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ON A RESULT OF WINKLER

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1. Introduction and results

I thank Professor O. Lehto for suggesting this subject to me.

We shall use the usual notations of the Nevanlinna theory.

Let a_k be a sequence of non-zero complex numbers such that $a_k \to \infty$ as $k \to \infty$. We denote by $n(r)$ the number of points a_k satisfying $|a_k| \leqq r$. It is well known that there exists an entire function of the form

$$F(z) = \prod_{k=1}^{\infty} E_{p_k}(z/a_k), \tag{1.1}$$

where

$$E_{p_k}(u) = (1-u)\exp\left(u+\left(\tfrac{1}{2}\right)u^2+\ldots+(1/p_k)u^{p_k}\right),$$

such that F has exactly the zeros a_k.

Let $[x]$ be the integer part of a non-negative real number x. Winkler [2] proved the following theorem.

Theorem A. *Let a_k be as above and suppose that $\sigma > 1$. Then the entire function F of the form* (1.1) *with*

$$p_k = [\log n^{\sigma}(|a_k|)]$$

satisfies

$$\log M(r, F) = O\left(N(\gamma r, 0, F)^{\sigma(1+\log r)}\right) \quad (r \to \infty) \tag{1.2}$$

for any $\gamma > \exp(\sqrt{1/\sigma})$.

There arise the following two questions. Does the rapid growth of $n(r)$ imply that we must take a rapidly growing sequence p_k in (1.1), and does the rapid growth of $n(r)$ imply that, for any entire function F of the form (1.1), $\log M(r,f)$ is essentially larger than $N(r,0,f)$?

We shall give a negative answer to both of these questions. We prove the following

Theorem. *Given any increasing function $\varphi(r)$ such that $\varphi(r) \to \infty$ as $r \to \infty$, there exists a sequence a_k, $a_k \to \infty$ as $k \to \infty$, such that the product*

$$\prod_{k=1}^{\infty}(1-z/a_k)$$

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converges uniformly on bounded subsets of the complex plane and that the entire function

$$f(z) = \prod_{k=1}^{\infty} (1 - z/a_k)$$

satisfies

$$\varphi(r) = O\big(n(r, 0, f)\big) \quad (r \to \infty) \tag{1.3}$$

and

$$\log M(r, f) = N(r, 0, f) + O(1) \quad (r \to \infty). \tag{1.4}$$

2. Proof of the Theorem

Let z_n be the following sequence constructed by Erdös [1, Problem 4.1]: Let $z_1 = 1$, $z_2 = -1$, and if z_n has already been defined for $1 \leqq p \leqq 2^k$, then we define for $1 \leqq p \leqq 2^k$,

$$z_{p+2^k} = z_p \exp(2^{-k} \pi i).$$

Lemma. *Let z_n be as above, $0 < d < 1/8$, and suppose that $s \geqq 1$ is an integer. Then*

$$\left|\log \prod_{n=1}^{s} (1 - z/z_n)\right| \leqq 4d \tag{2.1}$$

on $|z| \leqq d$, where $\log w$ is chosen so that $\log 1 = 0$.

Proof. It follows from the choice of z_n that for any integers $p \geqq 0$ and $k \geqq 1$ there exists a real φ such that

$$\prod_{n=p2^k+1}^{(p+1)2^k} (1 - z/z_n) = 1 - (ze^{i\varphi})^{2^k} \tag{2.2}$$

and that $\varphi = 0$ if $p = 0$.

Let k be chosen so that $2^k \leqq s < 2^{k+1}$. Then

$$s = \sum_{p=0}^{k} t_p 2^{k-p},$$

where $t_p(1 - t_p) = 0$ for any p, and we deduce from (2.2) that if $|z| \leqq d$, then

$$\left|\log \prod_{n=1}^{s} (1 - z/z_n)\right| \leqq 2 \sum_{p=0}^{k} t_p d^{2^{k-p}} \leqq 2 \sum_{p=1}^{\infty} d^p < 4d,$$

which proves the Lemma.

Proof of the Theorem. Let $\varphi(r)$ be as in the Theorem. We choose a positive integer k_1 such that

$$n_1 = 2^{k_1} > \varphi(16)$$

and set

$$a_n = 4z_n \quad \text{for} \quad n = 1, 2, \ldots, n_1,$$

and if a_n has already been defined for $n=1, \ldots, n_{p-1}$, we choose a positive integer k_p such that

$$(2.3) \qquad n_p = n_{p-1}+2^{k_p} > \varphi(4^{p+1})$$

and set

$$a_n = 4^p z_{n-n_{p-1}} \quad \text{for} \quad n = n_{p-1}+1, \ldots, n_p.$$

Let

$$f_s(z) = \Pi_{p=1}^{s}(1-(4^{-p}z)^{2^{k_p}})$$

and

$$f(z) = \lim_{s\to\infty} f_s(z).$$

Clearly $f_s(z)\to f(z)$ uniformly on bounded subsets of the complex plane. From (2.3) it follows that f satisfies (1.3).

We define

$$f_s(z) = \Pi_{n=1}^{n_s}(1-z/a_n)$$

for any s, and we deduce from the Lemma that if $|z|\leqq M$ and $n_s<t\leqq n_{s+1}$, then

$$|\log((1/f(z))\,\Pi_{n=1}^{t}(1-z/a_n))| \leqq |\log(f_s(z)/f(z))|+|\log \Pi_{n=n_s+1}^{t}(1-z/a_n)|$$

$$\leqq o(1)+O(M/4^s) = o(1) \quad (t\to\infty),$$

which implies that

$$\Pi_{n=1}^{t}(1-z/a_n) \to f(z)$$

uniformly on bounded subsets of the complex plane.

Suppose that $4^s/2\leqq r<4^{s+1}/2$. We get

$$\log M(r,f)-N(r,0,f) \leqq \sum_{q=1}^{\infty}\log(1+(r/4^q)^{2^{k_q}})-\sum_{q=1}^{s}2^{k_q}\log^+(r/4^q)$$

$$\leqq \sum_{q=1}^{s-1}\log(1+(4^q/r)^{2^{k_q}})+\log(1+(r/4^s)^{2^{k_s}})-2^{k_s}\log^+(r/4^s)+O(1)$$

$$= O(1)+\min\{\log(1+(r/4^s)^{2^{k_s}}),\ \log(1+(4^s/r)^{2^{k_s}})\} = O(1) \quad (r\to\infty).$$

This implies that

$$(2.4) \qquad \log M(r,f) \leqq N(r,0,f)+O(1) \quad (r\to\infty).$$

From the first main theorem of the Nevanlinna theory we deduce that

$$N(r,0,f) \leqq T(r,f)+O(1) \leqq \log M(r,f)+O(1) \quad (r\to\infty),$$

which together with (2.4) proves (1.4). The Theorem is proved.

References

[1] HAYMAN, W. K.: Research problems in function theory. - The Athlone Press, University of London, 1967.

[2] WINKLER, J.: Über minimale Maximalbeträge kanonischer Weierstrassprodukte unendlicher Ordnung. - Resultate Math. 4, 1980, 102—116.

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