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A REMARK ON THE CONFORMAL CAPACITY OF GRÖTZSCH'S CONDENSER IN SPACE

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1. Grötzsch's condenser

Grötzsch's condenser is a well-known extremal condenser in the complex plane and the corresponding configuration in space has certain applications e.g. for the theory of quasiconformal mappings.

By the conformal *capacity* of Grötzsch's condenser $(B^n, J^n(r))$, $B^n=\{x\in \mathbf{R}^n \mid |x|<1\}$ and $J^n(r)=\{(0, \ldots, 0, x_n) \mid 0\leqq x_n \leqq r\}$, we mean the quantity

$$v_n(r) = \inf_{\varphi} \int_{B^n} |\nabla\varphi|^n \, dm \quad (0<r<1). \tag{1.1}$$

Here the infimum is taken among all $\varphi\in C^1(B^n)$ with boundary values $\varphi|\partial B^n=0$ and $\varphi|J^n(r)=1$. A basic fact is that the integral $\int |\nabla\varphi|^n dm$ is conformally invariant. The function φ for which (1.1) is actually obtained is not known in space. Nevertheless, good approximations for $v_n(r)$ have been obtained.

Replacing $J^n(r)$ by a ball with $J^n(r)$ as a diameter one obtains the bound

$$v_n(r) < \omega_{n-1}\left(\overline{\mathrm{ar}} \cosh (1/r)\right)^{1-n} \quad (0<r<1), \tag{1.2}$$

which is equal to the conformal capacity of the enlarged condenser $\left(B^n, B^n((0, \ldots, 0, r/2), r/2)\right)$. Here ω_{n-1} is the area of the unit sphere in $\mathbf{R}^n$. However, (1.2) is accurate only for small values of r.

A lower bound of the form

$$v_n(r) > A_n \log\frac{1+r}{1-r} \quad (0<r<1) \tag{1.3}$$

is easily derived via a Möbius transformation that maps B^n onto the upper half space in $\mathbf{R}^n$, where the oscillation lemma of Gehring yields the desired result. See [3] and [6].

A difficult question has been how to achieve natural upper bounds for $v_n(r)$ as $r\to 1-0$. An estimate was given in 1974 by G. Andersson, who calculated that

$$v_n(r) < A_n \log\frac{1+r}{1-r}+C_n \quad (0<r<1). \tag{1.4}$$

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Here A_n is the same positive constant as in (1.3) and hence (1.4) is, indeed, relevant for r close to 1. See [2, Theorem 2, Theorem 3, Corollary 1] and [5, Lemma 1].

Andersson's method is based on the fact that in plane the infimum (1.1) is obtained for a well-known function $\varphi(z)$, described by the aid of Jacobi's elliptic sine function. Then one may say $\varphi(z)$ is rotated to give a function in space, admissible for the infimum in (1.1). The objective of our note is to achieve (1.4) more adequately. We are guided by an explicit expression for the n-harmonic measure of a diagonal in a ball; c.f. [4].*

2. An estimate

The following result, due to Andersson, will be proved by elementary calculus.

Theorem. *For* $0<r<1$ *the estimate*

$$v_n(r) < A_n \log\frac{1+r}{1-r} + C_n \tag{2.1}$$

holds. Here C_n *depends only on n and* $A_n = \omega_{n-2}/\varkappa_n^{n-1}$, ω_{n-2} *being the area of the unit sphere in* $\mathbf{R}^{n-1}$ *and*

$$\varkappa_n = \int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)}\, dt. \tag{2.2}$$

Proof. (We do not care about the best possible C_n but A_n is important in view of (1.3).)

Since the integral $\int |\nabla\varphi|^n dm$ is conformally invariant, i.e., it is preserved by Möbius transformations, we can carry over the problem to a situation in the upper half-space $H^n = \{(x_1, \ldots, x_n) | x_n > 0\}$. To begin with we note that

$$v_n(r) = \mu_n\left(\frac{1-(1-r^2)^{1/2}}{r}\right), \tag{2.3}$$

where $\mu_n(r')$ is the conformal capacity of the condenser $(B^n, I^n(r'))$, $I^n(r') = \{(0, \ldots, 0, x_n) | -r' \leqq x_n \leqq r'\}$. Thus we have to estimate $\mu_n(r)$. For a fixed r, $0<r<1$, there is a Möbius transformation γ from B^n onto H^n such that

$$\gamma(0, \ldots, 0, \pm r) = \left(0, \ldots, 0, 2\frac{1\mp r}{1\pm r}\right)$$

and $\gamma(I^n(r))$ is a certain line segment. If φ is admissible for the auxiliary condenser $(B^n, I^n(r))$, then

$$\int_{B^n} |\nabla\varphi|^n\, dm = \int_{H^n} |\nabla(\varphi\circ\gamma^{-1})|^n\, dm \tag{2.4}$$

by conformal invariance.

* A general idea of the principles involved is given in Granlund, S. Lindqvist, P., and Martio, O.: F-harmonic measure in space, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 7 (1982), 233—247.

Let θ denote the angle between the positive x_n-axis and $x \in \bar{H}^n$, i.e.,

$$\cos\theta = x_n/|x|, \quad 0 \leqq \theta \leqq \pi/2$$

for $x \neq 0$. The function

$$V(\theta) = \frac{1}{\varkappa_n}\int_\theta^{\pi/2} (\sin t)^{(2-n)/(n-1)}\,dt \quad (0 \leqq \theta \leqq \pi/2)$$

is the key to (2.1). Our explanation is that V is in fact the so-called n-harmonic measure of $X = \{(0, \ldots, 0, x_n) | x_n \geqq 0\}$ taken with respect to the domain $H^n \setminus X$, and hence V is somehow close to "the right function". (Especially, V is a solution of the n-harmonic equation $\operatorname{div}(|\nabla V|^{n-2}\nabla V) = 0$ in $H^n \setminus X$.) See [4].

A simple calculation in spherical coordinates gives

$$\int_{\substack{r_1 < |x| < r_2 \\ x_n > 0}} |\nabla V|^n\,dm = \frac{\omega_{n-2}}{\varkappa_n^{n-1}} \log\frac{r_2}{r_1}, \tag{2.5}$$

where we are interested in the choices

$$r_1 = 2\frac{1-r}{1+r}, \quad r_2 = 2\frac{1+r}{1-r}.$$

Unfortunately, the integral $\int |\nabla V|^n\,dm$ is infinite when taken over H^n. Therefore we are forced to adjust V near the points 0 and ∞. To this end, define

$$\zeta(x) = \begin{cases} 0 & \text{if} \quad |x| \geqq 2r_2 \quad \text{or} \quad |x| \leqq r_1/2 \\ 1 & \text{if} \quad r_1 \leqq |x| \leqq r_2 \\ \log(2|x|/r_1)/\log 2 & \text{if} \quad r_1/2 \leqq |x| \leqq r_1 \\ \log(|x|/2r_2)/\log(1/2) & \text{if} \quad r_2 \leqq |x| \leqq 2r_2 \end{cases}$$

for $x \in H^n$ and consider the admissible function ζV.

By (2.4)

$$\mu_n(r) \leqq \int_{H^n} |\nabla(\zeta V)|^n dm. \tag{2.6}$$

By virtue of (2.5) we obtain

$$\mu_n(r) \leqq \frac{\omega_{n-2}}{\varkappa_n^{n-1}} \log\left(\frac{1+r}{1-r}\right)^2 + \int_{0<\zeta<1} |\nabla(\zeta V)|^n\,dm$$

and a rough upper bound for the last integral is

$$2^{n-1}\left\{\int_{0<\zeta<1} |\nabla V|^n\,dm + \int_{0<\zeta<1} |\nabla\zeta|^n\,dm\right\} \leqq 2^n\left\{\frac{\omega_{n-2}\log 2}{\varkappa_n^{n-1}} + \frac{\omega_{n-1}}{(\log 2)^{n-1}}\right\} = C_n.$$

Thus we have

$$\mu_n(r) \leqq 2A_n \log\frac{1+r}{1-r} + C_n, \tag{2.7}$$

and a substitution in (2.3) gives (2.1). This concludes our proof.

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