Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 10, 1985, 47—56

Commentationes in honorem
Olli Lehto
LX annos nato

RAMIFICATION OF KLEIN COVERINGS

CABIRIA ANDREIAN CAZACU

The aim of this paper is to extend the Hurwitz formula on the ramification of the covering surfaces [7], [8], as well as its generalizations by S. Stoilow [11], [12], ourselves [3] and I. Bârză [6] from Riemann coverings to Klein coverings. In what follows we shall use definitions and notations due to N. L. Alling and N. Greenleaf [2].

1. Let $\mathscr{X}=(X, \mathscr{A})$ and $\mathscr{Y}=(Y, \mathscr{B})$ be connected *Klein surfaces,* orientable or non-orientable, with or without border: X and Y will be connected two-manifolds with countable bases (surfaces) $\mathscr{A}=\{(U, z)\}$ and $\mathscr{B}=\{(W, w)\}$ dianalytic atlasses, B_X and B_Y the border (boundary) of X and Y, respectively, [2], Section 2.

A *morphism of Klein surfaces* T: $\mathscr{X}\to\mathscr{Y}$ is a continuous mapping T: $X\to Y$, with the properties that $T(B_X)\subset B_Y$ and that for all points $P\in X$ there exist dianalytic charts $(U, z)\in\mathscr{A}$ and $(W, w)\in\mathscr{B}$ about P and $p=T(P)$, respectively, and an analytic function F on $z(U)\subset \boldsymbol{C}^+=\{z\in \boldsymbol{C}\colon y\geqq 0\}$, such that $T|U=w^{-1}\circ\varphi\circ F\circ z$. Here φ: $\boldsymbol{C}\to\boldsymbol{C}^+$ is the folding mapping $\varphi(z)=x+i|y|$ that folds $\boldsymbol{C}$ over $\boldsymbol{C}^+$, [2], Section 4.

Evidently, if $B_Y=\emptyset$, it follows $B_X=\emptyset$ and one may give up φ, but if $B_Y\neq\emptyset$, even for Riemann surfaces, i.e., for X and Y orientable surfaces, this concept of morphism differs from the classical one ([9], I, II) since it permits the folding over B_Y. Starting from Stoilow's topological theory of Riemann surfaces [12], it was natural to compare this concept with that of the *interior transformation* (continuous, open and 0-dimensional (light) mapping) and we proved [5]:

Theorem 1. *Non-constant morphisms of Klein surfaces are topologically equivalent to interior transformations in the sense of Stoilow.*

It is obvious that a non-constant morphism is an interior transformation, but Stoilow's methods extend, so that we generalized his local inversion theorem for interior mappings T: $X\to Y$ between surfaces X and Y as above. Namely, if $P\in X\setminus T^{-1}(B_Y)$, then in the neighbourhood of P the map T is topologically equivalent to $w=z^h$, and if $P\in T^{-1}(B_Y)$, to $w=\varphi\circ z^h$, which corresponds to the local *normal form* of a morphism ([2], p. 30). If T: $X\to Y$ is an interior map, and if we organize Y with a dianalytic structure as a Klein surface $\mathscr{Y}$, this structure is lifted by means of T in a unique way to X yielding a Klein surface $\mathscr{X}$ such that T becomes a morphism $\mathscr{X}\to\mathscr{Y}$.

doi:10.5186/aasfm.1985.1007

On the model of Stoilow's definition of Riemann covering ([12], Chapter II, I, 2 and Chapter V, III, 4) we call each triple, (X, T, Y) where T: $X \to Y$ is an interior transformation between the surfaces X and Y a *Klein covering.*

The problem of the ramification of a Klein covering being of topological nature we shall not further refer to the Klein structure of our surfaces X and Y, which we always suppose to be connected and of finite Euler characteristic ϱ_X and ϱ_Y, respectively.

2. Let us first shortly resume the previous results: the Hurwitz formula and its generalizations in the case of the coverings without folds, i.e., $T^{-1}(B_Y) = B_X$, in particular in the case $B_Y = \emptyset$.

1) *Hurwitz formula* ([7], p. 54) is proved in Kerékjártó's book ([8], p. 158—159) for both orientable and non-orientable, compact surfaces with or without border X and Y under the hypothesis $T^{-1}(B_Y) = B_X$. The ramification number r of the relatively unbordered n-sheeted covering T: $X \to Y$ satisfies the relation

$$r = \varrho_X - n\varrho_Y. \tag{1}$$

(See also Ahlfors' generalization of this formula [1], p. 168, and [10], p. 324.)

In Stoilow's theory X and Y are orientable surfaces without border and the Hurwitz formula holds for the *total* covering, which is realized by any interior mapping T: $X \to Y$ with the following property: for each infinite sequence of points $P_\nu \in X$ which tends to the ideal boundary ∂X of X (i.e., has no accumulation point in X, notation: $P_\nu \to \partial X$), its projection $p_\nu = T(P_\nu)$ tends to ∂Y. This is in particular the case when X and Y are compact ([12], Chapter VI, II—III).

2) In 1933, [11], Stoilow extended formula (1) by introducing the *partially regular* covering T: $X \to Y$, characterized by the existence of a finite family of mutually disjoint Jordan curves γ on Y with the following properties: (β_1) for each sequence $P_\nu \to \partial X$, its projection $p_\nu \to \gamma \cup \partial Y$ (i.e., p_ν has accumulation points only on γ), and (β_2) the set $T^{-1}(\gamma)$ is either compact or empty. Then the family γ decomposes Y into a finite number of regions Y_i of finite characteristic ϱ_i and totally covered with n_i sheets by $T^{-1}(Y_i)$. For such a partially regular covering Stoilow obtained the formula ([11], [12], Chapter VI, IV)

$$r = \varrho_X - \sum n_i \varrho_i. \tag{2}$$

3) In 1960, [3], we considered — again in the orientable, unbordered case — a family γ on Y, consisting of a finite number of mutually disjoint Jordan curves and of a finite number of Jordan arcs with the end points in well-determined points on Y, which we called knots, or elements of ∂Y. An arc of γ can meet another arc or curve of γ only in a knot. The set of knots will be denoted by $\mathcal{N}$.

We supposed that the interior transformation T: $X \to Y$ satisfies the condition (β_1) but we gave up the condition (β_2) and introduced the local condition $(L\beta_2)$: a point $p \in \gamma$ satisfies $(L\beta_2)$ if there exists a neighbourhood v of p such that the preimage of the component of $v \cap \gamma$, which contains p, is either relatively compact in X

or empty. In this way it was natural to define the *exceptional points* $p \in \gamma$ which do not satisfy $(L\beta_2)$, and we proved that the set $\mathscr{E}$ of these points is finite or empty.

The surface Y is again decomposed by γ into a finite number of regions Y_i of finite characteristic ϱ_i and totally covered by n_i sheets.

We chose a family γ' of mutually disjoint Jordan curves from γ, which contains all Jordan curves of $\gamma \setminus (\mathscr{E} \cup \mathscr{N})$*). Let $\mathscr{E}_{\gamma'}$ be the set of exceptional points of γ' with respect to itself (i.e., the set of the points $p \in \gamma'$ for which $T^{-1}(v \cap \gamma')$ is not relatively compact in X for every neighbourhood v of p). The curves γ' with $\mathscr{E}_{\gamma'} \neq \emptyset$ are decomposed by $\mathscr{E}_{\gamma'}$ into open Jordan arcs and similarly $(\gamma \setminus \gamma') \setminus (\mathscr{E} \cup \mathscr{N})$ consists of open Jordan arcs. We called these arcs *cross cuts* and denoted them by γ_j'', the family contained in γ' by $\{\gamma_j''\}^1$ and the rest by $\{\gamma_j''\}^2$. Every cross cut γ_j'' is covered by s_j sheets.

Further, we denoted by $\nu(p)$ the number of sheets of the covering over a point $p \in Y$**), designated by p_k the points of the set $\mathscr{E}_{\gamma'} \cup (\mathscr{E} \cup \mathscr{N} \setminus \gamma')$ and wrote $\nu(p_k) = \nu_k$.

Then we obtained the following generalization of the Hurwitz–Stoilow formula (2):

$$r = \varrho_X - \sum n_i \varrho_i - \sum s_j + \sum \nu_k. \tag{3}$$

Its importance is due to its wide application possibilities, for instance to the regions of the exhaustion of a Riemann surface and thus to the study of the ramification of Riemann coverings in general [4].

4) The formulae (2) and (3) remain valid for non-orientable surfaces without border, as it was proved by I. Bârză [6].

3. We shall now consider the *general case* of a Klein covering $T: X \to Y$ and obtain in Theorem 2 the formula (4), which, assuming the particular hypotheses presented above, reduces to the results 1)—4). It thus remains to concentrate on the case $T^{-1}(B_Y) \neq B_X$, X and Y being orientable or non-orientable, $B_Y \neq \emptyset$ but B_X being empty or not. It should be mentioned that even in the case $T^{-1}(B_Y) = B_X$ formula (4) will bring new information since the assertions 2)—4) have been established only for $B_Y = \emptyset$ but according to (4) they are also true if $B_Y \neq \emptyset$.

3.1. Since X and Y have finite characteristics, we can represent them by means of homeomorphisms which do not influence the ramification of the Klein covering $T: X \to Y$ as subsets of compact surfaces $\mathbf{X}$ and $\mathbf{Y}$ with the same genus as X and Y. Here $\mathbf{X}$ is orientable if and only if X is orientable; the same holds for $\mathbf{Y}$ and Y. Under these homeomorphisms the ideal boundaries of X and Y correspond to a finite set of points $F_X \subset x$ and $F_Y \subset y$ and their borders to a finite family of mutually disjoint Jordan curves and arcs B_X on $\mathbf{X}$ and B_Y on $\mathbf{Y}$, the arcs ending in points of F_X and F_Y, respectively. For X we shall write $B_X = B_X^1 \cup B_X^2$ and $F_X = F_X^1 \cup F_X^2$, where B_X^1 contains the Jordan curves of B_X and B_X^2 its Jordan arcs,

*) We denote by γ the family of curves and arcs, one curve or arc of the family as well as the set of the points of these curves and arcs; the same remark holds about γ'.

**) $\nu(p)$ is the number of points in $T^{-1}(p)$ counted with their multiplicities.

F_X^1 the isolated points of $B_X \cup F_X$ and F_X^2 the end points of the arcs from B_X^2. Moreover — without influencing the ramification of the covering — we can suppose ([5]) that $B_X^2 \cup F_X^2$ consists of a finite number of mutually disjoint Jordan curves on X, each point of F_X^2 giving exactly two end points of arcs of B_X^2. Similar notations will be used for Y.

3.2. On Y we shall consider a family γ like that of 3) but we let arcs from γ have end points on $B_Y \cup F_Y$. The points of $\gamma \cap B_Y$ will also be called knots and the set of all the knots will again be designated by $\mathcal{N}$. The covering (X, T, Y) will satisfy the condition (β_1) for $\gamma \cup B_Y \cup F_Y$. Further, $\mathcal{E}$ will designate the set of the exceptional points of γ. Obviously $\mathcal{E} \cap B_Y \subset \mathcal{N}$. Besides $\mathcal{E}$ the covering can present exceptional points on B_Y relatively to B_Y itself. For such a point $p \in B_Y$ the pre-image $T^{-1}(v \cap B_Y)$ is not relatively compact for any neighbourhood v of p. The set of these points will be denoted by E.

3.3. Another important set will be $\mathcal{R}$ — the set of the projections of the ramification points of the covering T: $X \to Y$.

The new type of coverings we are now considering may be characterized by one of the following equivalent conditions:

(i) $T^{-1}(B_Y) \neq B_X$, and

(ii) the covering presents folds over B_Y.

If $\mathcal{R} \cap B_Y \neq \emptyset$, then these conditions are fulfilled.

Before writing formula (4) we shall discuss in 3.4 and 3.5 the two new aspects that occur for the covering.

3.4. *The case* $\mathcal{R} \cap B_Y \neq \emptyset$. Let p be a point of B_Y. Its pre-image $T^{-1}(p)$ consists of points $Q_j \in \mathring{X} = X \setminus B_X$, $j = 1, \ldots, \mathfrak{i}(p)$, with the multiplicity h_j (i.e., where locally T is topologically equivalent to the mapping $w = \varphi \circ z^{h_j}$), and of points $P_j \in B_X$, $j = 1, \ldots, \mathfrak{b}(p)$, with the multiplicity k_j (i.e., where locally T is topologically equivalent to one of the mappings $w = \varphi \circ z^{k_j}$ or $w = \varphi \circ (-z^{k_j})$).

Let v be a sufficiently small open neighbourhood of p and l, l^* the two open Jordan arcs in which p divides the component of $v \cap B_Y$ that contains it. For each Q_j the corresponding component of $T^{-1}(v)$ is a normal region (in Stoilow's sense [12], Chapter V, II) which covers v under T with $2h_j$ sheets and $l \cup l^*$ with h_j folds. Similarly, for each P_j the corresponding component of $T^{-1}(v)$ is a normal region and covers v under T with k_j sheets and $l \cup l^*$ with $(k_j - 1)$ folds and 2 borders. More precisely, if k_j is odd, each of the arcs l and l^* will be covered by $(k_j - 1)/2$ folds and a border, and if k_j is even, one of them will be covered by $k_j/2$ folds and the other by $(k_j - 2)/2$ folds and 2 borders.

In order to preserve the form (1) of the Hurwitz formula for total coverings, we shall define the ramification order of a point Q_j as usual by $(h_j - 1)$ but the ramification order of a point P_j by $(k_j - 1)/2$,*) so that the ramification of the covering at

*) This definition has also an interpretation in connection with the double coverings.

p will be

$$r(p) = \sum_{j=1}^{i(p)} (h_j - 1) + \frac{1}{2} \sum_{j=1}^{b(p)} (k_j - 1).$$

A special role will be played in what follows by $\mathscr{R}_{1/2} = \{p \in \mathscr{R} \cap B_Y: h_j = 1$ for each $Q_j \in T^{-1}(p) \cap \mathring{X}$ and $k_j = 1$ or 2 for each $P_j \in T^{-1}(p) \cap B_X$, with at least one $k_j = 2\}$. If $p \in \mathscr{R}_{1/2}$, then the number of the points P_j with $k_j = 2$ in $T^{-1}(p) \cap B_X$ will be $2r(p)$. Thus $2r(p)$ represents also the number of the folds that cover one of the arcs l or l^* but transform at p into two borders covering the arc l^* or l, respectively.

At a point $p \in \mathring{Y} = Y \setminus B_Y$ the ramification will be taken as usual

$$r(p) = \sum_{j=1}^{i(p)} (h_j - 1),$$

where $T^{-1}(p) = \{Q_1, \dots, Q_{i(p)}\} \subset \mathring{X}$ and Q_j has the multiplicity h_j (i.e., locally T is topologically equivalent to $w = z^{h_j}$) and the ramification order $(h_j - 1)$.

Evidently, $p \in \mathscr{R}$ if and only if $r(p) > 0$ and the ramification number of the covering $r = \sum_{p \in \mathscr{R}} r(p)$.

3.5. Folds "ending at a point" of $E \cup F_Y^2$. If $p \in E$, there are folds or borders (at least one) which cover one of the arcs l or l^* without covering p. Let $f(p)$ and $b(p)$, respectively, be the numbers of these folds and borders. Such a fold (border) represents an asymptotic way in $\mathring{X}$ (on B_X) for T with the asymptotic value p and will be called fold or border "ending at p".

The same notations $f(p)$ and $b(p)$ will be introduced for the points $p \in F_Y^2$. (If we do not suppose, as indicated in 3.1, that $B_Y^2 \cup F_Y^2$ consists of Jordan curves and in a sufficiently small neighbourhood v of p one has exactly two arcs l and l^* of B_Y^2 ending at p, then p can be an end point for more than two arcs like l or for a single one, and $f(p)$ as well as $b(p)$ will be the numbers of folds or borders "ending at p" over all these arcs. The sum $\sum_{p \in F_Y^2} f(p)$ is independent of this [5].)

3.6. With these remarks and notations we can formulate

Theorem 2. *Let* T: $X \to Y$ *be a Klein covering,* γ *a family of curves and arcs as in* 3) *but which, if* $B_Y \neq \emptyset$, *may have end points on* B_Y, *too, and suppose that T satisfies the condition* (β_1) *with respect to* $\gamma \cup B_Y \cup F_Y$ *(if* $P_\nu \to B_X \cup F_X$, *then* $p_\nu \to \gamma \cup B_Y \cup F_Y$*). The family* γ *decomposes Y into the regions* Y_i *with the characteristics* ϱ_i *and number of the sheets* n_i: $Y \setminus (\gamma \cup B_Y) = \cup Y_i$. *We choose a family* γ' *of mutually disjoint Jordan curves from* γ, *so that it contains all those curves from* γ *without exceptional points with respect to themselves (i.e., if a curve* γ *has* $\mathscr{E}_\gamma = \emptyset$, *then it will be included in the family* γ'*).*) As in* 3) *we determine the cross cuts* γ_j'' *covered by* s_j *sheets. Further, let* p_k *be the points of the set* $\mathscr{E}_{\gamma'} \cup [(\mathscr{E} \cup \mathscr{N}) \setminus (\gamma' \cup B_Y)]$ *and* q_l *the points of* $E \cup F_Y^2$, $\nu_k = \nu(p_k)$

*) There is a difference between the actual construction of γ' and that of 3) but this does not influence the result.

and $f_l = f(q_l)$. *Then the ramification number of the covering is given by the formula*

$$r = \varrho_X - \sum n_i \varrho_i - \sum s_j + \sum \nu_k - \tfrac{1}{2} \sum f_l. \tag{4}$$

Remark 1. We can admit that γ contains, besides curves and arcs as before, a finite number of points $p \in \mathring{Y}$ such that for each neighbourhood v, its pre-image $T^{-1}(v)$ is not relatively compact in X. These points will be interpreted as curves γ' reduced to a point and be considered in $\mathscr{E}_{\gamma'}$, hence denoted by p_k. Formula (4) holds in this case too, with the mention that the sum $\Sigma \nu_k$ contains also the terms corresponding to these points p_k.

Remark 2. Evidently (4) refers also to the case of the total covering with folds: $\gamma = \emptyset$. If $f(p) = 0$ for each $p \in B_Y \cup F_Y^2$, then (4) takes the form (1). This case may also be proved directly by counting the simplexes of the covering as for the classical Hurwitz formula ([10], p. 324). This was done in [2, p. 43], for the case of the unramified double covering of a compact Klein surface.

The proof of (4) will be given in the next two sections, 4. and 5., by adapting the method we used in [3] in order to prove (3). Ahlfors' formula for the addition of the characteristics will remain the main tool.

The finiteness of the sets $\mathscr{E}$, E, $\mathscr{R}$ as well as the total covering and the finite numbers of sheets n_i, s_j, ν_k over each Y_i, γ_j'', p_k, respectively, and the finiteness of the numbers f_l are proved by similar devices as in [3]. (For details see [5].)

4. Let us first prove Theorem 2 in *the special case* $\mathscr{R} \cap (\gamma \cup B_Y) = \mathscr{R}_{1/2}$ and $\mathscr{E} \cup \mathscr{N} \subset \gamma' \cup B_Y$.

Besides the cross cuts $\{\gamma_j''\}$, we shall also consider a family of cross cuts $\{B_m''\}$ on B_Y, namely, the family of Jordan arcs which appear on B_Y when one takes out the points of $\mathscr{R}_{1/2} \cup E$. These arcs have their end points in the set $M = \mathscr{R}_{1/2} \cup E \cup F_Y^2$ and each B_m'' is covered by σ_m sheets.

We denote by X_u and X_{ui}^v the components of $X \setminus T^{-1}(\gamma' \cup B_Y)$ and $X \setminus T^{-1}(\gamma \cup B_Y)$, respectively, where X_{ui}^v are the components of $X_u \cap T^{-1}(Y_i)$, and remark that $(X_{ui}^v, T|, Y_i)$ is a total covering in Stoilow's sense, so that we can apply the Hurwitz formula (1) to get

$$r_{ui}^v = \varrho_{ui}^v - n_{ui}^v \varrho_i, \tag{5}$$

where r_{ui}^v is the ramification of the covering, n_{ui}^v the number of sheets and ϱ_{ui}^v the characteristic of X_{ui}^v.

If ϱ_u is the characteristic of X_u and N_1 or N_B, respectively, represents the number of cross cuts in the decomposition $X \setminus T^{-1}(\gamma' \cup B_Y) = \cup X_u$ which comes from $T^{-1}(\gamma')$ and $T^{-1}(B_Y)$, respectively (we remark that $\gamma' \cap B_Y = \emptyset$), then by Ahlfors' formula

$$\varrho_X = \sum \varrho_u + N_1 + N_B. \tag{6}$$

Further, let N_2 be the number of cross cuts which appear when we continue the decomposition: $\cup X_u \setminus T^{-1}(\{\gamma_j''\}^2) = \cup X_{ui}^v$. The same formula gives

$$(7) \qquad \sum \varrho_u = \sum \varrho_{ui}^v + N_2.$$

By repeating the device used in [3], we obtain again

$$(8) \qquad N_1 + N_2 = \sum s_j - \sum v_k$$

($N_1 = \sum^1 s_j - \sum v_k$, $N_2 = \sum^2 s_j$, where $\sum^1$ and $\sum^2$ extend to the families $\{\gamma_j''\}^1$ and $\{\gamma_j''\}^2$, respectively, and $p_k \in \mathscr{E}_{\gamma'}$).

In order to evaluate N_B we remark that a cross cut from $T^{-1}(\{B_m''\})$ determines a fold and counted with its end points contributes with 2 to the sum $\sum_{p \in M}(2r(p)+f(p))$. Therefore

$$(9) \qquad N_B = \tfrac{1}{2} \sum_{p \in M} (2r(p) + f(p)).$$

Since $r = \sum r_{ui}^v + \sum_{p \in M} r(p)$ and $\sum_{p \in M} f(p) = \sum f_l$, formula (4) follows from (5)—(9).

Remark 3. By the device used in the calculation of N_1, [3], one proves that

$$2(\sum \sigma_m - \sum_{p \in M} v(p)) = \sum_{p \in M} (2r(p) + f(p) + b(p)).$$

5. *The general case.* In order to obtain from the general Klein covering T: $X \to Y$ with the condition (β_1) for a family $\gamma \cup B_Y \cup F_Y$ a covering from the special case 4, we suitably modify the method used in [3], 8. Namely, we introduce the sets $\mathscr{R}^* = (\mathscr{R} \cap \gamma) \setminus (\mathscr{E} \cup \mathscr{N})$, $\mathscr{R}^{**} = (\mathscr{R} \cap B_Y) \setminus \mathscr{R}_{1/2}$ and $A = (\mathscr{R}^* \cup \mathscr{E} \cup \mathscr{N}) \setminus B_Y$, and we choose a set of sufficiently small open neighbourhoods v for the points $p \in A \cup \mathscr{R}^{**}$, such that the closed neighbourhoods $\bar{v}$ are mutually disjoint and the following conditions are fulfilled: For each $p \in A$, $\bar{v} \subset \mathring{Y}$, $\bar{v}$ is a Jordan domain bounded by a Jordan curve c; $(\bar{v} \setminus \{p\}) \cap (\mathscr{E} \cup \mathscr{N} \cup \mathscr{R}) = \emptyset$ and $\bar{v} \cap \gamma$ consists of a finite number of Jordan arcs with an end point at p and another on c; these arcs decompose v into sectors; any non-compact component of $T^{-1}(\bar{v})$ does not intersect $T^{-1}(p)$. For each $p \in \mathscr{R}^{**}$, $\bar{v} \setminus B_Y \subset \mathring{Y}$, $\bar{v} \cap B_Y$ is a Jordan arc apb while $\bar{v}$ is a Jordan domain bounded on y by $a\,p\,b$ and a Jordan arc c which is contained in $\mathring{Y}$ except for its end points a and b; $(\bar{v} \setminus \{p\}) \cap (\mathscr{E} \cup \mathscr{N} \cup \mathscr{R} \cup E) = \emptyset$, and $\bar{v} \cap \gamma$ has the same properties as in the case $p \in A$; any non-compact component of $T^{-1}(\bar{v})$ does not intersect $T^{-1}(p)$ and every compact component contains a single point of $T^{-1}(p)$.

First we take out of X the union of the non-compact components of $T^{-1}(\bar{v})$ for all $p \in A$ and obtain a surface X^*.

According to [3], 7, these components are either simply connected and separated from X^* by a cross cut or doubly connected and separated from X^* by a Jordan curve, such that we have

$$(10) \qquad \varrho_X = \varrho_{X^*} \quad \text{and} \quad r = r^*,$$

where ϱ_{X^*} means the characteristic of X^* and r^* the ramification number of the covering $(X^*, T|, Y)$.

Secondly, we take out of X^* the non-compact components of $T^{-1}(\bar{v})$ and the relatively compact components of $T^{-1}(v)$ for all $p\in\mathscr{R}^{**}$, obtaining a surface $\tilde{X}$ with the characteristic $\varrho_{\tilde{X}}$, and take out from Y the neighbourhoods v for all $p\in\mathscr{R}^{**}$, obtaining the surface $\tilde{Y}$.

By a direct computation we prove that the non-compact components of $T^{-1}(\bar{v})$, $p\in\mathscr{R}^{**}$, are again of the same type as described for $p\in A$.*) If $T^{-1}(p)=\{Q_1, \ldots, Q_{\mathrm{i}(p)}, P_1, \ldots, P_{\mathrm{b}(p)}\}$, $Q_j\in\mathring{X}^*$, $P_j\in B_{X^*}$, as in 3.4, the component of $T^{-1}(\bar{v})$ containing Q_j will be a Jordan domain included in $\mathring{X}^*$ and the component containing P_j will be homeomorphic to a half disc separated from $\tilde{X}$ by a Jordan arc projected by T on c. Therefore with Alhfors' formula we have

$$\varrho_{\tilde{X}} = \varrho_X + \textstyle\sum_{p\in\mathscr{R}^{**}} \mathrm{i}(p). \tag{11}$$

On the other hand, taking out a relatively compact component of $T^{-1}(v)$ for $p\in\mathscr{R}^{**}$ which contains a point Q_j with the multiplicity h_j, we have $2h_j$ new ramification points of order 1/2 projected over a and b. Similarly, if the component contains a point P_j with the multiplicity k_j, we have (k_j-1) new ramification points of order 1/2 projected over a and b.**) Therefore the ramification number $\tilde{r}$ of the covering $(\tilde{X}, T|, \tilde{Y})$ will be given by the relation

$$\tilde{r} = r + \textstyle\sum_{p\in\mathscr{R}^{**}} \mathrm{i}(p). \tag{12}$$

However, the covering $(\tilde{X}, T|, \tilde{Y})$ satisfies the condition (β_1) with respect to the family of curves and arcs $\tilde{\gamma}=(\gamma\setminus\bigcup_{p\in A\cup\mathscr{R}^{**}} v)\cup(\bigcup_{p\in A} c)$ and to $B_{\tilde{Y}}\cup F_{\tilde{Y}}$, the family $\tilde{\gamma}'$ consisting of the curves c for $p\in A$ and the former curves γ' which do not intersect A. Let us remark that $\mathscr{E}_c=\emptyset$ for each $p\in A$, hence $\mathscr{E}_{\tilde{\gamma}'}=\emptyset$. Further, $B_{\tilde{Y}}$ is obtained from B_Y replacing the arc apb by the corresponding arc c for each $p\in\mathscr{R}^{**}$. It follows that $F_{\tilde{Y}}^2=F_Y^2$. We denote by $\tilde{E}$ the set of the points of $B_{\tilde{Y}}$ which are exceptional with respect to $B_{\tilde{Y}}$. Obviously $\tilde{E}$ is obtained from E replacing each point $p\in\mathscr{R}^{**}\cap E$ by the corresponding pair $\{a, b\}$.

In order to apply (4) to the covering $(\tilde{X}, T|, \tilde{Y})$, let us denote by $\tilde{Y}_\lambda$ the components of $\tilde{Y}\setminus(\tilde{\gamma}\cup B_{\tilde{Y}})$, by $\tilde{\gamma}_\mu$ the cross cuts (of the type $\{\gamma_j''\}^2$) determined on $\tilde{\gamma}\setminus\tilde{\gamma}'$, by $\tilde{\varrho}_\lambda$ the characteristic of $\tilde{Y}_\lambda$, by $\tilde{n}_\lambda$ and $\tilde{s}_\mu$ the number of sheets over $\tilde{Y}_\lambda$ and $\tilde{\gamma}_\mu$, by $\tilde{q}$ a point of $\tilde{E}\cup F_{\tilde{Y}}^2$ and by $\tilde{f}(\tilde{q})$ the number of folds of $(\tilde{X}, T|, \tilde{Y})$ "ending at $\tilde{q}$". In this

---

*) Let $\mathscr{V}$ be the interior of such a component. The covering $(\mathscr{V}, T|, v)$ satisfies (β_1) with respect to $(\gamma\cap v)\cup\partial v$ and v is decomposed by the arcs from $\gamma\cap v$ into sectors. The number of the sheets over a sector is at least equal to the number of the sheets over each arc from $\gamma\cap v$ on its boundary and at least twice the number of the folds over one of the arcs pa and pb on its boundary. The preimages of these arcs and of pa and pb give cross cuts of $\mathscr{V}$. The assertion follows using Ahlfors' formula for the characteristic of $\mathscr{V}$ and the Hurwitz formula (1) for each covering of the sectors.

**) One sees why it was necessary to consider $\mathscr{R}_{1/2}$ in Section 4.

way formula (4) gives

$$\tilde{r} = \varrho_{\tilde{X}} - \sum \tilde{n}_\lambda \tilde{\varrho}_\lambda - \sum \tilde{s}_\mu - \tfrac{1}{2} \sum \tilde{f}(\tilde{q}). \tag{13}$$

It remains to express (13) in terms of the covering (X, T, Y).

One sees immediately that

$$\sum \tilde{f}(\tilde{q}) = \sum f_l. \tag{14}$$

Indeed, if $q_l \in (E \cup F_Y^2) \setminus \mathscr{R}^{**}$, then it is a point $\tilde{q}$ with $\tilde{f}(\tilde{q}) = f_l$, and if $q_l = p \in \mathscr{R}^{**}$, then it is replaced in $\tilde{E} \cup F_{\tilde{Y}}^2$ by the corresponding pair $\{a, b\}$ and $f_l = \tilde{f}(a) + \tilde{f}(b)$.

A region $\tilde{Y}_\lambda$ is either a neighbourhood v for a point $p \in A$ and in this case $\tilde{\varrho}_\lambda = -1$ while $\tilde{n}_\lambda = v(p)$, or it is included in a uniquely determined region Y_i. Then $\tilde{\varrho}_\lambda = \varrho_i$, $\tilde{n}_\lambda = n_i$ and there exists a bijection between the regions $\tilde{Y}_\lambda$ of this last type and the regions Y_i, [3], 8. Consequently

$$\sum \tilde{n}_\lambda \tilde{\varrho}_\lambda = \sum n_i \varrho_i - \sum_{p \in A} v(p). \tag{15}$$

The curves γ' which do not intersect A have no contribution to $\sum \tilde{s}_\mu$ nor to $\sum s_j$. Let γ_1' be the family of curves γ' with $\mathscr{E}_{\gamma'} = \emptyset$ but which intersect A. Write $A_1 = A \cap \gamma_1'$. The curves γ_1' have no contribution to $\sum s_j$ but yield $\sum_{p \in A_1} v(p)$ in $\sum \tilde{s}_\mu$. The other curves γ' with $\mathscr{E}_{\gamma'} \neq \emptyset$ decompose into cross cuts of the family $\{\gamma_j''\}^1$ and these cross cuts as well as those from the family $\{\gamma_j''\}^2$ yield cross cuts $\tilde{\gamma}_\mu$. Namely, if a cross cut γ_j'' contains the points $p \in A$, then γ_j'' contributes with $s_j + \sum v(p)$ to $\sum \tilde{s}_\mu$. Hence

$$\sum \tilde{s}_\mu = \sum s_j + \sum_{p \in A_1 \cup A_2} v(p), \tag{16}$$

where $A_2 = A \cap \gamma''$ and γ'' is the family of all the cross cuts γ_j''.

Therefore (11)—(16) imply

$$r = \varrho_X - \sum n_i \varrho_i - \sum s_j + \sum_{p \in A \setminus (A_1 \cup A_2)} v(p) - \tfrac{1}{2} \sum f_l \tag{17}$$

and it is easy to verify that $A \setminus (A_1 \cup A_2) = \{\mathscr{E}_{\gamma'} \cup [(\mathscr{E} \cup \mathscr{N}) \setminus \gamma']\} \setminus B_Y$, so that $\sum v(p)$ in (17) is equal to $\sum v_k$ in (4).

6. Finally, let us establish Theorem 2 in the case mentioned in Remark 1 in 3.4 when γ also contains a finite number M of points of $\mathring{Y}$. As indicated in 3.4, we consider these points in $\mathscr{E}_{\gamma'}$, denote them by p_k and suppose (if necessary by a change of numeration) that they correspond to the indices $k = 1, \dots, M$. Set for each of these k, $T^{-1}(p_k) = \{Q_{k1}, \dots, Q_{ki_k}\} \subset \mathring{X}$, every point Q_{kj} having the multiplicity h_{kj}, $\hat{X} = X \setminus \bigcup_{k=1}^M T^{-1}(p_k)$ and $\hat{Y} = Y \setminus \{p_1, \dots, p_M\}$.

We can apply formula (4) to the covering $(\hat{X}, T|, \hat{Y})$ so that the corresponding ramification number $\hat{r}$ is given by

$$\hat{r} = \varrho_{\hat{X}} - \sum n_i \varrho_i - \sum s_j + \sum_{k > M} v_k - \tfrac{1}{2} \sum f_l,$$

where $\varrho_{\hat{X}}$ is the characteristic of $\hat{X}$ and Y_i are the components of $Y \setminus (\gamma \cup B_Y)$, the points $p_1, \dots, p_M$ being included in γ.

On the other hand,

$$r = \hat{r} + \sum_{k=1}^{M} \left(\sum_{j=1}^{\mathfrak{i}_k} (h_{kj} - 1)\right) = \hat{r} + \sum_{k=1}^{M} \nu_k - \sum_{k=1}^{M} \mathfrak{i}_k$$

and $\varrho_{\hat{X}} = \varrho_X + \sum_{k=1}^{M} \mathfrak{i}_k$ so that the formula (4) is true for the covering (X, T, Y), too.

References

[1] AHLFORS, L.: Zur Theorie der Überlagerungsflächen. - Acta Math. 65, 1935, 157—194.

[2] ALLING, N. L., and N. GREENLEAF: Foundations of the theory of Klein surfaces. - Lecture Notes in Mathematics 219. Springer-Verlag, Berlin—Heidelberg—New York, 1971.

[3] ANDREIAN CAZACU, C.: Über eine Formel von S. Stoilow. - Rev. Roumaine Math. Pures Appl. 5, 1960, 59—74.

[4] ANDREIAN CAZACU, C.: Clase de acoperiri riemanniene I, II. - An. Univ. Bucureşti, Şti. Natur. 13:2, 1964, 55—75; 14:1, 1965, 71—95 (Romanian).

[5] ANDREIAN CAZACU, C.: Interior transformations between Klein surfaces. - To appear.

[6] BÂRZĂ, I.: Suprafeţe riemanniene neorientabile. - Teză de doctorat, Bucureşti, 1980. (Sur la ramification des recouvrements riemanniens non orientable. To appear).

[7] HURWITZ, A.: Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. - Math. Ann. 39, 1891, 1—61.

[8] KERÉKJÁRTÓ, B.: Vorlesungen über Topologie. I. - Die Grundlehren der mathematischen Wissenschaften 8. Springer-Verlag, Berlin, 1923.

[9] LEHTO, O.: Riemann surfaces. - Lecture notes, University of Minnesota, Minneapolis, Minnesota, 1970.

[10] NEVANLINNA, R.: Analytic functions. - Die Grundlehren der mathematischen Wissenschaften 162. Springer-Verlag, Berlin—Heidelberg—New York, 1970.

[11] STOÏLOW, S.: Du nombre des points de ramification des transformations intérieures sur une variété topologique à deux dimensions. - Bull. Sci. Math. (2) 57, 1933, 355—376.

[12] STOÏLOW, S.: Leçons sur les principes topologiques de la théorie des fonctions analytiques. — Gauthier-Villars, Paris, 1938, and ed 1956.

University of Bucharest
Faculty of Mathematics
Bucharest 1
Romania

Received 2 April 1984