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ENTIRE FUNCTIONS WITH SPIRAL LIMITS

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It is (perhaps) well-known that given $0=t_0<t_1<\ldots<t_n=2\pi$ and complex numbers $c_1, \ldots, c_n$, there exists an entire function f such that $\lim_{r\to\infty} f(re^{it})=c_j$ for each $t\in(t_{j-1}, t_j)$ and each $j=1, 2, \ldots, n$. (See G. Pólya [7] and Exercises IV 185—IV 186 of G. Pólya and G. Szegő [8].) In a similar vein K. Grandjot [5] proved the existence of a non-zero entire function f such that $f(z)\to 0$ as $z\to\infty$ along any algebraic curve (cf. also H. Bohr [2]). The review Mathematical Reviews 52 # 8433 gives a brief historical account of this subject; it should be supplemented by Paragraphs 21 and 43 of the encyclopedia article [1] of L. Bieberbach. For some related results, we mention the anecdotal paper [10] by W. J. Schneider. In this note, we shall construct an entire function with "spiral" limits, where the limits are finitely many preassigned polynomials. Our method uses the well-known technique of shoving poles to infinity.

For each $p>0$, let S_p denote the class of all continuously differentiable real-valued functions θ on $(0, \infty)$ such that

(i) $$\int_1^\infty |\theta'(t)|\cdot t^{-p}\,dt<\infty.$$

Thus every function of the form $P(t)\cos Q(t)+R(t)\sin S(t)$ belongs to S_p for some $p>1$, where P, Q, R, S are polynomials with real coefficients. By a *spiral region* we mean an open set in the complex plane $\boldsymbol{C}$ of the form

(ii) $$\Omega=\{re^{it}\colon r>0 \quad\text{and}\quad \theta_1(r)<t<\theta_2(r)\},$$

where $\theta_1, \theta_2\in S_p$ for some $p>0$ and $\theta_1(t)<\theta_2(t)\leqq\theta_1(t)+2\pi$ for all $t>0$. As is customary, we shall often identify a curve with its image set.

Theorem. *Let $\Omega_1, \ldots, \Omega_k$ be pairwise disjoint spiral regions, $E_j\subset\Omega_j$ unbounded closed subsets of $\boldsymbol{C}$, $P_j(z)$ polynomials in $z\in\boldsymbol{C}$ for $j=1, 2, \ldots, k$, and N a natural number. Then there exists an entire function g such that*

$$g(z)=P_j(z)+o(|z|^{-N}) \quad \text{as} \quad z\in E_j \quad \text{tends to } \infty$$

for each $j=1, 2, \ldots, k$.

Corollary. *Let P be a polynomial, and N a natural number. Then there exists a non-polynomial entire function h such that $h(z)=P(z)+o(|z|^{-N})$ as $z\to\infty$ along any algebraic curve.*

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To prove these results, we need three lemmas.

Lemma 1. *Suppose that* $\theta_1, \theta_2 \in S_p$ *for some* $p>1$, *and that* $\theta_1<\theta_2<\theta_3$ *on* $(0, \infty)$, *where* $\theta_3=\theta_1+2\pi$. *For* $j=1, 2$, *let*

(a) $$\Omega_j = \{re^{it}\colon r>0 \ \text{ and } \ \theta_j(r)<t<\theta_{j+1}(r)\},$$

and let g_j *be a holomorphic function on* Ω_j *such that*

(b) $$z\in\Omega_j \ \text{ and } \ |z|>C \Rightarrow |g_j'(z)|<C/|z|^{2N}$$

for some C *and* $N>p+1$. *Then there exist* $c_j\in C$ *such that*

(c) $$g_j(z) = c_j+o(|z|^{-N}) \ \text{ as } \ z\in\Omega_j \ \text{ tends to } \ \infty \ (j=1, 2).$$

Proof. For $j=1,2$ and $t\geqq 0$, define

(1) $$\tau_j(t) = 2^{-1}\{\theta_j(t)+\theta_{j+1}(t)\} \ \text{ and } \ \gamma_j(t) = te^{i\tau_j(t)}.$$

We fix j, and write $\tau=\tau_j$, $\gamma=\gamma_j$, etc. Since $N-1>p>1$, we have

(2) $$\int_1^\infty (1+t|\tau'(t)|)t^{-N}\,dt \leqq \int_1^\infty (1+|\theta_1'(t)|+|\theta_2'(t)|)\,t^{1-N}\,dt<\infty$$

by (i).

Notice that $\gamma((0,\infty))\subset\Omega_j$, $|\gamma(t)|=t$, and $\gamma'(t)=\{1+it\tau'(t)\}e^{i\tau(t)}$ for all $t>0$ by (1). It follows from (b) that $s>r>C$ implies

(3) $$|g(\gamma(s))-g(\gamma(r))| = \left|\int_r^s g'(\gamma(t))\gamma'(t)\,dt\right| \leqq \int_r^s Ct^{-2N}(1+t|\tau'(t)|)\,dt$$
$$\leqq Cr^{-N}\int_r^\infty (1+t|\tau'(t)|)t^{-N}\,dt.$$

From (2) and (3) we infer that $g(\gamma(s))$ converges to some complex number $c=c_j$ as $s\to\infty$, and that

(4) $$|c-g(\gamma(r))| = o(r^{-N}) \ \text{ as } \ r\to\infty.$$

Now suppose that $z=re^{is}\in\Omega_j$, where $r>C$ and that $\theta_j(r)<s<\theta_{j+1}(r)$. Then we have

$$|g(\gamma(r))-g(z)| = \left|\int_s^{\tau(r)} g'(re^{it})ire^{it}\,dt\right| < 2\pi C/r^{2N-1}$$

by (1) and (b), so

$$|c-g(z)| < |c-g(\gamma(r))|+2\pi C/r^{2N-1}.$$

This inequality, combined with (4), yields the desired conclusion.

Lemma 2. *Let* θ_j, p, *and* Ω_j $(j=1,2)$ *be as in Lemma 1. Fix a positive real number* r_0 *and a natural number* $N>p+1$. *Let*

(a) $$\alpha = r_0\exp[i\theta_1(r_0)] \ \text{ and } \ \beta = r_0\exp[i\theta_2(r_0)],$$

and let $g(z)$ *be a holomorphic antiderivative of* $[(z-\alpha)(z-\beta)]^{-N}$ *in the simply connected region*

(b) $$\Omega_0 = C\setminus\bigcup_{j=1}^{2}\{r\exp[i\theta_j(r)]\colon\ r \geqq r_0\}.$$

Then there exist complex numbers c_1, c_2 *such that*

(c) $$g(z) = c_j + o(|z|^{-N}) \quad as \quad z\in\Omega_j \quad tends\ to\ \infty$$

for $j=1, 2$. *Moreover,* $c_1 \neq c_2$.

Proof. The existence of c_j satisfying (c) is an immediate consequence of Lemma 1. So we only need to check that $c_1 \neq c_2$.

Let γ_1 and γ_2 be the two infinite curves defined as in the proof of Lemma 1. Notice that both γ_1 and γ_2 lie in Ω_0, that $\gamma_2=-\gamma_1$ and that

(1) $$g(v)-g(u) = \int_\Gamma (z-\alpha)^{-N}(z-\beta)^{-N}\,dz \quad (u, v\in\Omega_0),$$

where $\Gamma=\Gamma(u, v)$ is any smooth curve in Ω_0 from u to v. Now pick any $r>r_0$, and consider the closed curve γ consisting of the following three pieces: $\gamma_2(r-t)$ for $0\leqq t\leqq r$, $\gamma_1(t-r)$ for $r\leqq t\leqq 2r$, and the semicircle

$$C_r(t) = \gamma_1(r)\exp[i(t-2r)] \quad \text{for} \quad 2r \leqq t \leqq 2r+\pi.$$

It is easy to check that α and β lie "outside" and "inside" of γ, respectively.

It follows from (1) and Cauchy's residue theorem that

(2) $$g(\gamma_1(r))-g(\gamma_2(r))+\int_{C_r}(z-\alpha)^{-N}(z-\beta)^{-N}\,dz = 2\pi i\,\mathrm{Res}\,(\beta),$$

where

(3) $$\mathrm{Res}\,(\beta) = \frac{[2(N-1)]!}{[(N-1)!]^2}(-1)^{N-1}(\beta-\alpha)^{1-2N} \neq 0.$$

But it is routine to show that $\lim r\to\infty \int_{C_r}=0$. Letting $r\to\infty$ in (2), we therefore conclude from (c) and (3) that $c_1-c_2+0\neq 0$, as desired.

Now we write $P^*(w)=P(1/w)$ for a polynomial P and $w\neq 0$.

Lemma 3. *Let* E *be a closed subset of* C, K *a compact connected subset of* $C\setminus E$, *and* $u, v\in K$. *If* N *is a nonnegative integer,* $\varepsilon>0$, *and* R_1 *is a polynomial, then there exists a polynomial* R_2 *such that*

$$|R_1^*(z-u)-R_2^*(z-v)| < \varepsilon/(2+|z|)^N \quad \forall\, z\in E.$$

Proof. For $N=0$, this is a consequence of Runge's theorem. (Indeed, it can be proved by an elementary method.) See, for example, Chapter IV, Paragraph 1 of S. Saks and A. Zygmund [9].

So assume that $N\geqq 1$ and that the result is true with N replaced by $N-1$. Apply this inductive hypothesis to $R_1(z)=(u-v)z$ to find a polynomial Q such that

(1) $$|(u-v)(z-u)^{-1}-Q^*(z-v)| < \varepsilon/(2+|z|)^{N-1} \quad \forall\, z\in E.$$

Divide both sides of this inequality by $|z-v|$ to obtain

(2) $$|(z-u)^{-1}-\{(z-v)^{-1}+(z-v)^{-1}Q^*(z-v)\}| < \frac{\varepsilon}{|z-v|(2+|z|)^{N-1}} \quad \forall z\in E.$$

But $(2+|z|)/|z-v|$ is bounded on E and $\varepsilon>0$ is arbitrary. Thus we conclude that there exists a polynomial R such that

(3) $$|(z-u)^{-1}-R^*(z-v)| < \varepsilon/(2+|z|)^N \quad \forall z\in E,$$

which establishes the desired result for $R_1(z)=z$. Since $(z-u)^{-1}$ is bounded on E, (3) shows that $R^*(z-v)$ is bounded on E. Therefore the general case follows from this special case combined with the elementary formula $A^n-B^n=(A-B)(A^{n-1}+\ldots+ +B^{n-1})$. This completes the induction and hence the proof.

Proof of the Theorem. First consider the case $k=1$. In this case we may assume that Ω_1 is the complement of a curve Γ of the form $\Gamma(t)=te^{i\theta(t)}$ for $t\geqq 0$, where θ is in S_p for some $p>0$. Then the hypothesis on the closed unbounded set E_1 is that it be disjoint from Γ. For such an E_1, it is easy to construct an infinitely differentiable function δ on $(0,\infty)$ such that $0<\delta(t)<2\pi$ and $|\delta'(t)|<1$ for all $t>0$ and such that

$$E_1\subset\{re^{it}\colon r>0 \text{ and } \theta(r)<t<\theta(r)+2\pi-\delta(r)\}.$$

Therefore the case $k=1$ can be reduced to the case $k=2$. Also the desired result for $k\geqq 3$ follows from k applications of the result for $k=2$ as follows: for each pair of complementary spiral regions Ω_j and $C\setminus\bar\Omega_j$ and respective polynomials P_j and 0, the result for $k=2$ supplies us with an appropriate entire function g_j and for the desired function g we take $g_1+g_2+\ldots+g_k$. Thus it will be sufficient to deal with the case $k=2$.

So assume that $k=2$ and also, without loss of generality, that Ω_1 and Ω_2 are defined by (a) in Lemma 1. Let E_j be a closed unbounded set contained in Ω_j for $j=1,2$. Choose and fix an infinitely differentiable function δ on $(0,\infty)$, with bounded derivative, such that

(1) $$0<\delta<4^{-1}\min\{\theta_2-\theta_1,\theta_3-\theta_2\} \quad\text{on}\quad (0,\infty), \quad\text{and for}\quad j=1,2$$

(2) $$E_j\subset\{re^{it}\colon r>0 \text{ and } \theta_j^+(r)<t<\theta_{j+1}^-(r)\}\overset{\text{def}}{=}\Omega_j^*,$$

where

(3) $$\theta_j^+=\theta_j+\delta \quad\text{and}\quad \theta_j^-=\theta_j-\delta \quad (j=1,2,3).$$

Now let $\varepsilon>0$ and a natural number $q>p+1$ be given. Put

(4) $$U_n=\{re^{it}\colon r>n-1/2,\ |t-\theta_1(t)|<\delta(r)\},$$

(5) $$V_n=\{re^{it}\colon r>n-1/2,\ |t-\theta_2(r)|<\delta(r)\},$$

(6) $$\alpha_n=ne^{i\theta_1(n)} \quad\text{and}\quad \beta_n=ne^{i\theta_2(n)} \quad\text{for}\quad n=1,2,\ldots.$$

We shall construct two sequences of polynomials Q_n and R_n as follows. The rational function $f_1(z)=[(z-\alpha_1)(z-\beta_1)]^{-q}$ admits a representation of the form

$$f_1(z) = Q_1^*(z-\alpha_1)+R_1^*(z-\beta_1) \tag{7}$$

for some polynomials Q_1 and R_1. Suppose that polynomials Q_n and R_n have been chosen for some $n \geqq 1$. Notice that α_n and α_{n+1} are contained in an arc which is disjoint from the closed set $C \setminus U_n$. It follows from Lemma 3 that there exists a polynomial Q_{n+1} such that

$$|Q_{n+1}^*(z-\alpha_{n+1})-Q_n^*(z-\alpha_n)| < \varepsilon/(2+|z|)^{2qn} \quad \forall z \in C \setminus U_n. \tag{8}$$

Similarly there exists a polynomial R_{n+1} such that

$$|R_{n+1}^*(z-\beta_{n+1})-R_n^*(z-\beta_n)| < \varepsilon/(2+|z|)^{2qn} \quad \forall z \in C \setminus V_n. \tag{9}$$

This completes our induction.

Now set $f_n(z)=Q_n^*(z-\alpha_n)+R_n^*(z-\beta_n)$ for $n=1, 2, \ldots$. Then (8) and (9) yield

$$|f_{n+1}(z)-f_n(z)| < 2\varepsilon/(2+|z|)^{2qn} \quad \forall z \in C \setminus (U_n \cup V_n). \tag{10}$$

It follows from (4), (5) and (10) that the rational functions f_n converge to an entire function f uniformly on each compact set. Moreover, we have

$$|f(z)-f_1(z)| < \sum_{n=1}^{\infty} \frac{2\varepsilon}{(2+|z|)^{2qn}} \leqq \frac{4\varepsilon}{(2+|z|)^{2q}} \quad \forall z \in C \setminus (U_1 \cup V_1). \tag{11}$$

Let g be the antiderivative of f with $g(0)=0$, and let g_1 be the antiderivative of f_1 with $g_1(0)=0$ in

$$C \setminus \bigcup_{j=1}^{2} \{t \exp[i\theta_j(t)]\colon\ t \geqq 1\}.$$

By Lemma 2, there exist distinct complex numbers c_1 and c_2 such that

$$g_1(z) = c_j+o(|z|^{-q}) \quad \text{as} \quad z \in \Omega_j \quad \text{tends to} \quad \infty \tag{12}$$

for $j=1, 2$. By the definition of f_1 and (11), we can find $C>1$ so large that $|f(z)| < C/|z|^{2q}$ for all z in $C \setminus (U_1 \cup V_1)$ with $|z|>C$. It follows from (two applications of) Lemma 1 and (1)—(5) that there exist two complex numbers b_1 and b_2 such that

$$g(z) = b_j+o(|z|^{-q}) \quad \text{as} \quad z \in \Omega_j^* \quad \text{tends to} \quad \infty \tag{13}$$

for $j=1, 2$. We claim that $b_1 \neq b_2$, provided that $\varepsilon>0$ is small enough.

Indeed, let τ_j and γ_j be as in the proof of Lemma 1:

$$\tau_j(t) = \{\theta_j(t)+\theta_{j+1}(t)\}/2 \quad \text{and} \quad \gamma_j(t) = te^{i\tau_j(t)} \quad \text{for} \quad t \geqq 0.$$

According to (1)—(5) we have

$$\gamma_j(0, \infty) \subset \Omega_j^* \subset C \setminus (U_1 \cup V_1), \quad j = 1, 2. \tag{14}$$

Since $g(0)=g_1(0)=0$, it follows from (11) and (14) that $r>0$ implies

$$(15)\qquad |g(\gamma_j(r))-g_1(\gamma_j(r))| = \left|\int_0^r \{f(\gamma_j(t))-f_1(\gamma_j(t))\}\gamma_j'(t)\,dt\right|$$
$$< 4\varepsilon\int_0^\infty \frac{1+t(|\theta_1'(t)|+|\theta_2'(t)|)}{(2+|t|)^{2q}}\,dt = 4\varepsilon B, \quad \text{say.}$$

Notice that B is a finite constant which is independent of ε. Letting $r\to\infty$ in (15) we obtain from (12) and (13) that $|b_j-c_j|\leqq 4\varepsilon B$ for $j=1,2$. Hence $|b_1-b_2|\geqq |c_1-c_2|-8\varepsilon B>0$, provided that $\varepsilon>0$ is small enough, which confirms our claim. Upon setting $h=\alpha+\beta g$ for appropriate coefficients α and β, we therefore obtain an entire function h such that

$$(16)\qquad h(z)=\begin{cases} a+o(|z|^{-q}) & \text{as } z\in E_1 \text{ tends to } \infty, \text{ and}\\ b+o(|z|^{-q}) & \text{as } z\in E_2 \text{ tends to } \infty,\end{cases}$$

where a and b are arbitrary, but preassigned, complex numbers.

Finally let P_1 and P_2 be two given polynomials. Write

$$P_1(z)=\sum_{k=0}^M a_k z^k \quad\text{and}\quad P_2(z)=\sum_{k=0}^M b_k z^k.$$

Choose a natural number $q>M+N+p$ and entire functions h_k which behave as in (16) with $a=a_k$ and $b=b_k$ $(k=0,1,\dots,M)$. Put $F(z)=\sum_0^M z^k h_k(z)$. It is evident that F has the required properties.

Proof of the Corollary. Let f be any continuously differentiable, positive real-valued function on $(0,\infty)$ for which there exists $p>0$ such that

$$(1)\qquad x+f'(x)f(x)>0 \quad\text{for all}\quad x\geqq p,$$

and

$$(2)\qquad \int_p^\infty |f'(x)x-f(x)|\cdot(x^2+f(x)^2)^{-p/2-2}\,dx<\infty.$$

The graph of such a function is essentially a curve of the kind which bounds spiral regions. To see this, define two functions t and θ of x by setting $t(x)=(x^2+f(x)^2)^{1/2}$ and $\theta(x)=\arctan[f(x)/x]$. Then $dt/dx=[x+f'(x)f(x)]/t>0$ for all $x\geqq p$ by (1), so t has a continuously differentiable inverse on $[b,\infty)$, where $b=t(p)$. Accordingly, θ may be regarded as a function of $t\geqq b$. A simple calculation shows that

$$\int_b^\infty \left|\frac{d\theta}{dt}\right| t^{-p}\,dt = \int_p^\infty \left|\frac{d\theta}{dx}\right| t^{-p}\,dx$$
$$= \int_p^\infty |f'(x)x-f(x)|\,(x^2+f(x)^2)^{-p/2-2}\,dx,$$

which is finite by (2). Thus (after an extension to $(0,b)$) the function θ belongs to S_p.

It is evident that e^x satisfies (1) and (2) for all $p>0$. Put

$$(3)\qquad H=\{(x,y)\colon x\geqq 1 \text{ and } 2^{-1}e^x\leqq y\leqq 2e^x\}.$$

By the result established in the preceding paragraph, $C\setminus H$ is essentially contained in a spiral region Ω. (More precisely, $C\setminus H$ is contained in the union of Ω and a bounded set.) Given a natural number N, our theorem yields an entire function g such that $g(z)=o(|z|^{-N})$ as $z\in\Omega$ tends to ∞. Indeed, such a g can be chosen so that $g(z)=1+o(|z|^{-N})$ as $z\to\infty$ along the curve $y=e^x$ for $x>0$.

Now let Γ be an arbitrary algebraic curve. Thus, by definition, there exist finitely many polynomials $Q_0, Q_1, \ldots, Q_n$ in x, with $Q_n\neq 0$, such that each point (x, y) of Γ satisfies

$$Q_0(x)+Q_1(x)y+\ldots+Q_n(x)y^n = 0. \tag{4}$$

It is easily seen from (3) that if (x, y) is in H and x or y is large enough, then (x, y) does not satisfy (4). In other words, only a bounded portion of Γ lies outside Ω; hence $g(z)=o(|z|^{-N})$ as $z\to\infty$ along Γ. It follows that, given a polynomial P, the entire function $g+P$ has the required properties.

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