(2r,f) \leqq T(r,f)\left(1+\frac{m\delta}{\log\log r}\right) \tag{1}$$

*for all large values of $r$, then $f(z)\to\infty$ as $z\to\infty$ outside an $\varepsilon$-set.*

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The growth condition (1) with $m<\log 2$ is more or less the best possible. The question whether $m$ can be replaced by $\log 2$ in (1) remains open, but we shall show that $m$ cannot be larger than $\log 2$.

Theorem 2. *For any* $\delta$ *and* $m$, $0<\delta\leqq 1$, $m>\log 2$, *there exists a transcendental meromorphic function* $f$ *(*$f$ *entire if* $\delta=1$*) such that* $\delta(\infty, f)=\delta$, $f$ *has no radial asymptotic values, and*

$$T(2r, f) \leqq T(r, f)\left(1+\frac{m\delta}{\log\log r}\right) \tag{2}$$

*for all large values of* $r$.

On the other hand, Goldberg and Eremenko [3] and the author [7] have proved that if $\varphi(r)\to\infty$ as $r\to\infty$, then there exists an entire function $f$ satisfying

$$T(r, f) = O(\varphi(r)(\log r)^2) \tag{3}$$

such that if $\Gamma$ is any asymptotic path of $f$, then $l(r, \Gamma)\neq O(r)$. There arises the question whether an entire function $f$ satisfying (3) can be constructed such that

$$\liminf_{r\to\infty}\frac{l(r, \Gamma)}{r} > 1$$

for any asymptotic path $\Gamma$. We shall give a negative answer to this question.

Theorem 3. *If* $f$ *is an entire function satisfying*

$$T(r, f) = O((\log r)^M) \tag{4}$$

*for some* $M>0$, *then there exists an asymptotic path* $\Gamma$ *such that*

$$\liminf_{r\to\infty}\frac{l(r, \Gamma)}{r} = 1. \tag{5}$$

Furthermore, we prove the following two theorems.

Theorem 4. *Let* $f$ *be an entire function of order zero. Then there exists an asymptotic path* $\Gamma$ *such that*

$$l(r, \Gamma) = o(r^{1+\varepsilon}) \tag{6}$$

*for any* $\varepsilon>0$.

Theorem 5. *If* $f$ *is an entire function of order zero satisfying*

$$\liminf_{r\to\infty}\frac{n(r^{1+\varepsilon}, 0, f)}{T(r, f)} = 0 \tag{7}$$

*for some* $\varepsilon>0$, *then there exists an asymptotic path* $\Gamma$ *satisfying* (5).

## 2. Proof of Theorem 1

Let $f$ satisfy the hypotheses of Theorem 1. Following Anderson [1], we choose a finite $b$ such that

$$N(r, b) = T(r, f) + O((T(r, f))^{3/4}).$$

Then it follows from (1) that

$$\text{(i)} \qquad \begin{aligned} n(r, b)\log 2 &\leqq N(2r, b) - N(r, b) \\ &\leqq T(2r, f) - T(r, f) + O((T(r, f))^{3/4}) \\ &\leqq (1+o(1)) m\delta T(r, f)(\log\log r)^{-1}. \end{aligned}$$

Let $k$ be a positive integer. We choose $\varrho$ such that

$$\text{(ii)} \qquad \log\left(\frac{\varrho}{2^{k+3}}\right) = -\lambda \log\log 2^k,$$

where $\lambda = ((\log 2)/m)^{1/2} > 1$. From the proof of Theorem 2 of Anderson [1] it follows that for $2^k \leqq |z| < 2^{k+1}$

$$\text{(iii)} \qquad \log|f(z)-b| \geqq (\delta + o(1)) T(|z|, f) + n(2^{k+2}, b)\log(\varrho/2^{k+3})$$

outside a set of circles in $2^{k-1} \leqq |z| \leqq 2^{k+2}$, the sum of whose radii is at most $32\varrho$. Let $z$, $2^k \leqq |z| < 2^{k+1}$, lie outside these discs. From (i) and (ii) it follows that

$$n(2^{k+2}, b)\log(\varrho/2^{k+3}) \geqq -(1+o(1))\lambda^{-1}\delta T(2^{k+2}, f),$$

and since $T(8r, f) = (1+o(1)) T(r, f)$, we conclude from (iii) that

$$\log|f(z)| > (\delta(1-\lambda^{-1}) + o(1)) T(|z|, f) > (1+o(1)) \log|z|.$$

Thus $\log|f(z)| \geqq (1+o(1)) \log|z|$ outside a set of circles the sum of whose radii taken over circles meeting the set $2^k \leqq |z| \leqq 2^{k+1}$ is at most

$$O(2^k \exp(-\lambda \log\log 2^k)) = O(2^k k^{-\lambda}).$$

These circles subtend angles at the origin whose sum is $O(k^{-\lambda})$. Since $\lambda > 1$, the series $\sum k^{-\lambda}$ converges, and we conclude that $f(z) \to \infty$ as $z \to \infty$ outside an $\varepsilon$-set. This proves Theorem 1.

## 3. Proof of Theorem 2

Let $m > \log 2$ and $\delta$, $0 < \delta \leqq 1$, be given. Let $M > 10$ be a positive integer such that

$$\text{(i)} \qquad m(1-e^{-M})(1-2^{-M}) > \log 2.$$

We denote

$$\text{(ii)} \qquad \alpha = 2^M \delta^{-1}(1-e^{-M})(1-2^{-M})^2,$$

and for $k=1, 2, \ldots$, we set $r_k=k^{\alpha k}$ and $a_k=r_k e^{i\varphi_k}$, where the angles $\varphi_k$ will be specified later. Let $t_1=2^M$ and $t_k=2^{Mk}-2^{M(k-1)}$ for $k\geqq 2$.

We shall consider the function

$$f_1(z) = \Pi_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)^{t_k}.$$

The sequence $t_k$ is chosen such that $n(r, 0, f_1)=2^{Mk}$ for $r_k\leqq r<r_{k+1}$, and we see from the choice of $r_k$ that

$$N(r_{k+1}, 0, f_1)-N(r_k, 0, f_1) = 2^{Mk}\log(r_{k+1}/r_k) = (1+o(1))2^{Mk}\alpha\log k.$$

This implies that

(iii) $$N(r_{k+1}, 0, f_1) = (1+o(1))2^{Mk}\alpha\log k(1+2^{-M}+2^{-2M}+\ldots) = (1+o(1))\alpha 2^{Mk}(1-2^{-M})^{-1}\log k.$$

Now we see that $n(r, 0, f_1)=o(N(r, 0, f_1))$ and therefore $N(2r, 0, f_1)=(1+o(1))\cdot N(r, 0, f_1)$. Using Lemma 1 of Anderson [1] we conclude that

$$\log M(r, f_1) = (1+o(1))N(r, 0, f_1) = (1+o(1))T(r, f_1).$$

Then $f$ satisfies the condition

$$\log M(2r, f_1) = (1+o(1))\log M(r, f_1),$$

and it follows from Theorem 2 of Anderson [1] that

$$\log|f_1(z)| = (1+o(1))\log M(r, f_1) = (1+o(1))N(r, 0, f_1)$$

$(r=|z|)$ outside the union of the discs $U(a_n, r_n/4)$.

If $\delta=1$, we set $f_2(z)\equiv 1$, and if $0<\delta<1$, then

$$f_2(z) = \Pi_{k=1}^{\infty}\left(1+\frac{z}{a_k}\right)^{s_k},$$

where the sequence $s_k$ of positive integers is chosen such that

$$|n(r_k, 0, f_2)-(1-\delta)2^{Mk}| \leqq \tfrac{1}{2}$$

for any $k$. Then

$$|n(r, 0, f_2)-(1-\delta)n(r, 0, f_1)| \leqq \tfrac{1}{2}$$

and, as above, we see that

$$\log|f_2(z)| = (1+o(1))(1-\delta)N(|z|, 0, f_1)$$

outside the union of the discs $U(-a_n, r_n/4)$.

We set $f(z)=f_1(z)/f_2(z)$. Then $f$ is entire if $\delta=1$, and for any $\delta$, $0<\delta\leqq 1$, $f$ satisfies

(iv) $$\log|f(z)| \geqq (1+o(1))\delta N(|z|, 0, f)$$

outside the union of the discs $U(a_n, r_n/4)$, and on the boundary of these discs

(v) $$\log|f(z)| = (1+o(1))\delta N(|z|, 0, f).$$

We choose a finite $b$ such that

(vi) $$N(r, b, f) = T(r, f)+O((T(r, f))^{3/4}).$$

Using Rouche's theorem, we see from (iv) that for all large values of $k$,

$$n(r, b, f) = n(r, 0, f) = 2^{M(k-1)}$$

if $(5/4)r_{k-1} \leqq r \leqq (3/4)r_k$, and

$$2^{M(k-1)} \leqq n(r, b, f) \leqq 2^{Mk}$$

if $(3/4)r_k \leqq r \leqq (3/4)r_{k+1}$. Therefore we see from (iii) that if $(5/4)r_{k-1} \leqq r < (5/4)r_k$, then

$$N(2r, b, f) - N(r, b, f) \leqq n(2r, b, f)\log 2$$

$$\leqq (1+o(1))N(r, 0, f)\frac{2^M(1-2^{-M})\log 2}{\alpha\log k}.$$

This implies together with (vi) and (ii) that

$$T(2r, f) - T(r, f) \leqq (1+o(1))T(r, f)\frac{\delta\log 2}{(1-e^{-M})(1-2^{-M})\log k},$$

and we see from (i) that $f$ satisfies the condition (2) for all large values of $r$.

Comparing the growth of $N(r, b, f)$ and $N(r, 0, f)$ we see easily that

$$N(r, b, f) = (1+o(1))N(r, 0, f).$$

Then we have $N(r, 0, f) = (1+o(1))T(r, f)$, and since

$$N(r, \infty, f) = N(r, 0, f_2) = (1+o(1))N(r, 0, f)(1-\delta),$$

we conclude that $\delta(\infty, f) = \delta$.

The function $h_n(z) = f(z)(1-z/a_n)^{-t_n}$ is analytic in $|z-a_n| \leqq r_n/4$, and we see from (v) that

(vii) $$\log|h_n(z)| \leqq (1+o(1))\delta N(r_n, 0, f)+t_n\log 4 \leqq (1+o(1))\delta N(r_n, 0, f)$$

on the boundary of $U(a_n, r_n/4)$. Then it follows from the maximum principle that $h_n$ satisfies (vii) in $U(a_n, r_n/4)$. We define $\varrho_n$ by the equation

(viii) $$\log(r_n/\varrho_n) = (1-e^{-2M})\log n.$$

We see from (vii) that if $z$ lies on the boundary of $U(a_n, \varrho_n)$, then

$$\log|f(z)| = \log|h_n(z)| - t_n\log(r_n/\varrho_n)$$

$$\leqq (1+o(1))\delta N(r_n, 0, f) - t_n\log(r_n/\varrho_n).$$

This implies together with (iii) and (viii) that

$$\log|f(z)| \leqq -(1+o(1))(e^{-M}-e^{-2M})2^{Mn}(1-2^{-M})\log n \leqq -(1+o(1))\log n$$

in $U(a_n, \varrho_n)$, and we see that if $z$ tends to infinity through the union of the discs $U(a_n, \varrho_n)$, then $f(z)$ tends to zero.

We assume now that the angles $\varphi_k$ are chosen such that $\varphi_1=0$ and $\varphi_{k+1}=\varphi_k+\varrho_k/r_k$ for $k\geqq 1$. If $\varphi_k\leqq\varphi\leqq\varphi_{k+1}$ and $|z_0|<\varrho_k/8$, then the ray $z=z_0+re^{i\varphi}$ meets at most one of the discs $U(a_k, \varrho_k)$ and $U(a_{k+1}, \varrho_{k+1})$. It follows from (viii) that $\varrho_n\to\infty$ as $n\to\infty$ and that the series $\sum\varrho_n/r_n$ diverges. Therefore any ray $z=z_0+re^{i\varphi}$ meets infinitely many of the discs $U(a_n, \varrho_n)$ and so

$$\liminf_{r\to\infty}|f(z_0+re^{i\varphi})| = 0$$

for any fixed complex $z_0$ and real $\varphi$. On the other hand, it follows from (iv) that

$$\limsup_{r\to\infty}|f(z_0+re^{i\varphi})| = \infty$$

for any fixed $z_0$ and $\varphi$, and we conclude that $f$ has no radial asymptotic values. This completes the proof of Theorem 2.

## 4. Proof of Theorem 4

Let $f$ be an entire function of order zero. We may suppose that $f$ has no radial asymptotic values because otherwise we could choose a ray for the desired path $\Gamma$. We choose a continuous path $\gamma$: $[0, 1)\to C$ such that $\gamma(0)=0$, $\gamma(t)\to\infty$ as $t\to 1$ and

(i) $$\log|f(z)| \geqq 3\log|z|$$

on $\gamma$ for all large values of $|z|$. We denote

$$B = \{z\in C\colon \log|f(z)| \leqq \log|z|\}.$$

Using (i), we choose $t_0$, $0<t_0<1$, such that

(ii) $$\log|f(\gamma(t))| \geqq 3\log|\gamma(t)| > 9$$

for $t\geqq t_0$. We choose $\varrho_0>0$ such that $U(\gamma(t_0), \varrho_0)$ is contained in the complement of $B$ and that the circle $|z-\gamma(t_0)|=\varrho_0$ contains at least one point of $B$. Inductively, if $t_{k-1}$ and $\varrho_{k-1}$ $(k\geqq 1)$ are determined, we choose $t_k$ to be the greatest value of $t$ such that the open disc

$$U(\gamma(t_k), |\gamma(t_k)-\gamma(t_{k-1})|-\varrho_{k-1})$$

does not contain any point of $B$ and that the boundary of this disc contains at least one point of $B$. The radius of this disc is denoted by $\varrho_n$ and, for the sake of simplicity, we write $C_k=U(\gamma(t_k), \varrho_k)$. Since $\infty$ is not a radial asymptotic value of $f$, we see that our process gives a denumerable collection of discs $C_k$. From the continuity of $f$ we conclude that the points $\gamma(t_k)$ cannot have any finite point $z$ as a limit point. Then $\gamma(t_k)\to\infty$ as $k\to\infty$ and, using again the fact that $\infty$ is not a radial asymptotic value of $f$, we note that

(iii) $$\lim_{k\to\infty} (|\gamma(t_k)| - \varrho_k) = \infty.$$

The open discs $C_k$ are mutually disjoint and the boundary circles of $C_k$ and $C_{k+1}$ have exactly one point in common. Since all the discs $C_k$ are contained in the complement of $B$, we deduce now that $\log|f(z)| \geqq \log|z|$ on that segment which joins the points $\gamma(t_k)$ and $\gamma(t_{k+1})$. Let $\Gamma$ be the path consisting of these segments. It follows from (iii) that $\Gamma$ is a path going from $\gamma(t_0)$ to $\infty$, and since $\log|f(z)| \geqq \log|z|$ on $\Gamma$, we deduce that $f(z) \to \infty$, as $z \to \infty$ along $\Gamma$.

We denote by $a_n$, $n=1, 2, \ldots$, the zeros of $f$, and for any finite $z$ we set $\omega(z) = \min\{|z-a_n|: n=1, 2, \ldots\}$. Let $r>4$ and $|z|<4r$. Then the logarithmic derivative of $f$ satisfies

$$\left|\frac{f'(z)}{f(z)}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{z-a_n}\right| \leqq \omega(z)^{-1} n(r^4, 0) + 2\sum_{|a_n|>r^4} |a_n|^{-1}.$$

Since $f$ is of order zero,

$$\sum_{|a_n|>r^4}^{\infty} |a_n|^{-1} \leqq \sum_{k=4}^{\infty} r^{-k} n(r^{k+1}, 0) = o(r^{-2}),$$

and we deduce that

(iv) $$\left|\frac{f'(z)}{f(z)}\right| \leqq \omega(z)^{-1} n(r^4, 0) + o(r^{-2})$$

in $|z| \leqq 4r$.

We denote by $l(A)$ the length measure of $A$ if $A$ is a set consisting of a finite number of rectifiable arcs. For $k \geqq 1$, the set $\Gamma \cap C_k$ consists of two radii of $C_k$. These radii are denoted by $\alpha_k$ and $\beta_k$; then clearly $l(\alpha_k) = l(\beta_k) = \varrho_k$ and $l(\Gamma \cap C_k) = 2\varrho_k$. We denote

$$\Gamma_r = \Gamma \cap \{z: r \leqq |z| \leqq 2r\}.$$

If $\Gamma_r \cap \beta_k \neq \emptyset$ and $\varrho_k \geqq r/4$, we choose an open disc $D_k$ with radius $d_k \geqq r/8$ such that

$$\Gamma_r \cap \beta_k \subset D_k \subset C_k \cap U(0, 3r).$$

Then $l(\Gamma_r \cap \beta_k) \leqq 2d_k$ and the area $\pi d_k^2$ of $D_k$ satisfies the inequality

(v) $$16\pi d_k^2 \geqq \pi r l(\Gamma_r \cap \beta_k).$$

The discs $D_k$ are mutually disjoint and all of them are contained in $U(0, 3r)$. Therefore the sum of the areas of $D_k$ is at most $9\pi r^2$, and we deduce from (v) that

$$\sum_{\beta_k \geqq r/4} l(\Gamma_r \cap \beta_k) \leqq 144r = O(r).$$

In the same manner, we get the estimate

$$\sum_{\alpha_k \geqq r/4} l(\Gamma_r \cap \alpha_k) = O(r)$$

and conclude that

(vi) $$l(\Gamma_r \cap \bigcup_{\varrho_k \geqq r/4} C_k) = O(r).$$

Let $C_p$ contain at least one point of $r \leqq |z| \leqq 2r$ and let the radius $\varrho_p$ satisfy

(vii) $$r/2^{k+2} \leqq \varrho_p < r/2^{k+1}$$

for some positive integer $k$. Then we have $l(\Gamma_r \cap C_p) \leqq l(\Gamma \cap C_p) = 2\varrho_p < r/2^k$ and

(viii) $$\varrho_p^2 \geqq l(\Gamma_r \cap C_p) r/2^{k+3}.$$

On the boundary of $C_p$ there exists a point $b$ such that $\log |f(b)| = \log |b|$. Let $J$ be the segment joining $b$ and the centre $\gamma(t_p)$ of $C_p$. Then it follows from (ii) and (iv) that

$$\log r \leqq 3 \log |\gamma(t_p)| - \log |b| \leqq \log |f(\gamma(t_p))| - \log |f(b)|$$

$$\leqq \left| \int_J \frac{f'(z)}{f(z)} dz \right| \leqq \varrho_p \left( \frac{n(r^4, 0)}{\omega(\gamma(t_p)) - \varrho_p} + o(r^{-2}) \right).$$

For large values of $r$ this implies that $\omega(\gamma(t_p)) \leqq (1/8) \varrho_p n(r^4, 0)$, and we conclude from (vii) that there exists $n$, $1 \leqq n \leqq n(r^4, 0)$, such that

$$C_p \subset U(a_n, n(r^4, 0) r/2^{k+1}).$$

Since the discs $C_p$ are mutually disjoint, we see by comparing the areas from (viii) that

$$\frac{r}{2^{k+3}} \sum l(\Gamma_r \cap C_p) \leqq n(r^4, 0) \left( \frac{n(r^4, 0) r}{2^{k+1}} \right)^2,$$

where the sum is taken over those $p$ which satisfy (vii). This implies that

(ix) $$l(\Gamma_r \cap \bigcup_{\varrho_p < 1/4} C_p) \leqq 4r (n(r^4, 0))^3 \sum_{k=1}^{\infty} 2^{-k-1}$$

for all large values of $r$.

Let $\varepsilon > 0$ be given. We choose $\alpha$ such that $1 < \alpha < 1 + \varepsilon$. Since $f$ is of order zero, we see from (vi) and (ix) that $l(\Gamma_r) = o(r^\alpha)$. We choose $r_0$ such that $l(\Gamma_r) < r^\alpha$ for $r \geqq r_0$. We get for $r > r_0$

$$l(r, \Gamma) \leqq l(2r_0, \Gamma) + \sum_{k=0}^{\infty} \left( \frac{r}{2^k} \right)^\alpha \leqq l(2r_0, \Gamma) + 2r^\alpha,$$

and therefore we have $l(r, \Gamma) = o(r^{1+\varepsilon})$. This completes the proof of Theorem 4.

## 5. Proof of Theorems 5 and 3

Let $f$ be an entire function of order zero, $\varepsilon > 0$, and let there exist a sequence $r_n$ such that $\lim r_n = \infty$ and $n(r_n^{1+\varepsilon}, 0) = o(T(r_n, f))$. We choose $\alpha$ by the equality $\alpha^5 = 1 + \varepsilon$. The method used by Anderson [1] in the proofs of Theorems 1 and 2 is directly applicable in the ring domains $r_n^\alpha \leqq |z| \leqq r_n^{\alpha^4}$, and we may conclude that there exist circles $C_n$: $|z| = \varrho_n$, $r_n^\alpha \leqq \varrho_n \leqq 2r_n^\alpha$, $C_n'$: $|z| = R_n$, $r_n^{\alpha^3} \leqq R_n \leqq 2r_n^{\alpha^3}$, and a path $\gamma_n$ joining the circles $C_n$ and $C_n'$ with $l(\gamma_n) = (1 + o(1)) R_n$ such that $\log |f(z)| > (1/2 + o(1)) T(|z|, f)$ on $C_n' \cup C_n \cup \gamma_n$. From Theorem 4 it follows that there exists an asymptotic path $\Gamma_0$ such that $l(r, \Gamma_0)=o(r^\alpha)$. Using $C_n$, $C_n'$ and $\gamma_n$ we may modify $\Gamma_0$ into a new asymptotic path $\Gamma$ such that

$$l(R_n-1, \Gamma) \leqq l(\Gamma_0, \varrho_n)+2\pi\varrho_n+l(\gamma_n) = o\big((2r_n^\alpha)^\alpha\big)+O(r_n^\alpha)+\big(1+o(1)\big)R_n.$$

Since $R_n \geqq r_n^{\alpha^3}$, we conclude now that $l(R_n-1, \Gamma)=\big(1+o(1)\big)(R_n-1)$. This proves Theorem 5.

Let us suppose that $f$ is an entire function satisfying (4). We choose $\delta>0$ such that $T(r,f)=O\big((\log r)^{\delta+1/2}\big)$ and $T(r,f)\neq O\big((\log r)^\delta\big)$. Then $n(r,0,f)=O\big((\log r)^{\delta-1/2}\big)$, and there exist arbitrarily large values of $r$ such that $T(r,f)>(\log r)^\delta$. For these values of $r$ we have

$$\frac{n(r^2, 0, f)}{T(r, f)} \leqq \frac{O\big((\log r)^{\delta-1/2}\big)}{(\log r)^\delta} = O\big((\log r)^{-1/2}\big) = o(1),$$

and Theorem 3 follows from Theorem 5.

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