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ON FUNCTIONS WITH A FINITE OR LOCALLY BOUNDED DIRICHLET INTEGRAL

MATTI VUORINEN

1. Introduction

The modulus of a curve family is an effective tool in the geometric function theory and in the theory of quasiconformal and quasiregular mappings (cf. [1], [13], [18], [19]). The aim of this paper is to show that the modulus method applies to the study of real-valued functions as well. We shall give an approach to the theory of Dirichlet finite functions, which is based on the well-known connection between the Dirichlet integral of an ACL^n function u and the modulus of the family of curves joining given level sets of u. In this approach the modulus method has a role similar to that of the length-area principle in [7] and [17]. We shall extend some results, which were previously known in the case of quasiconformal mappings, to the case of Dirichlet finite functions (cf. [21], [24]).

A function $u\colon R^n_+ \to R$ is said to have a finite Dirichlet integral (or to be Dirichlet finite) if u is ACL^n and if

$$\int_{R^n_+} |\nabla u|^n \, dm < \infty. \tag{1.1}$$

In this paper all functions are required to be continuous. We shall consider the following weaker condition: u is ACL^n and there are numbers $M, B \in (0, \infty)$ such that

$$\int_{D(x,M)} |\nabla u|^n \, dm \leqq B \tag{1.2}$$

for all $x \in R^+_n$, where $D(x, M)$ is the hyperbolic ball in R^n_+ with the centre x and radius M. If (1.2) holds, u is said to have a locally bounded Dirichlet integral.

In the preliminary Section 2 we prove, using the modulus method, that functions of R^n_+ which are monotone (in the sense of Lebesgue) and have a locally bounded Dirichlet integral are uniformly continuous with respect to the hyperbolic metric of R^n_+. In Section 3 we prove that a monotone function satisfying (1.1) and having a limit α at 0 through a set $E \subset R^n_+$, has in fact an angular limit α at 0 provided that the lower capacity density of E at 0 is positive, $\mathrm{cap}\, \underline{\mathrm{dens}}\, (E, 0) > 0$. Similar results for some other classes of functions were given in [21] and [22]. An example is given to show that the condition $\mathrm{cap}\, \overline{\mathrm{dens}}\, (E, 0) > 0$ is not sufficient here. As an application of the results of Section 3 we can prove that a Dirichlet finite quasiregular mapping

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$f\colon R^n_+ \to R^n$, $f=(f_1, \ldots, f_n)$, will have an angular limit $(\alpha_1, \ldots, \alpha_n)$ at 0 if each coordinate function f_j has a limit α_j through a set E_j with cap $\underline{\text{dens}}\,(E_j, 0)>0$. A related result for bounded analytic functions was proved by F. W. Gehring and A. J. Lohwater [4]. In Section 4 we shall give an example illustrating the behaviour of a monotone function satisfying (1.2). The topic of Section 5 is the behaviour of a monotone function at an isolated singularity. In the final section, Section 6, we prove a variant of the Iversen—Tsuji theorem for monotone Dirichlet finite functions.

2. Preliminary results

We shall follow, as a rule, the notation and terminology of Väisälä's book [18], which the reader is referred to for some definitions etc. Some notation will be introduced at first.

2.1. For $x\in R^n, n\geq 2$, and $r>0$, let $B^n(x, r)=\{z\in R^n\colon |z-x|<r\}$, $S^{n-1}(x, r)=\partial B^n(x, r)$, $B^n(r)=B^n(0, r)$, $S^{n-1}(r)=\partial B^n(r)$, $B^n=B^n(1)$, and $S^{n-1}=\partial B^n$. If $x\in R^n$ and $b>a>0$, then we write $R(x, b, a)=B^n(x, b)\setminus \bar{B}^n(x, a)$ and $R(b, a)=R(0, b, a)$. The standard coordinate unit vectors are $e_1, \ldots, e_n$. If $A\subset R^n$, let $A_+=\{x=(x_1, \ldots, x_n)\in A\colon x_n>0\}$. The hyperbolic metric ϱ in R^n_+ is defined by the element of length $d\varrho=|dx|/x_n$. If $x\in R^n_+$ and $M>0$, we write $D(x, M)=\{z\in R^n_+\colon \varrho(z, x)<M\}$. A well-known fact is that the hyperbolic balls are balls in the euclidean geometry as well, for instance

$$D(te_n, M) = B^n\big((t\cosh M)e_n, t\sinh M\big), \quad t>0. \tag{2.2}$$

For $x, y\in R^n_+$ the following formula holds [2, (3.3.4) p. 35]:

$$\cosh \varrho(x, y) = 1+\frac{|x-y|^2}{2x_n y_n}. \tag{2.3}$$

Sometimes we shall regard B^n as a hyperbolic space as well and use the same symbols as in the case of R^n_+. The hyperbolic metric ϱ is then defined by $d\varrho=2|dx|/(1-|x|^2)$. The counterpart of (2.2) for B^n is

$$D(x, M) = B^n(y, r)\begin{cases} y = \dfrac{x(1-\tanh^2(M/2))}{1-|x|^2\tanh^2(M/2)} \\[2ex] r = \dfrac{(1-|x|^2)\tanh(M/2)}{1-|x|^2\tanh^2(M/2)}. \end{cases} \tag{2.4}$$

2.5. *Monotone functions.* Let $G\subset R^n$ be an open set. A continuous function $u\colon G\to R$ is said to be *monotone* (in the sense of Lebesque) if

$$\max_{\bar{D}} u(x) = \max_{\partial D} u(x), \quad \min_{\bar{D}} u(x) = \min_{\partial D} u(x),$$

whenever D is a domain with compact closure $\bar{D}\subset G$.

2.6. Remark. It follows from the above definition that if $t\in R$, then each component $A\neq\emptyset$ of the set $\{z\in G\colon u(z)>t\}$ is not relatively compact, i.e., $\bar{A}\cap(\partial G\cup\{\infty\})\neq\emptyset$. A similar statement holds if $>$ is replaced by $\geqq$, $<$, or $\leqq$. Hence monotone functions obey a sort of maximum principle.

2.7. *The modulus of a curve family.* Let Γ be a family of curves in R^n. The *modulus* of Γ is defined by

$$M(\Gamma)=\inf_{\varrho\in F(\Gamma)}\int_{R^n}\varrho^n\,dm,$$

where $F(\Gamma)$ is the family of all non-negative Borel functions $\varrho\colon R^n\to R^1\cup\{\infty\}$ with $\int_\gamma\varrho\,ds\geqq 1$ for all locally rectifiable $\gamma\in\Gamma$. For the properties of the modulus the reader is referred to Väisälä's book [18]. If $E, F, G\subset R^n$, then $\Delta(E, F; G)$ is the family of all curves $\gamma\subset G$ joining E to F in the following sense: $\bar{\gamma}\cap E\neq\emptyset\neq\bar{\gamma}\cap F$.

2.8. *Condenser and its capacity.* A pair (A, C) is said to be a *condenser* if $A\subset R^n$ is open and $C\subset A$ is compact. The capacity of $E=(A, C)$ is defined by

(2.9) $$\operatorname{cap} E=\operatorname{cap}(A, C)=\inf_u\int_{R^n}|\nabla u|^n\,dm,$$

where u runs through the set of all ACL^n functions with compact support in A and with $u(x)\geqq 1$ for $x\in C$. An alternative definition is

(2.10) $$\operatorname{cap}(A, C)=M(\Delta(C, \partial A; R^n))=M(\Delta(C, \partial A; A)).$$

Ziemer [26] proved that (2.9) and (2.10) agree for bounded A, from which the same conclusion for unbounded A follows easily. A compact set E in R^n is said to be of *capacity zero* if $\operatorname{cap}(B^n(t), E)=0$ for some $t>0$ such that $E\subset B^n(t)$.

The following lower bound for the Dirichlet integral of an ACL^n function will be applied several times in what follows. In fact, it is the easy part in the proof of (2.10). The proof is standard (cf. [1, p. 65], [9, p. 577], [26, Lemma 3.1]).

2.11. Lemma. *Let $u\colon G\to R$ be an* ACL^n *function,* $-\infty<a<b<\infty$, *and let $A, B\subset G$ be such that $u(x)\leqq a$ for $x\in A$ and $u(x)\geqq b$ for $x\in B$. Then*

$$M(\Delta(A, B; G))\leqq(b-a)^{-n}\int_G|\nabla u|^n\,dm.$$

2.12. Remark. In classical complex analysis one often uses a different capacity. The connection between the classical capacity and the above one has been studied in [1, p. 70] and in [13].

We next give a proof of the fact that a monotone ACL^n function with a locally bounded Dirichlet integral is uniformly continuous with respect to the hyperbolic metric. For the sake of technical reasons we shall consider functions defined in B^n, but with small modifications one can prove a similar result for functions defined in R^n_+.

2.13. Lemma. *Let* $u: B^n \to R$ *be a monotone* ACL^n *function satisfying* (1.2). *Then*

$$|u(x)-u(y)|^n \leqq C\left(\log\frac{1}{r}\right)^{-1}(1-r)^{1-n},$$

where $r=\tanh\left(\varrho(x,y)/4\right)$ *and* C *is a positive constant depending only on* n, M, *and* B *in* (1.2). *In particular,* $u: (B^n, \varrho) \to (R, |\ |)$ *is uniformly continuous.*

Proof. Clearly we may assume $u(x)<u(y)$. Since the right side depends only on x and y through the Möbius invariant quantity $\varrho(x,y)$, we may assume $x=re_1=-y$, $r=\tanh\left(\varrho(x,y)/4\right)$ (cf. (2.4)). Let

$$E=\{z\in B^n\colon\ u(z)\leqq u(x)\},$$

$$F=\{z\in B^n\colon\ u(z)\geqq u(y)\},$$

$$\Gamma_r=\Delta\left(E, F; B^n(\sqrt{r})\right).$$

Then by Remark 2.6 and [18, 10.12]

$$M(\Gamma_r)\geqq c_n\log\frac{1}{\sqrt{r}}.$$

Lemma 2.11 yields

$$M(\Gamma_r)\leqq |u(x)-u(y)|^{-n}\int_{B^n(\sqrt{r})}|\nabla u|^n\,dm.$$

In view of (1.2) the integral can be estimated from above in terms of B and the following number

$$k=\inf\left\{p\colon\ \overline{B^n(\sqrt{r})}\subset\bigcup_{j=1}^{p}D(x_j, M)\right\}.$$

An upper bound for k can be found by a method involving estimation of the hyperbolic volume of $\cup\{D(x,M)\colon\ |x|\leqq\sqrt{r}\}$. (For details see [25, Section 9]). This method yields the estimate

$$\int_{B^n(\sqrt{r})}|\nabla u|^n\,dm\leqq d(1-\sqrt{r})^{1-n},$$

where d is a positive number depending only on n, M, and B. The desired estimate with $C=2^n d/c_n$ follows from this and the preceding estimates.

The uniform continuity in 2.13 can also be proved with the help of an oscillation inequality of Gehring [3, p. 355]. (See also Lelong—Ferrand [7, p. 7]). For n-dimensional version of the oscillation inequality see Mostow [12]. The oscillation inequality is applied in the proof of [18, 10.12], which was exploited above.

2.14. Remark. For large values of $\varrho(x,y)$ the above upper bound is not sharp. Indeed, one can replace the factor $(\log(1/r))^{-1}(1-r)^{1-n}$ in 2.13 by $(1+\log((1+r)/(1-r)))\cdot\mathrm{const.}$, which yields a better estimate for large values of

$\varrho(x, y)$. Such an estimate can be deduced from the above version of Lemma 2.13, in particular, from the fact that u is uniformly continuous.

A continuous function $v\colon R^n_+ \to R_+ \cup \{0\}$, $R_+ = \{x \in R\colon x > 0\}$, is said to satisfy a *Harnack inequality* if there exist numbers $\lambda \in (0, 1)$ and $C_\lambda \geqq 1$ such that (cf. [23])

$$\max_{B^n(x,\lambda r)} v(z) \leqq C_\lambda \min_{B^n(x,\lambda r)} v(z) \tag{2.15}$$

whenever $B^n(x, r) \subset R^n_+$. The next result should be compared with [5, p. 200].

2.16. Corollary. *If $u\colon R^n_+ \to R$ is a monotone function satisfying* (1.2), *then e^u satisfies* (2.15) *for every $\lambda \in (0, 1)$ with*

$$\log C_\lambda = C^{1/n}\left(\log\frac{1}{\lambda}\right)^{-1/n}\left(\frac{1+\lambda}{1-\lambda}\right)^{(n-1)/(2n)},$$

where C is the number in 2.13.

Proof. Fix $\lambda \in (0, 1)$. Choose $B^n(x, r) \subset R^n_+$. Then $r \leqq x_n$, where $x = (x_1, \ldots, x_n)$, and $\varrho(\bar{B}^n(x, \lambda r)) \leqq \log((1+\lambda)/(1-\lambda))$ by (2.2) or (2.3). For $z, y \in \bar{B}^n(x, \lambda r)$ we get by 2.13 and by some elementary inequalities that

$$|u(z) - u(y)| \leqq C^{1/n}\left(\log\frac{1}{\lambda}\right)^{-1/n}\left(\frac{1+\lambda}{1-\lambda}\right)^{(n-1)/(2n)},$$

from which (2.15) for e^u follows.

We shall now give examples of functions satisfying (1.2) in B^n.

2.17. *The function u_F.* Let F be a relatively closed subset of B^n. For $x \in B^n$ set (cf. [23, 3.6])

$$u_F(x) = \exp(-\varrho(x, F)).$$

As shown in [23], the function u_F has some extremal properties for appropriate choices of F. For $x, y \in B^n$ and $\varrho(x, F) \leqq \varrho(y, F)$ we get

$$|u_F(x) - u_F(y)| \leqq |e^{-\varrho(x,F)} - e^{-\varrho(y,F)}| = e^{-\varrho(x,F)}|1 - e^{-a}|$$
$$\leqq e^{-\varrho(x,F)} a \leqq e^{-\varrho(x,F)} \varrho(x, y) \leqq \varrho(x, y),$$

where $a = \varrho(y, F) - \varrho(x, F)$. Similarly $|u_F(x) - u_F(y)| \leqq \varrho(x, y)$ for $\varrho(x, F) \geqq \varrho(y, F)$ as well. Next we apply [22, 2.11] to get

$$\varrho(x, y) \leqq \log\left(1 + \frac{|x-y|}{d(x) - |x-y|}\right) \leqq \frac{|x-y|}{d(x) - |x-y|},$$

for $y \in B^n(x, d(x))$ where $d(x) = d(x, \partial B^n)$. Therefore

$$\limsup_{y \to x} \frac{|u_F(x) - u_F(y)|}{|x-y|} \leqq \limsup_{y \to x} \frac{1}{d(x) - |x-y|} = \frac{1}{d(x)}.$$

It follows that u_F is locally Lipschitz continuous and hence ACL^n. Fix $M>0$. Then we get by (2.4) that

$$\int_{D(x,M)} |\nabla u_F(x)|^n \, dm(x) \leqq \int_{D(x,M)} d(x)^{-n} \, dm(x) \leqq c_1(n, M).$$

In conclusion, u_F satisfies (1.2). Note, however, that u_F is usually not monotone.

2.18. Remark. It follows from [5, p. 23, formula (2.32)] and (2.4) that all bounded harmonic functions satisfy (1.2).

3. Behaviour at an individual boundary point

For $\varphi \in (0, \pi/2)$ let $C(\varphi)=\{y \in R^n_+ : (y|e_n) > |y| \cos \varphi\}$, where $(z|u)$ is the inner product $\sum z_i u_i$. A function $u: R^n_+ \to R$ is said to have an *angular limit* α at 0 if, for each $\varphi \in (0, \pi/2)$, $\lim_{x \to 0, x \in C(\varphi)} u(x) = \alpha$. A function u is said to have an *asymptotic value* α at 0 if there exists a continuous curve γ: $[0, 1) \to R^n_+$ with $u(\gamma(t)) \to \alpha$ and $\gamma(t) \to 0$ as $t \to 1$.

3.1. Lemma. *Let* u: $R^n_+ \to R$ *be a monotone* ACL^n *function satisfying* (1.2) *and let* $E \subset R^n_+$ *be a measurable set such that*

$$\lim_{r \to 0} m((R^n_+ \setminus E) \cap B^n(r)) r^{-n} = 0.$$

If $u(x) \to \alpha$ *as* $x \to 0$ *and* $x \in E$, *then* u *has an angular limit* α *at* 0.

Proof. The proof follows from 2.13 and [23, 6.13].

3.2. Remarks. (1) It is not difficult to show that the assumption in 3.1 is equivalent to the condition that u has an approximate limit α at 0 [23, 6.3]. For the definition of an approximate limit see [23, 6.1]. For a related result see J. Lelong—Ferrand [7, p. 16] and [21, 5.9].

(2) The monotone ACL^2 function u: $R^2_+ \to R$, $u(x, y) = \overline{\text{arc}} \tan(y/x)$, satisfies (1.2) (but not (1.1)) and has infinitely many distinct asymptotic values at 0 but no angular limit at 0. Hence an approximate limit in the hypotheses of 3.1 cannot be replaced by an asymptotic value.

For $E \subset R^n$, $x \in R^n$, and $t > r > 0$ set

$$M_t(E, r, x) = M(\Delta(S^{n-1}(x, t), \ E \cap \bar{B}^n(x, r); \ R^n)),$$
$$M(E, r, x) = M_{2r}(E, r, x).$$

The *lower* and *upper capacity densities* of E at x are defined by

$$\text{cap}\,\underline{\text{dens}}\,(E, x) = \liminf_{r \to 0} M(E, r, x),$$
$$\text{cap}\,\overline{\text{dens}}\,(E, x) = \limsup_{r \to 0} M(E, r, x).$$

3.3. Lemma. *Let* $u: (R^n_+, \varrho) \to (R, |\,|)$ *be uniformly continuous, let* $b_k \in R^n_+$, $b_k \to 0$, *and let* $u(b_k) \to \beta \neq \infty$. *For every* $\varepsilon > 0$ *there exists* $M \in (0, \infty)$ *and* $p \geqq 1$ *such that*

$$|u(x) - \beta| < \varepsilon, \quad x \in E_M = \bigcup_{k \geqq p} D(b_k, M).$$

Moreover, if there exists $\varphi \in (0, \pi/2)$ *such that* $b_k \in C(\varphi)$ *for all k, then there exists a positive number c depending only on n,* φ, *and M such that*

$$\text{cap}\, \overline{\text{dens}}\, (E_M, 0) \geqq c > 0.$$

Proof. The first part follows from the definition of uniform continuity. For the proof of the second part we assume $b_k \in C(\varphi)$, $k = 1, 2, \ldots$. It follows from (2.2) that $F_M \subset E_M$,

$$F_M = \bigcup_{k \geqq p} B^n(b_k, b_{kn}(1 - e^{-M})),$$

where $b_{kn} \geqq |b_k| \cos \varphi$ is the n-th coordinate of b_k. Hence the proof follows from [18, 10.12]. For more details see [21, 2.5 (2)].

3.4. Theorem. *Let* $u: (R^n_+, \varrho) \to (R, |\,|)$ *be a uniformly continuous and Dirichlet finite function and let* $E \subset R^n_+$ *be a set with* cap $\underline{\text{dens}}\, (E, 0) > 0$. *If* $u(x) \to \alpha$ *as* $x \to 0$, $x \in E$, *then u has an angular limit* α *at* 0.

Proof. Fix $\varphi \in (0, \pi/2)$. Suppose, on the contrary, that there exists a sequence (b_k) in $C(\varphi)$ with $b_k \to 0$ and $u(b_k) \to \beta \neq \alpha$. We shall assume that $-\infty < \alpha < \beta < \infty$; in other cases the proof is similar. Let B_k be the b_k-component of the set $B = \{z \in R^n_+ : u(z) > (\alpha + 2\beta)/3\}$ and let $A = \{z \in R^n_+ : u(z) < (2\alpha + \beta)/3\}$. Since u is uniformly continuous, there exist by Lemma 3.3 numbers $M > 0$ and $p \in \mathbf{N}$ such that $D(b_k, M) \subset B_k$ for $k \geqq p$ and

$$\text{cap}\, \overline{\text{dens}}\, (B, 0) \geqq c > 0.$$

Since cap $\underline{\text{dens}}\, (E, 0) > 0$, it follows from [21, 4.3] and [20, 3.8] that

$$M(\Delta(A, B; R^n_+)) \geqq M(\Delta(A, B; R^n))/2 = \infty.$$

A contradiction follows from (1.1) and 2.11.

3.5. Remark. The condition cap $\underline{\text{dens}}\, (E, 0) > 0$ is satisfied for instance if E is a curve terminating at 0. This fact follows from [18, 10.12]; for details and other sufficient conditions for cap $\underline{\text{dens}}\, (E, 0) > 0$ see [21]. On the other hand, there are compact sets E of zero Hausdorff dimension with cap $\underline{\text{dens}}\, (E, 0) > 0$ (cf. [21, 2.5 (3)]).

We shall now construct an example showing that the condition cap $\overline{\text{dens}}\, (E, 0) > 0$ would not suffice in 3.4.

3.6. Example. There exists a monotone ACL^n function $u\colon R^n_+\to R$ with a finite Dirichlet integral such that for a sequence $r_k\to 0$ with $0<r_{k+1}<r_k$

(a) $$u\Big|\bigcup_{k=1}^{\infty} S^{n-1}_+(r_{2k-1}) = 1,$$

(b) $$u\Big|\bigcup_{k=1}^{\infty} S^{n-1}_+(r_{2k}) = 0.$$

Let $r_1=1$. Select $r_{k+1}\in(0, r_k/2)$, $k=1,2,\ldots$, such that

(3.7) $$\left(\log\frac{r_k}{r_{k+1}}\right)^{1-n} < 2^{-k}.$$

Define $u\colon R^n_+\to R$ by $u(x)=1$ for $x\in R^n_+\setminus B^n$ and

$$u(x) = \frac{\log|x|-\log r_{2k}}{\log(r_{2k-1}/r_{2k})};\quad r_{2k-1}>|x|\geqq r_{2k},\ x\in R^n_+,$$

$$u(x) = \frac{\log r_{2k}-\log|x|}{\log(r_{2k}/r_{2k+1})};\quad r_{2k}>|x|\geqq r_{2k+1},\ x\in R^n_+,$$

for $k=1,2,\ldots$. It follows from (3.7) that (1.1) holds (cf. [18, 7.5]). Clearly u is monotone and satisfies (a) and (b). In addition, $\operatorname{cap}\overline{\operatorname{dens}}\,(B,0)>0$, $B=\bigcup_{k=1}^{\infty}S^{n-1}_+(r_{2k})$ and $u(z)\to 0$, as $z\to 0$, $z\in B$, and u fails to have an asymptotic value and hence an angular limit at 0. Therefore the condition $\operatorname{cap}\underline{\operatorname{dens}}\,(E,0)>0$ in Theorem 3.4 cannot be replaced by $\operatorname{cap}\overline{\operatorname{dens}}\,(E,0)>0$.

3.8. Theorem. *Let $u\colon R^n_+\to R$ be a monotone and Dirichlet finite function, let $(b_k)\subset R^n_+$ be a sequence with $\varrho(b_k,b_h)\geqq 4M$ for $k\neq h$, and let $a_k\in D(b_k,M)$. For each $\varepsilon>0$ let $P_\varepsilon=\{k\in N\colon |u(a_k)-u(b_k)|\geqq\varepsilon\}$. Then*

$$\operatorname{card} P_\varepsilon \leqq A\varepsilon^{-n}\int_{R^n_+}|\nabla u|^n\,dm,$$

where A is a positive number depending only on n and M.

Proof. Fix $\varepsilon>0$. Let

$$A_k = \{z\in R^n_+\colon |u(z)-u(a_k)|<\varepsilon/3\},$$
$$B_k = \{z\in R^n_+\colon |u(z)-u(b_k)|<\varepsilon/3\},$$
$$\Gamma_k = \Delta(A_k, B_k; D(b_k, 2M)),\quad k\in P_\varepsilon.$$

From 2.6 it follows that the a_k-component of A_k meets $\partial D(b_k,2M)$, and that so does the b_k-component of B_k, when $k\in P_\varepsilon$. It follows from the conformal invariance of the modulus [18, 8.1], from $\varrho(a_k,b_k)<M$, and from (2.4) that

$$M(\Gamma_k)\geqq c_n\log\frac{\tanh M}{\tanh(M/2)}\geqq c_n(\log 2)e^{-M}$$

holds for $k\in P_\varepsilon$. Let $A=\bigcup_{k\in P_\varepsilon} A_k$, $B=\bigcup_{k\in P_\varepsilon} B_k$ and $\Gamma=\Delta(A, B; R^n_+)$. Since $\varrho(b_k, b_h)\geqq 4M$ for $k\neq h$, it follows that $\{\Gamma_k\colon k\in P_\varepsilon\}$ are separate subfamilies of Γ. Hence we get by [18, 6.7]

$$\sum_{K\in P_\varepsilon} M(\Gamma_k) \leqq M(\Gamma) \leqq \left(\frac{3}{\varepsilon}\right)^n \int_{R^n_+} |\nabla u|^n \, dm.$$

The desired upper bound for card P_ε follows from this and the preceding inequality with $A=(3^n e^M)/(c_n \log 2)$.

3.9. Remark. A similar result holds for a uniformly continuous Dirichlet finite function $u\colon (R^n_+, \varrho)\to(R, |\;|)$ as well, but with a more complicated dependence on the Dirichlet integral and the modulus of continuity of the function.

3.10. Theorem. *Let* $u\colon (R^n_+, \varrho)\to(R, |\;|)$ *be a uniformly continuous and Dirichlet finite function, let* $(b_k)\subset R^n_+$, $b_k\to 0$, $u(b_k)\to\beta$ *and let* $M\in(0, \infty)$. *Then* $u(x)\to\beta$ *as* $x\to 0$, $x\in \bigcup D(b_k, M)$.

Proof. In the case of monotone functions the proof follows from 3.8. The general case follows from 3.9.

The next theorem has its roots in [24], where a similar result was proved for quasiconformal mappings. We shall omit the proof since it parallels the proofs of Theorems 3.8 and that of [24, 4.9]. For the statement of the theorem the following condition is needed. Let (a_k) and (b_k) be sequences in R^n_+ tending to 0 and let $J_k=J[a_k, b_k]$ be the closed geodesic segment in the hyperbolic geometry joining a_k with b_k. Thus J_k is the arc between a_k and b_k on a circle through a_k and b_k, which is orthogonal to ∂R^n_+. Suppose that there exists a positive number M such that

$$\varrho(J_k, J_h) \geqq M > 0 \quad \text{for} \quad k \neq h. \tag{3.11}$$

As in [24, 4.9], condition (3.11) is needed to guarantee that some curve families are separate.

3.12. Theorem. *Let* (a_k) *and* (b_k) *be sequences in* R^n_+ *tending to* 0 *and satisfying* (3.11), *let* $u\colon (R^n_+, \varrho)\to(R, |\;|)$ *be uniformly continuous and Dirichlet finite, and let* $u(a_k)\to\alpha$, $u(b_k)\to\beta$. *If* $\sum \varrho(a_k, b_k)^{1-n}=\infty$, *then* $\alpha=\beta$.

Proof. The proof is similar to the proof of 3.8 and [24, 4.9]. The details are left to the reader.

3.13. Remarks. (1) Theorem 3.4 was proved by V. M. Mikljukov [10] in the case when the set E is a curve and the mapping is vector-valued and of class BL. One variant of Theorem 3.10 for these mappings was given by G. D. Suvorov [17, p. 122].

(2) It should be observed that Theorem 3.12 follows from Theorem 3.10 in the case $\liminf \varrho(a_k, b_k)<\infty$.

(3) We shall next show that condition (3.11) cannot be removed from the hypotheses of Theorem 3.12 even in the case of continuous mappings (and, a fortiori, in the case of uniformly continuous Dirichlet finite functions). Let $u\colon R^n_+\to R$ be a continuous function such that $u(\tilde{a}_k)\to 0$, $u(\tilde{b}_k)\to 1$, where $\tilde{a}_k, \tilde{b}_k\to 0$. We shall construct two new sequences (a_k) and (b_k) with $\{a_j\}=\{\tilde{a}_k\}$ and $\{b_j\}=\{\tilde{b}_k\}$ for which (3.11) fails to hold and for which $\sum \varrho(a_k, b_k)^{1-n}=\infty$. To this end, let $p_1=1$, $p_{j+1}>p_j$ be an increasing sequence of integers such that

$$(p_{k+1}-p_k)\varrho(\tilde{a}_k, \tilde{b}_k)^{1-n} > 1/k$$

for all $k=1, 2, \ldots$. Set

$$a_j = \tilde{a}_i \quad \text{and} \quad b_j = \tilde{b}_i \quad \text{if} \quad p_i \leqq j < p_{i+1}.$$

Then $a_k, b_k\to 0$ and $\sum \varrho(a_k, b_k)^{1-n}=\infty$ hold while $u(a_k)\to 0$, $u(b_k)\to 1$.

We show that Theorem 3.12 is sharp in a sense.

3.14. Theorem. *Let (b_k) be a sequence in R^n_+ with $|b_{k+1}|<|b_k|$, $b_k\to 0$, let $a_k=|b_k|e_n$, and suppose that (3.11) holds. If $\sum \varrho(a_k, b_k)^{1-n}<\infty$, then there exists a monotone Dirichlet finite function $u\colon R^n_+\to R$ having an angular limit 0 at 0 and satisfying $u(b_k)\to 1$.*

Proof. Since $a_k=|b_k|e_n$, we obtain by (3.11), (2.2), and (2.3) that

$$\varrho(J_k, J_{k+1}) = \log\frac{|b_k|}{|b_{k+1}|} \geqq M > 0.$$

It follows that the annuli $R(|b_k|\lambda, |b_k|/\lambda)$ are disjoint when $\lambda=e^{M/2}$. Let $w_k\in\partial R^n_+$ be a unit vector such that $b_k-b_{kn}e_n=cw_k$, where b_{kn} is the n-th coordinate of b_k and c is a positive number such that $|b_k|^2=c^2+b_{kn}^2$. The balls $B_k=B^n(|b_k|w_k, r_k)$, $r_k=|b_k|(1-1/\lambda)$ are then disjoint. Let $t_k=\big|b_k-|b_k|w_k\big|$. It follows from (2.3) that

$$t_k < 2|b_k|\exp\left(-\varrho(a_k, b_k)\right)$$

(for more details see [23, (2.4)]). Since $\sum\varrho(a_k, b_k)^{1-n}<\infty$, it follows that $\varrho(a_k, b_k)\to\infty$ as $k\to\infty$. By relabelling and passing to a subsequence if necessary we may hence assume, in view of the above estimate for t_k, that $t_k<r_k$ for all k. Choose now a monotone ACL^n function u_k such that (cf. 3.6)

$$u_k\,|\,R^n_+\setminus B_k = 0, \quad u_k\,|\,R^n_+\cap\bar{B}^n(|b_k|w_k, t_k) = 1,$$

$$d_k = \int_{R^n_+}|\nabla u_k|^n\,dm = \frac{\omega_{n-1}}{2}\left(\log\frac{r_k}{t_k}\right)^{1-n}.$$

There exist numbers k_0 and $c(n, M)$ such that for $k\geqq k_0$

$$d_k \leqq c(n, M)\varrho(a_k, b_k)^{1-n}.$$

Set $u=\sum_{k\geqq k_0}u_k$. Then u is monotone, ACL^n, $u(b_k)\to 1$, $u(te_n)=0$, $t>0$ and u has a finite Dirichlet integral, as desired.

We shall next give some applications of the preceding results to the theory of quasiregular mappings. A continuous ACL^n mapping $f\colon R^n_+ \to R^n$ is called *quasiregular* (qr) if there exists a constant $K \in [1, \infty)$ such that

$$\sup_{|h|=1} |f'(x)\,h|^n \leqq K J_f(x)$$

a.e. in R^n_+, where J_f is the Jacobian determinant of f. A sense-preserving homeomorphism is quasiregular if and only if it is quasiconformal (qc). For the basic parts of the theory of qc and qr mappings the reader is referred to [15], [18], [19]. For the following result see Rešetnjak's book [15, p. 118].

3.15. Lemma. *The coordinate functions $f_1, \dots, f_n$ of a* qr *mapping $f\colon R^n_+ \to R^n$, $f=(f_1, \dots, f_n)$, are monotone.*

3.16. Theorem. *Let $f\colon R^n_+ \to R^n$ be a* qr *mapping with*

$$\int_{R^n_+} |\nabla f_j|^n \, dm < \infty, \quad j = 1, \dots, n.$$

If $f_j(x) \to \alpha_j$ as $x \to 0$, $x \in E_j$ and cap $\underline{\mathrm{dens}}\,(E_j, 0) > 0$, *then f_j has an angular limit α_j at 0, $j = 1, 2, \dots, n$.*

Proof. The proof follows from 3.15, 2.13, and 3.4.

3.17. Remarks. For bounded analytic functions a result similar to 3.16 holds without a condition about finite Dirichlet integral (Gehring—Lohwater [4]). In the case of bounded qr mappings $f\colon R^n_+ \to R^n$ such a condition is, however, necessary if $n \geqq 3$. This fact follows from an example due to Rickman [16].

4. On the behaviour at a typical boundary point

In this section we shall study the behaviour of a Dirichlet finite function at a "typical" boundary point. We shall employ the following result of Rešetnjak [14].

4.1. Lemma. *Let $u\colon R^n_+ \to R$ be an* ACL^n *function with a finite Dirichlet integral. Then there exists a set $E \subset \partial R^n_+$ such that every compact set F in E is of zero n-capacity and such that u has an essential value at every point of $\partial R^n_+ \setminus E$, i.e., for every $x \in \partial R^n_+ \setminus E$ there exists a number α with*

$$\lim_{r \to 0} r^{-n} \int_{B^n_+(x, r)} |f(y) - \alpha| \, dm = 0.$$

4.2. Theorem. *Let $u\colon R^n_+ \to R$ be a monotone* ACL^n *function with a finite Dirichlet integral. Then u has an angular limit at every point of $\partial R^n_+ \setminus E$, where E is as in* 4.1.

Proof. Since u has an essential value at the points of $\partial R^n_+ \setminus E$, it has an approximate limit as well by [23, 6.7 (1)]. By 3.1 and 3.2 (1) it has an angular limit, too.

We next show that 4.2 fails to hold for monotone functions satisfying (1.2) but not (1.1).

4.3. Example. There exists a bounded monotone ACL^2 function $u\colon R^2_+ \to R$ satisfying condition (1.2), having an asymptotic value at each point of a dense subset of ∂R^n_+, but having no angular limits.

Divide the square $Q=[0,1]\times(0,1]\subset R^2_+$ into four equal squares by joining the midpoints of opposite sides with (euclidean) segments. Repeat the division in those resulting squares which have one side on the x-axis. By continuing this process we get a division of Q into closed squares Q^j_i: $i=1,2,\dots$, $j=1,\dots,2^i$ of constant hyperbolic size, where Q^j_i has euclidean side-length 2^{-i}. Join the center of Q^j_i by (euclidean) segments to the centres of those two adjacent squares in $\{Q^j_{i+1}\colon j=1,2,\dots,2^{i+1}\}$ each of which has a side lying on a side of Q^j_i, for each i and j. As a result we get two distinct "treelike" infinite polygonal curves approaching the x-axis. The union of these curves will be denoted by T.

Define $u(x)=0$ if x is located on a side A of a square Q^j_i and $A\cap T=\emptyset$ and $u(y)=1$ if $y\in T$. In $(\operatorname{int} Q)\setminus T$ define u in such a way that $u\colon Q\to R\cup[0,1]$ will be monotone, have all partial derivatives, continuous in $\bigcup(\operatorname{int} Q^j_k)\setminus T$ and

$$|\nabla u(z)| \leqq 2^{k+3} \quad \text{for} \quad z\in(\operatorname{int} Q^j_k)\setminus T, \tag{4.4}$$

$j=1,2,\dots,2^k$. Extend the domain of definition of u to R^2_+ as follows. If $\operatorname{Im} z>1$, set $u(z)=0$. If $p\in Z$ and $z\in Q+\{(p,0)\}$, then $z-(p,0)\in Q$; set $u(z)=u(z-(p,0))$. Then u is defined in R^2_+, has an asymptotic value 1 at the points of $\overline{T}\cap\partial R^2_+\setminus\{\infty\}$ through the set T and is monotone, and it follows from (4.4) that (1.2) holds. Moreover, it is clear that u has no angular limits.

5. On isolated singularities and Phragmén—Lindelöf-type behaviour

A function with a finite Dirichlet integral need not have a limit at an isolated singularity. To see this fact we may consider the function in Example 3.5 and extend it by reflection in ∂R^n_+ to a map $v\colon R^n\setminus\{0\}\to R$ with a finite Dirichlet integral and with no limit at 0. This function is not, however, monotone although $v|R^n_+$ indeed is monotone.

5.1. Theorem. *Let $u\colon R^n\setminus\{0\}\to R$ be a monotone ACL^n function. If u has no limit at* 0, *then*

$$\liminf_{t\to 0} \int_{R(1,t)} |\nabla u|^n \, dm/\log\frac{1}{t} > 0.$$

Proof. Suppose that there are sequences $\{a_k\}$, $\{b_k\}$ in $B^n\setminus\{0\}$ with $a_k, b_k\to 0$ and $u(a_k)\to\alpha$, $u(b_k)\to\beta\neq\alpha$. We may assume $-\infty<\alpha<\beta<\infty$. Let A_k be the a_k-component of the set

$$A=\{z\in R^n\setminus\{0\}\colon\ u(z)\leqq(2\alpha+\beta)/3\},$$

and B_k the b_k-component of the set

$$B=\{z\in R^n\setminus\{0\}\colon\ u(z)\geqq(2\beta+\alpha)/3\}.$$

By 2.6, $\bar{A}_k\cap\{0,\infty\}\neq\emptyset\neq\bar{B}_k\cap\{0,\infty\}$ for all large k. There is a sequence (j_k) such that either $0\in\bar{A}_{j_k}$ for all j_k or $\infty\in\bar{A}_{j_k}$ for all j_k. Consider the first case, the proof being similar in the second case. For $t\in(0,1)$ set

$$\Gamma_t=\Delta(B, A_{j_1};\ R(1,t)).$$

Suppose that $0\in\bar{B}_k$ for some k such that $|b_k|<|a_{j_1}|$. Then we get by [18, 10.12]

$$M(\Gamma_t)\geqq c_n\log\frac{|b_k|}{t}=c_n\log|b_k|+c_n\log\frac{1}{t};\quad t<|b_k|. \tag{5.2}$$

Otherwise $\infty\in\bar{B}_k$ for all k such that $|b_k|<|a_{j_1}|$ and thus $S^{n-1}(r)\cap B\neq\emptyset$ for all $r\in(0,|a_{j_1}|)$, because $b_k\to 0$ (cf. 2.6). Hence (5.2) holds in this case for all $t\in(0,|a_{j_1}|)$ by [18, 10.12]. Lemma 2.11 yields

$$M(\Gamma_t)\leqq\left(\frac{3}{\beta-\alpha}\right)^n\int_C|\nabla u|^n\,dm,$$

where $C=R(1,t)$. This estimate together with (5.2) gives the desired lower bound.

5.3. Corollary. *Let $u\colon R^n\setminus\{0\}\to R$ be a bounded monotone* ACL^n *function and let $\alpha=\liminf_{x\to 0}u(x)$, $\beta=\limsup_{x\to 0}u(x)$. Then*

$$\liminf_{t\to 0}\Big(\int_{R(1,t)}|\nabla u|^n\,dm\Big)\Big/\log\frac{1}{t}\geqq c_n(\beta-\alpha)^n,$$

where c_n is the positive constant in the proof of 5.1.

A counterpart of condition (1.2) for the ACL^n function $u\colon R^n\setminus\{0\}\to R$ is the following one. There are constants $\mu\in(0,1)$ and $A>0$ such that

$$\int_{B_x}|\nabla u|^n\,dm\leqq A,\quad B_x=\bar{B}^n(x,\mu|x|) \tag{5.4}$$

for all $x\in R^n\setminus\{0\}$. From a standard covering argument (cf. [25]) and from (5.4) it follows that

$$\int_{R(t,t/2)}|\nabla u|^n\,dm\leqq d(n,A,\mu) \tag{5.5}$$

for $t\in(0,1)$, where $d(n,A,\mu)$ depends only on n, A and μ. Furthermore, it follows from (5.5) that for $t\in(0,1/2)$

$$(5.6)\qquad \int_{R(1,t)} |\nabla u|^n \, dm \leqq c(n, A, \mu) \log\frac{1}{t}.$$

A direct calculation shows that the monotone ACL^2 function $v(x,y)=y^2/(x^2+y^2)$, $(x,y)\in R^2\setminus\{0\}$ satisfies (5.4) and (5.6), but v fails to have a limit at 0. This example should be compared with 5.1.

According to Theorem 5.1 a monotone function with a finite Dirichlet integral has a limit at an isolated singularity. A natural question is whether a similar result holds for a countable sequence of isolated singularities.

5.7. Example. There is a monotone ACL^n function $u\colon R^n\setminus\{2^{-k}e_1\colon k=1,2,\ldots\}\setminus\{0\}\to R$ with

$$\lim_{x\to 2^{-k}e_1} u(x) = 1, \quad \lim_{t\to 0+} u(-te_1) = 0$$

$k=1,2,\ldots$ with a finite Dirichlet integral. The existence of such a function u can be seen by a direct construction. Clearly u has no limit at 0.

The next result is a Phragmén—Lindelöf type theorem.

5.8. Theorem. *Let $G\subset R^n$ be a domain such that $M(R^n\setminus G, r, 0)\geqq\delta>0$ for all $r\geqq r_0$, and let $u\colon G\to R$ be a monotone ACL^n function. If*

$$\limsup_{x\to y} u(x) \leqq 1$$

for all $y\in\partial G\setminus\{\infty\}$, then either $u(x)\leqq 1$ for all $x\in G$ or

$$\liminf_{t\to\infty} \int_{G\cap B^n(t)} |\nabla u|^n \, dm/\log t > 0.$$

Proof. Suppose that $u(x_0)=c>1$ for some $x_0\in G$. Let $E=\{x\in G\colon u(x)<(2+c)/3\}$. Then $\partial G\subset\bar{E}$ by the assumption. Let F be the x_0-component of $\{z\in G\colon u(z)>(1+2c)/3\}$. Then $\infty\in\bar{F}$ by 2.6. Let

$$\Gamma_t = \Delta(E, F; G\cap B^n(t)), \quad t\geqq r_0,$$

$$\tilde{\Gamma}_t = \Delta(E, F; B^n(t)), \quad t\geqq r_0.$$

By the geometry of the situation it follows that $M(\Gamma_t)=M(\tilde{\Gamma}_t)$ (cf. [18, 11.3] and (2.10)). From [20, 3.5] we obtain

$$M(\tilde{\Gamma}_t) \geqq c(n,\delta)\log t$$

for large values of t. The proof follows from Lemma 2.11.

6. Some properties of boundary values

Next we shall compare the limit values of a monotone Dirichlet finite function on the closure of its domain of definition to the limit values on the boundary.

6.1. Theorem. *Let* $u\colon R^n_+ \to R$ *be a monotone Dirichlet finite function and let* $E \subset \partial R^n_+$ *be a compact set of capacity zero with* $0 \in E$. *Then*

$$\limsup_{x\to 0} u(x) = \limsup_{x\to 0} \Big(\limsup_{\substack{y\to x \\ x\in \partial R^n_+ \setminus E}} u(y) \Big).$$

Proof. Since $\operatorname{cap} E = 0$, it follows that $0 \in \overline{(\partial R^n_+ \setminus E)}$ ([15, p. 72]) and hence the right side of the above equality makes sense. Denote the left and right sides by $\tilde{a}$ and $\tilde{b}$, respectively. Clearly $\tilde{a} \geqq \tilde{b}$. Hence it remains to be shown that $\tilde{a} > \tilde{b}$ is impossible. Choose a and b such that $\tilde{b} < b < a < \tilde{a}$. Let $r > 0$ be such that

$$\limsup_{y\to x} u(y) < b \tag{6.2}$$

for all $x \in (\partial R^n_+ \setminus E) \cap B^n(r)$. Choose a sequence (a_k) in $B^n_+(r)$ with $u(a_k) > a$ and $|a_k| < r/k$. Let A_k be the a_k-component of the set $\{z \in R^n_+ \colon u(z) > a\}$. It follows from 2.6 that $\bar{A}_k \cap (\partial R^n_+ \cup \{\infty\}) \neq \emptyset$ for all k. From (6.2) it follows that $\bar{A}_k \cap (E \cup (\partial R^n_+ \setminus B^n(r))) \neq \emptyset$ for all k. Let $B = \{z \in R^n_+ \colon u(z) < b\}$ and $\Gamma_k = \Delta(A_k, B; R^n_+)$. It follows from 2.11 that

$$M(\Gamma_k) \leqq (a-b)^{-n} \int_{R^n_+} |\nabla u|^n \, dm < \infty. \tag{6.3}$$

If $\bar{A}_k \cap (E \cap B^n(r)) \neq \emptyset$, then $M(\Gamma_k) = \infty$ because A_k is a connected set and $\operatorname{cap} E = 0$ (cf. [18, 10.12]). Otherwise $\bar{A}_k \cap (\partial R^n_+ \setminus B^n(r)) \neq \emptyset$, and since $\operatorname{cap} E = 0$ and A_k is connected, we get by [18, 10.12] that

$$M(\Gamma_k) \geqq c_n \log k.$$

In either case we obtain a contradiction with (6.3) when $k \to \infty$.

6.4. Remark. By inspecting the above proof we see that the condition $\operatorname{cap} E = 0$ can be weakened. In fact, it suffices to assume that $E \subset \partial R^n_+$ is a compact set which has no interior points (in the topology of ∂R^n_+) and which satisfies $M(y, \partial R^n_+ \setminus E) = \infty$ for all $y \in E$ in the sense of [9].

6.5. *A bound for a Dirichlet finite function.* Let $u\colon R^n_+ \to R$ be a monotone Dirichlet finite function, let $E \subset \bar{R}^n_+$, and let u have a continuous extension, denoted by u, to the points $E \cap \partial R^n_+$ such that $u(x) \leqq b$ for $x \in E$. Define

$$\sigma(x, E) = \inf_C M(\Delta(C, E; R^n_+)), \tag{6.6}$$

where the infimum is taken over all continua C with $x \in C$ and $C \cap (\partial R^n_+ \cup \{\infty\}) \neq \emptyset$.

It follows then that

$$u(x) \leqq b + \left[\left(\int_{R^n_+} |\nabla u|^n \, dm\right) / \sigma(x, E)\right]^{1/n} \tag{6.7}$$

for all $x \in R^n_+$. This estimate follows directly from Lemma 2.11 and Remark 2.6. In fact, this idea has been applied several times in this paper. The inequality (6.7) suggests that the quantity $\sigma(x, E)$ is of some interest in the theory of Dirichlet finite functions.

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University of Helsinki
Department of Mathematics
SF—00100 Helsinki 10
Finland

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