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A PROBLEM OF BURKHOLDER AND THE EXISTENCE OF HARMONIC MAJORANTS OF $|x|^p$ IN CERTAIN DOMAINS IN R^d

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1. Introduction and results

Let R^d (where $d\geqq 2$) denote a d-dimensional Euclidean space with $x=(x_1, x_2, \ldots$ $\ldots, x_d)$, $|x|=(\sum_{k=1}^d x_k^2)^{1/2}$ and $x_1=|x|\cos\varphi$, $0\leqq\varphi\leqq\pi$. Let $\delta>0$ be given and consider the "approximately conical" domain

$$T=\{x\colon |x|>1, \quad 0\leqq\varphi<\arctan(\delta\log|x|)\}.$$

For what values of δ does $|x|$ have a harmonic majorant in T?

When $d=2$, there is an equivalent question in terms of H^p-theory (cf. [3, p. 28]): let F_δ be a univalent analytic function mapping the unit disc U onto T. For what values of δ is F_δ in $H^1(U)$? This is an example of Burkholder [2, pp. 115—116], who showed that

(1) $$F_\delta\in H^1, \quad \delta<2/\pi,$$

(2) $$F_\delta\notin H^1, \quad \delta>2/\pi,$$

the result (1) by exhibiting an explicit harmonic majorant and (2) by using his "generalized subordination". In fact, (1) and (2) as well as

(2a) $$F_\delta\notin H^1, \quad \delta\geqq 2/\pi$$

follow easily from well-known estimates of harmonic measures (cf. [7], and [4]).

The purpose of this paper is to study a corresponding problem in R^d. We need some known results on harmonic majorization in R^d, $d\geqq 2$. Assuming that a given point x_0 belongs to the unbounded domain D, let D_r denote the component of $D\cap\{x\colon |x|<r\}$ which contains x_0. Consider the harmonic measures with the following boundary values:

$$\omega_r(x)=\begin{cases}1 & \text{on } \partial D_r\cap\{x\colon |x|=r\}\cap D,\\ 0 & \text{on the rest of } \partial D_r,\end{cases}$$

$$v_r(x)=\begin{cases}1 & \text{on } \partial D\cap\{x\colon |x|>r\},\\ 0 & \text{on } \partial D\cap\{x\colon |x|\leqq r\}.\end{cases}$$

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It is clear that if $|x|^p$, $0<p<\infty$, has a harmonic majorant in D then

(3) $$\int^{\infty} v_r(x_0)r^{p-1}\,dr < \infty$$

and that if

(4) $$\int^{\infty} \omega_r(x_0)r^{p-1}\,dr < \infty$$

then $|x|^p$ has a harmonic majorant in D. Burkholder's results on Brownian motion imply that (3)$\Leftrightarrow$(4), see [1, Theorem 2.2, p. 189, Theorem 3.1, p. 191], provided that the complement of D is not thin at ∞.

Thus, to find an answer to our problem, it is sufficient to study the harmonic measure ω_r. We can use a method of Carleman and give an analogue of (1) for $d\geqq 3$ (for this we refer to [8]). To prove an analogue of (2a) we have to prove that, for remaining values of δ, the integral in (4) will be divergent. For this we need estimates from below of ω_r, which for the plane case follow from some version of Ahlfors' "Second distortion inequality", (see [6, pp. 3—4]). However, thanks to the nice geometry of T, we can prove the necessary estimate of ω_r in R^d also for $d\geqq 3$. In fact, T is "approximately conical" and we can work with harmonic minorants in successive inscribed cones. This is analogous to an "approximately cylindrical" case, treated in [6, pp. 24—26], where the derivative of the "Nevanlinna mean" of a harmonic measure, weighted with a first eigenfunction, is considered.

To state the results we use the following notation. Let, for $\delta>0$,

(5) $$T = T_{\delta,d} = \{x\in R_d\colon\ |x|>1,\quad 0\leqq\varphi<\varphi(|x|)\},$$

where

(6) $$\varphi(r) = \tfrac{1}{2}\pi-\delta^{-1}(\log r)^{-1}+O\left((\log r)^{-2}\right),\quad r\to\infty,$$

(6a) $$\varphi'(r) = \delta^{-1}(\log r)^{-2}r^{-1}+O\left((\log r)^{-3}r^{-1}\right),\quad r\to\infty.$$

We assume that φ is increasing and twice continuously differentiable and satisfies the following convexity condition:

(6b) $\psi(r)=\varphi(r)+\arctan\left(r\varphi'(r)\right)$ is increasing, $1<r<\infty$, $\psi'(r)\approx\varphi'(r)$, $r\to\infty$.

To explain this condition geometrically, consider the curve $\Gamma=\{(r,\varphi(r)),\ r>1\}$ in polar coordinates in the plane. Then the slope of the tangent of Γ at $(r,\varphi(r))$ is an increasing function of r. The boundary of T is obtained by rotating Γ around the x_1-axis (for $d=2$ by reflecting Γ to the x_1-axis).

On the spherical cap $\{x\colon |x|=r,\ 0\leqq\varphi<\varphi_0\}$, where $0<\varphi_0<\pi$ is given, consider the Laplace—Beltrami equation and the corresponding *characteristic constant* α_0. Namely let λ_0 be the first eigenvalue of

(7) $$\begin{cases}\dfrac{d}{d\varphi}\left((\sin\varphi)^{d-2}f'(\varphi)\right)+\lambda_0(\sin\varphi)^{d-2}f(\varphi)=0,\\ f(0)=1,\quad f'(0)=0,\quad f(\varphi_0)=0,\quad f(\varphi)>0\quad\text{for}\quad 0\leqq\varphi<\varphi_0.\end{cases}$$

Then α_0 satisfies

(8) $$\alpha_0(\alpha_0+d-2) = \lambda_0,\ \alpha_0 > 0.$$

In particular

$$\lambda_0 = d-1, \quad \alpha_0 = 1, \quad \text{for} \quad \varphi_0 = \tfrac{1}{2}\pi;$$

the corresponding first eigenfunction $f(\varphi)$ is $\cos\varphi$ for $\varphi_0=\pi/2$. From [5, (3.3), Theorem 2, Theorem 3 p. 140 and $\alpha(S)=1/(2S)$ p. 153] it follows that

(9) $$\alpha_0' = \frac{d\alpha_0}{d\varphi_0}(\varphi_0) = -2\sigma_{d-1}/\sigma_d \quad \text{for} \quad \varphi_0 = \frac{1}{2}\pi,$$

where σ_d is the area of the unit sphere in R_d, i.e. $\sigma_d=2\pi^{d/2}/\Gamma(d/2)$, for $d\geqq 2$, and $\sigma_1=2$.

Let $\omega_r(x)$ denote the harmonic measure of $T_{\delta,d}\cap\{x\colon |x|=r\}$ with respect to $T_{\delta,d}\cap\{x\colon |x|<r\}$ and let $\omega_r(\varrho)=\omega_r((\varrho, 0, \ldots, 0))$.

Theorem 1. *Let $\gamma>1$ be given and let ω_r be defined as above. Then*

(10) $$\omega_r(\varrho) \geqq C(\varrho/r)\left((\log\varrho)/\log r\right)^{|\alpha_0'|/\delta}, \quad r > \varrho \geqq \gamma > 1,$$

where $|\alpha_0'|=2\sigma_{d-1}/\sigma_d$ and the constant C depends only on γ and constants of $T_{\delta,d}$.

Corollary 1. *There exists a harmonic majorant of $|x|$ in $T_{\delta,d}$ if and only if*

$$\delta < 2\sigma_{d-1}/\sigma_d.$$

Proof of Corollary 1. If $\delta\geqq 2\sigma_{d-1}/\sigma_d=|\alpha_0'|$, it follows from (10) that (4) and thus also (3) do not hold. Consequently, there is no harmonic majorant. The "if" part of the corollary follows from an inequality opposite to (10) proved by a method of Carleman as in [8].

For simplicity, we have first restricted ourselves to the case when $T_{\delta,d}$ is almost a half-space in R^d. Similar results are true for circular cones. Let $\varphi_0\in]0,\pi[$ be given and let $T_{\delta,d}(\varphi_0)$ be defined as before, but with

$$\varphi(r) = \varphi_0-\delta^{-1}(\log r)^{-1}+O\left((\log r)^{-2}\right), \quad r\to\infty.$$

Let $\omega_r(x)$ denote the harmonic measure of $T_{\delta,d}(\varphi_0)\cap\{x\colon |x|=r\}$ with respect to $T_{\delta,d}(\varphi_0)\cap\{x\colon |x|<r\}$ and let $\omega_r(\varrho)=\omega_r((\varrho, 0, \ldots, 0))$.

Theorem 2. *Let $\gamma>1$ be given and let ω_r be defined as above. Then, with α_0 given by* (8) *and $\alpha_0'=(d\alpha_0/d\varphi_0)(\varphi_0)$,*

(11) $$\omega_r(\varrho) \geqq C(\varrho/r)^{\alpha_0}(\log\varrho/\log r)^{|\alpha_0'|/\delta}, \quad r > \varrho \geqq \gamma > 1,$$

where the constant C depends only on γ and constants of $T_{\delta,d}(\varphi_0)$.

Corollary 2. *Let $\varphi_0\in]0,\pi[$ be given. Then $|x|^{\alpha_0}$ has a harmonic majorant in $T_{\delta,d}(\varphi_0)$ if and only if $\delta<|\alpha_0'|$.*

Remark 1. It follows from standard results on the Legendre equation that $\alpha_0'<0$. In general, explicit expressions for α_0 and α_0' are not available; however

$$\alpha_0 = \tfrac{1}{2}\pi\varphi_0^{-1} \quad \text{for} \quad d = 2,$$

$$\alpha_0 = \pi\varphi_0^{-1}-1 \quad \text{for} \quad d = 4,$$

$$\alpha_0 = 1, \quad \varphi_0 = \tfrac{1}{2}\pi, \quad d = 2, 3, \ldots,$$

$$\alpha_0' = -2\sigma_{d-1}/\sigma_d, \quad \varphi_0 = \tfrac{1}{2}\pi, \quad d = 2, 3, \ldots, \tag{9}$$

$$\sigma_d = 2\pi^{d/2}/\Gamma(d/2), \quad d \geqq 2; \quad \sigma_1 = 2.$$

Remark 2. In the plane case Corollary 2 can be proved from Corollary 1 with conformal mapping (cf. [7]) and φ_0 can equal π.

Remark 3. Estimates from above and below of the least harmonic majorant $u(x)$ of $|x|^p$ in $T_{\delta,d}(\varphi_0)$ are given in [8, Example 1]. In fact, $u(x)\leqq C|x|^p \log |x|$ for sufficiently large x and this order of growth can be attained.

Remark 4. We have considered the most interesting, "logarithmic" case of "approximate cones" $\{x\colon 0\leqq\varphi<\varphi(r)\}$. The method of proof can be applied to the case $\varphi(r)=\varphi_0-\varepsilon(r)$, for more general $\varepsilon(r)$. If $\int^\infty \varepsilon(r)r^{-1}dr<\infty$, $\varphi(r)\nearrow\varphi_0$, $\psi(r)=\varphi(r)+\arctan\big(r\varphi'(r)\big)\nearrow\varphi_0$, $\psi'(r)\approx\varphi'(r)\searrow 0$, the result corresponding to Theorem 2 will be $\omega_r(\varrho)\geqq C(\varrho/r)^{\alpha_0}$.

2. Proofs of Theorem 1 and Theorem 2

Some basic facts to be kept in mind are the following. Let φ_0 be given and let f be the corresponding first eigenfunction of (7) and α_0 the characteristic constant as in (8). Then

$$v(x) = |x|^{\alpha_0}f(\varphi)$$

is harmonic in $\{x\in R_d\colon 0\leqq\varphi<\varphi_0\}$ with boundary values zero.

For $d=2$ we know that $f(\varphi)=\cos\big((\pi/2)\varphi/\varphi_0\big)$, so, for a fixed φ, the value of f increases as φ_0 increases. A corresponding result for the first eigenfunctions is true in general. We include a proof given by Dr. J. B. McLeod in Lemma 1 at the end of this paper. However, due to discrepancies in angles, this fact alone is not sufficient in our proof, and we have to consider error terms.

Proof of Theorem 1. We need some elementary Euclidean estimates. Let the tangent at the point $Q=\big(R, \varphi(R)\big)$ on the curve Γ (see (5) and (6)) intersect the polar axis at the point $P=P(R)$ with radius $p(R)$. Let $r(R)$ be the distance between P and Q and let $\psi(R)$ be the angle between the line PQ and the polar axis. See Figure 1.

Then

(12) $$p(R) = R((\log R)^{-2}/\delta + O((\log R)^{-3})), \quad R \to \infty$$

(13) $$r(R) = R(1-\delta^{-2}(\log R)^{-3} + O((\log R)^{-4})), \quad R \to \infty$$

(14) $$\psi(R) = \tfrac{1}{2}\pi - (\delta \log R)^{-1} + O((\log R)^{-2}), \quad R \to \infty.$$

Rotating the figure around the axis, we obtain an open circular cone $C(R)$ in R^d with vertex at P and with the angle $\psi(R)$ between the axis and the line PQ.

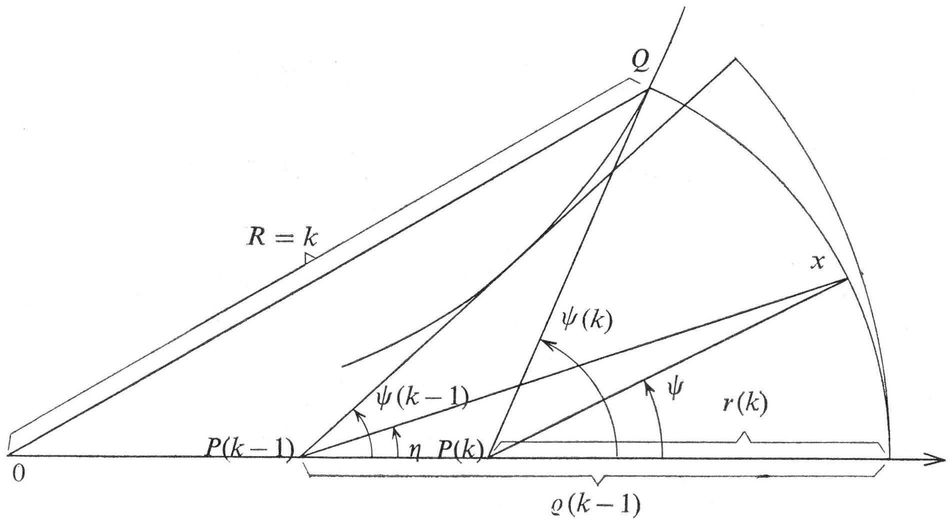

Figure 1.

We shall now consider radii $R=k\in Z_+$ and corresponding cones $C(k)$ and estimate harmonic measures in successive cones. We shall write T instead of $T_{\delta,d}$.

Let t be a given (large) number and let u be the harmonic measure of $T\cap\{x\colon |x|=t\}$ with respect to $T\cap\{x\colon |x|<t\}$. Let us assume that for some $k\in Z_+$, $k<t$, we have

(15) $$u(x) \geqq m_k f_k(\psi), \quad x\in T, \quad |x-P(k)| = r(k),$$

where m_k is some positive constant, f_k is the first eigenfunction associated with the zero $\psi(k)$ (see (7)), and ψ is the polar angle between the segment from $P(k)$ to x and the axis. Let $\alpha(k)$ be the characteristic constant corresponding to $\psi(k)$.

We shall now pass from k to $k-1$. Let η denote the polar angle between the segment from $P(k-1)$ to x and the axis. Then $\eta<\psi$ and $\psi(k-1)<\psi(k)$. Also let

(16) $$\varrho(k-1) = p(k)+r(k)-p(k-1),$$

see Figure 1. Now we compare $u(x)$ to a suitable harmonic minorant. Assume that

(17) $$f_k(\psi) \geqq s_k f_{k-1}(\eta), \quad x\in C(k-1), \quad |x-P(k)| = r(k),$$

with a positive s_k to be estimated later, and ψ and η being defined as above. In fact, for $\psi=\eta$ one can take $s_k=1$ in (17); however, since in our case $\psi>\eta$ we have to use $s_k=1$ – an error term, for large k. Using (17) and (15) we now compare u to

$$v(x) = m_k s_k(|x-P(k-1)|/\varrho(k-1))^{\alpha(k-1)} f_{k-1}(\eta),$$

in $C(k-1)\cap\{x\colon |x-P(k)|<r(k)\}$. By the comparison principle for harmonic functions it follows that $u(x)\geqq v(x)$. In particular,

$$u(x) \geqq m_k s_k(r(k-1)/\varrho(k-1))^{\alpha(k-1)} f_{k-1}(\eta), \quad x\in T, \quad |x-P(k-1)| = r(k-1).$$

Now define

$$m_{k-1} = m_k s_k (r(k-1)/\varrho(k-1))^{\alpha(k-1)}. \tag{18}$$

Then

$$u(x) \geqq m_{k-1} f_{k-1}(\eta), \; x\in T, \quad |x-P(k-1)| = r(k-1),$$

an analogue to (15), and we can go on iterating the procedure.

We recall that u is the harmonic measure of $T\cap\{x\colon |x|=t\}$ with respect to T_t. Choose the largest integer M such that $p(M-1)+\varrho(M-1)\leqq t$. Then, by the comparison principle in $C(M)\cap\{x\colon |x|<t\}$,

$$u(x) \geqq (r(M)/t)^{\alpha(M)} f_M(\psi), \quad x\in C(M), \quad |x-P(M)| = r(M),$$

where now ψ is the polar angle between the segment from $P(M)$ to x and the axis. Since $t\approx\varrho(M)$ this gives us the following value for m_M:

$$m_M \approx (r(M)/\varrho(M))^{\alpha(M)}. \tag{19}$$

We note that

$$M \approx t(1-\delta^{-1}(\log t)^{-2}). \tag{20}$$

By iteration we obtain from (15), (18) and (19) the following estimate for $q\in Z^+$, $q\geqq 3$:

$$\begin{aligned} u(x) &\geqq \prod_{k=q}^{M} (r(k)/\varrho(k))^{\alpha(k)} \prod_{k=q+1}^{M} s_k \\ &= \prod_{k=q}^{M} (r(k)/r(k+1))^{\alpha(k)} \prod_{k=q}^{M} (r(k+1)/\varrho(k))^{\alpha(k)} \prod_{k=q+1}^{M} s_k \\ &= A_1 A_2 A_3, \quad x = (p(q)+r(q), \; 0, \ldots, 0). \end{aligned} \tag{21}$$

Here, the first factor A_1 in the product will yield the desired estimate of $u(x)$, while A_2 and A_3 contain error terms due to discrepancies between radii and angles in the different inscribed cones; these can be collected into a constant factor. To be exact, we have $u(x)=\omega_t(p(q)+r(q))$ while we want to exhibit $\omega_t(\varrho)$ and require ϱ rather than q and t rather than M in the final estimate. However, we shall see from (20) that it is possible to substitute t for M by changing a constant in the estimate of the harmonic measure and a similar situation applies at the other end. Thus (21) will yield Theorem 1, after A_1, A_2 and A_3 have been discussed.

Estimate of A_1. Using (14) and

$$\alpha(k) = 1+\alpha_0'\left(\psi(k)-\tfrac{1}{2}\pi\right)+O\left(\left(\psi(k)-\tfrac{1}{2}\pi\right)^2\right)$$

where $\alpha_0'=(d\alpha_0/d\varphi_0)(\pi/2)$, we obtain

$$\alpha(k) = 1-\alpha_0'\delta^{-1}(\log k)^{-1}+O((\log k)^{-2}).$$

For $r(k)$ use (13). Thus

$$\log A_1 = \sum_q^M \alpha(k)\left(\log r(k)-\log r(k+1)\right)$$

$$\approx -\sum_q^M \left(1-\alpha_0'\delta^{-1}(\log k)^{-1}\right)k^{-1} \approx \log(q/M)+\alpha_0'\delta^{-1}(\log\log M-\log\log q)$$

with error terms $O\left(\sum_q^\infty (\log k)^{-2}k^{-1}\right)$ that can be estimated by constants.

Estimate of A_2. With $\alpha(k)$ given above and $r(k)$ and $\varrho(k)$ as in (12), (13) and (16) we obtain

$$\log A_2 = \sum_q^M \alpha(k)\left(\log r(k+1)-\log \varrho(k)\right)$$

$$= -\sum_q^M \alpha(k)\log\left(1+(p(k+1)-p(k))/r(k+1)\right) \approx \delta^{-1}\sum_q^M (\log k)^{-2}k^{-1},$$

which again can be estimated by a constant.

Estimate of A_3. Here we are concerned with Πs_k, where s_k should be a sufficiently good estimate from below of the quotient of eigenfunctions (see (17)):

$$\frac{f_k(\psi)}{f_{k-1}(\eta)} \geqq s_k \tag{22}$$

where ψ and η are angles as in Figure 1 and $|x-P(k)|=r(k)$, $0\leqq\eta<\psi(k-1)$. Using (12), (13), (16) we obtain

$$\beta(k) = \psi-\eta = \delta^{-1}(\sin\psi)(\log k)^{-2}k^{-1}+O\left((\log k)^{-3}k^{-1}\right), \quad k\to\infty. \tag{23}$$

Also $\psi(k)-\psi(k-1)>\beta(k)$ and (from (6a), (6b))

$$\psi(k)-\psi(k-1) = \delta^{-1}(\log k)^{-2}k^{-1}+O\left((\log k)^{-3}k^{-1}\right). \tag{24}$$

We denote the first eigenvalue in (7) corresponding to $\varphi_0=\psi(k)$ by $\lambda_0=\lambda(k)$ and the first eigenfunction in (7) by $f(\varphi, \lambda)$, which is analytic in φ and in λ. We need an estimate of $\lambda(k-1)-\lambda(k)$, obtained from

$$0 = f(\psi(k-1),\ \lambda(k-1)) = f(\psi(k),\ \lambda(k))+(\psi(k-1)-\psi(k)) \tag{25}$$

$$\cdot\frac{\partial f}{\partial\varphi}(\psi(k), \lambda(k))+(\lambda(k-1)-\lambda(k))\frac{\partial f}{\partial\lambda}(\psi(k), \lambda(k))+\text{error terms}$$

with $f(\psi(k), \lambda(k))$ zero and partial derivatives to be estimated. It follows from (7) that

$$(\sin\varphi)^{d-2}\frac{\partial f(\varphi, \lambda)}{\partial\varphi} = -\lambda\int_0^\varphi (\sin\varphi)^{d-2}f(\varphi, \lambda)\,d\varphi$$

and thus $\partial f/\partial\varphi<0$, $0<\varphi\leqq\varphi_0$, and $(\partial f/\partial\varphi)(\psi(k), \lambda(k))$ is uniformly bounded for say $k\geqq 3$.

By a result about eigenfunctions, stated in Lemma 1 at the end of this paper, $\partial f/\partial\lambda<0$, $0<\varphi\leqq\varphi_0$. Thus $(\partial f/\partial\lambda)(\psi(k), \lambda(k))$ is uniformly bounded away from zero for $k\geqq 3$ and it follows from (25) that

(26) $$\lambda(k-1)-\lambda(k) = O\big(k^{-1}(\log k)^{-2}\big), \quad k\to\infty.$$

We shall now return to the quotients (22). By monotonicity of f (with respect to φ in $[0, \varphi_0]$) and by Cauchy's mean value theorem, for $0\leqq\eta<\psi(k-1)$, we obtain

$$\frac{f_k(\psi)}{f_{k-1}(\eta)} = \frac{f(\eta+\beta(k), \lambda(k))}{f(\eta, \lambda(k-1))} \geqq \frac{f(\eta+\psi(k)-\psi(k-1), \lambda(k))-f(\psi(k), \lambda(k))}{f(\eta, \lambda(k-1))-f(\psi(k-1), \lambda(k-1))}$$

$$= \frac{\partial f(\xi(k)+\psi(k)-\psi(k-1), \lambda(k))}{\partial\varphi} \Big/ \frac{\partial f(\xi(k), \lambda(k-1))}{\partial\varphi}$$

$$\approx \Big(\frac{\partial f(\xi(k), \lambda(k-1))}{\partial\varphi}+(\psi(k)-\psi(k-1))\frac{\partial^2 f}{\partial\varphi^2}$$

$$+(\lambda(k)-\lambda(k-1))\frac{\partial^2 f}{\partial\varphi\partial\lambda}\Big)\Big/\frac{\partial f(\xi(k), \lambda(k-1))}{\partial\varphi}, \quad \xi(k)\in]\eta, \psi(k-1)[.$$

Using uniform boundedness of the second partial derivatives, uniform boundedness away from zero of the denominator, for say $(1/2)\psi(k-1)\leqq\xi(k)\leqq\psi(k-1)$, $k\geqq 3$, and (24) and (26) we obtain, for $(1/2)\psi(k-1)\leqq\eta\leqq\psi(k-1)$,

(27) $$\frac{f_k(\psi)}{f_{k-1}(\eta)} \geqq 1+O\big(k^{-1}(\log k)^{-2}\big), \quad k\to\infty.$$

We need the estimate in (27) also for $0\leqq\eta\leqq(1/2)\psi(k-1)$. This follows by writing $f_k(\psi)=f(\eta+\beta(k), \lambda(k))$ as $f(\eta, \lambda(k-1))+$terms containing first partial derivatives, since for $0\leqq\eta\leqq(1/2)\psi(k-1)$, $k\geqq 3$, the denominator $f_{k-1}(\eta)=f(\eta, \lambda(k-1))$ is uniformly bounded away from zero. Thus we have (27) for $0\leqq\eta<\psi(k-1)$, and using this estimate to define s_k for large k (cf. (22)), we see that $A_3=\Pi s_k$ can be estimated from below by a positive constant, there being no problem with a finite number of factors at the beginning. This concludes the proof of Theorem 1.

Proof of Theorem 2. It is sufficient to indicate changes in the basic approximations, for $0<\varphi_0<\pi$:

(12a) $$p(R) = R\big((\log R)^{-2}\delta^{-1}(\sin\varphi_0)^{-1}+O((\log R)^{-3})\big), \quad R\to\infty$$

(13a) $$r(R) = R\big(1-(\log R)^{-2}\delta^{-1}(\tan\varphi_0)^{-1}+O((\log R)^{-3})\big), \quad \varphi_0\neq\tfrac{1}{2}\pi, \quad R\to\infty$$

(14a) $$\psi(R) = \varphi_0-\delta^{-1}(\log R)^{-1}+O\big((\log R)^{-2}\big), \quad R\to\infty$$

(23a) $$\beta(k) = \delta^{-1}(\sin\psi)(\sin\varphi_0)^{-1}(\log k)^{-2}k^{-1}+O\big((\log k)^{-3}k^{-1}\big), \quad k\to\infty.$$

The proof follows as in the previous case, with some slight modifications, e.g. for obtuse φ_0.

Finally, we state a lemma the proof of which is due to Dr. J. B. McLeod, Oxford. This lemma is used in our proof of Theorem 1 to prove that $\partial f/\partial\lambda$ is bounded away from zero on a suitable set.

Lemma 1 (J. B. McLeod). *Consider the differential equation*

$$\text{(i)} \qquad \frac{d}{d\varphi}\left(p(\varphi)\frac{df}{d\varphi}\right)+\lambda p(\varphi)f(\varphi)=0$$

with initial conditions $f(0)=1, f'(0)=0$. *Let* λ *be a positive parameter and* $p(\varphi)=\sin^{n-2}\varphi,\ n=2,3,4,\ldots$ *Denote the first positive zero of* f *by* $\varphi_0=\varphi_0(\lambda),\ 0<\varphi_0<\pi$, *and* $f(\varphi)$ *by* $f(\varphi,\lambda)$ *to emphasize the dependence of* f *on* λ. *If* $\lambda_2>\lambda_1$, *then* $f(\varphi,\lambda_2)\leqq f(\varphi,\lambda_1)$ *for* $0\leqq\varphi\leqq\varphi_0(\lambda_1)$, *with equality only at* $\varphi=0$.

Proof. The result certainly follows if $(\partial/\partial\lambda)f(\varphi,\lambda)\leqq 0$, with equality only at $\varphi=0$. Differentiating (i) with respect to λ, we obtain

$$\text{(ii)} \qquad \frac{d}{d\varphi}\left(p(\varphi)\frac{d}{d\varphi}\left(\frac{\partial f}{\partial\lambda}\right)\right)+\lambda p(\varphi)\frac{\partial f}{\partial\lambda}=-p(\varphi)f$$

with

$$\text{(iii)} \qquad \frac{\partial f(0,\lambda)}{\partial\lambda}=0, \quad \frac{\partial f'(0,\lambda)}{\partial\lambda}=0.$$

Multiplying (i) by $\partial f/\partial\lambda$ and (ii) by f and subtracting, we have

$$\frac{d}{d\varphi}\left(p\left(\frac{\partial f}{\partial\lambda}f'-f\frac{\partial f'}{\partial\lambda}\right)\right)=pf^2>0, \quad 0<\varphi<\varphi_0,$$

so that

$$\text{(iv)} \qquad p\left(\frac{\partial f}{\partial\lambda}f'-f\frac{\partial f'}{\partial\lambda}\right)$$

is strictly increasing and so (from (iii)) strictly positive, except at $\varphi=0$. Further, (i) implies that

$$f''+(n-2)(\cot\varphi)f'+\lambda f=0,$$

$$f''(0)=-\lambda(n-1)^{-1}$$

and so $\partial f/\partial\lambda<0$ for sufficiently small positive φ. Now suppose, contrary to what we have to prove, that $\partial f/\partial\lambda=0$ first at $\varphi=\varphi_1,\ 0<\varphi_1\leqq\varphi_0$. Then necessarily $\partial f'/\partial\lambda\geqq 0$ at $\varphi=\varphi_1$, so that (iv) is non-positive, giving the required contradiction.

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