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SEQUENTIAL FOURIER–FEYNMAN TRANSFORMS

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In a forthcoming Memoir of the American Mathematical Society [3], the authors give a simple sequential definition of the Feynman integral which is applicable to a rather large class of functionals. In Corollary 2 to Theorem 3.1 of [3] we showed that the elements of the Banach algebra $\hat{S}$ defined in [3] (and below) are all sequentially Feynman integrable.

In the present paper we use the sequential Feynman integral to define a set of sequential Fourier–Feynman transforms. We also show that they form an abelian group of isometric transformations of $\hat{S}$ onto $\hat{S}$.

Notation. Let $C \equiv C[a, b]$ be the space of continuous functions $x(t)$ on $[a, b]$ such that $x(a)=0$, and let $C^\nu[a, b] = \times_{j=1}^{\nu} C[a, b]$.

Let a subdivision σ of $[a, b]$ be given:

$$\sigma\colon\ [a = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots < \tau_m = b].$$

Let $\vec{X} \equiv \vec{X}(t)$ be a polygonal curve in C^ν based on a subdivision σ and the matrix of real numbers $\vec{\vec{\xi}} \equiv \{\xi_{j,k}\}$, and defined by

$$\vec{X}(t) \equiv \vec{X}(t, \sigma, \vec{\vec{\xi}}) = [X_1(t, \sigma, \vec{\vec{\xi}}), \ldots, X_\nu(t, \sigma, \vec{\vec{\xi}})]$$

where

$$X_j(t, \sigma, \vec{\vec{\xi}}) = \frac{\xi_{j,k-1}(\tau_k - t) + \xi_{j,k}(t - \tau_{k-1})}{\tau_k - \tau_{k-1}}$$

when

$$\tau_{k-1} \leqq t \leqq \tau_k; \quad k = 1, 2, \ldots, m; \quad \text{and} \quad \xi_{j,0} \equiv 0.$$

(We note that as $\vec{\vec{\xi}}$ ranges over all of νm dimensional real space, the polygonal functions $\vec{X}((\cdot), \sigma, \vec{\vec{\xi}})$ range over all polygonal approximations to the functions $C^\nu[a, b]$ based on the subdivision σ. Specifically if $\vec{x}$ is a particular element of $C^\nu[a, b]$ and we set $\xi_{j,k} = x_j(\tau_k)$, the function $\vec{X}((\cdot), \sigma, \vec{\vec{\xi}})$ is the polygonal approximation of $\vec{x}$ based on the subdivision σ.) Where there is a sequence of subdivisions $\sigma_1, \sigma_2, \ldots$, then σ, m and τ_k will be replaced by σ_n, m_n and $\tau_{k,n}$.

Definition. Let $q \neq 0$ be a given real number and let $F(\vec{x})$ be a functional defined on a subset of $C^\nu[a, b]$ containing all the polygonal elements of $C^\nu[a, b]$.

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Let $\sigma_1, \sigma_2, \ldots$ be a sequence of subdivisions such that norm $\sigma_n \to 0$ and let $\{\lambda_n\}$ be a sequence of complex numbers with $\operatorname{Re} \lambda_n > 0$ such that $\lambda_n \to -iq$. Then if the integral in the right hand side of (1.0) exists for all n and if the following limit exists and is independent of the choice of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the *sequential Feynman integral with parameter* q *exists and is given by*

(1.0)

$$\int^{sf_q} F(\vec{x})\, d\vec{x} \equiv \lim_{n\to\infty} \gamma_{\sigma_n, \lambda_n} \int_{R^{\nu m_n}} \exp\left\{-\frac{\lambda_n}{2}\int_a^b \left\|\frac{d\vec{X}}{dt}(t, \sigma_n, \vec{\xi})\right\|^2 dt\right\} F(\vec{X}((\cdot), \sigma_n, \vec{\xi}))\, d\vec{\xi},$$

where

$$\gamma_{\sigma,\lambda} = \left(\frac{\lambda}{2\pi}\right)^{\nu m/2} \Pi_{k=1}^{m} (\tau_k - \tau_{k-1})^{-\nu/2}.$$

Definition. Let $D[a, b]$ be the class of elements $x \in C[a, b]$ such that x is absolutely continuous on $[a, b]$ and $x' \in L_2[a, b]$. Let $D^\nu = \times_1^\nu D$.

Definition. Let $\mathcal{M} \equiv \mathcal{M}(L_2^\nu[a, b])$ be the class of complex measures of finite variation defined on $B(L_2^\nu)$, the Borel measurable subsets of $L_2^\nu[a, b]$. We set $\|\mu\| =$ var μ. (In this paper, L_2 always means *real* L_2.)

Definition. The functional F defined on a subset of C^ν that contains D^ν is said to be an element of $\hat{S} \equiv \hat{S}(L_2^\nu)$ if there exists a measure $\mu \in \mathcal{M}$ such that for $\vec{x} \in D^\nu$

$$F(\vec{x}) \equiv \int_{L_2^\nu} \exp\left\{i \sum_{j=1}^{\nu} \int_a^b v_j(t) \left(\frac{dx_j(t)}{dt}\right) dt\right\} d\mu(\vec{v}). \tag{1.1}$$

We also define $\|F\| \equiv \|\mu\|$.

Lemma. *If* $F \in \hat{S}$ *and* $\vec{y} \in D^\nu$ *, then the translate of* F *by* $\vec{y}$ *is in* $\hat{S}$; *i.e.* $F((\cdot) + \vec{y}) \in \hat{S}$. *Moreover if for* $\vec{x} \in D^\nu$, $F(\vec{x})$ *is given by equation* (1.1) *where* $\mu \in \mathcal{M}$, *it follows that*

$$F(\vec{x} + \vec{y}) = \int_{L_2^\nu} \exp\left\{i \sum_{j=1}^{\nu} \int_a^b v_j(t) \frac{dx_j(t)}{dt}\right\} d\sigma(\vec{v}) \tag{1.2}$$

where $\sigma \in \mathcal{M}$ *and for each Borel subset* E *of* L_2^ν, $\sigma(E)$ *is given by*

$$\sigma(E) = \int_E \exp\left\{i \sum_{j=1}^{\nu} \int_b^a v_j(t) \frac{dy_j(t)}{dt} dt\right\} d\mu(\vec{v}). \tag{1.3}$$

Proof of the Lemma. If σ is given by equation (1.3) it is clearly in $\mathcal{M}$ and $\|\sigma\| \leq \|\mu\|$. For $\vec{x} \in D^\nu$, it follows from (1.1) that

$$F(\vec{x} + \vec{y}) = \int_{L_2^\nu} \exp\left\{i \sum_{j=1}^{\nu} \int_a^b v_j(t) \frac{dx_j(t)}{dt}\right\} \exp\left\{i \sum_{j=1}^{\nu} \int_a^b v_j(t) \frac{dy_j(t)}{dt} dt\right\} d\mu(\vec{v})$$

$$= \int_{L_2^\nu} \exp\left\{i \sum_{j=1}^{\nu} v_j(t) \frac{dx_j(t)}{dt} dt\right\} d\sigma(\vec{v}),$$

and hence $F((\cdot)+\vec{y})\in\hat{S}$ and the Lemma is proved. (Cf. also Corollary 2 of Theorem 4.1 of [3].)

Definition. If $p\neq 0$ and if for each $y\in D^{\nu}[a,b]$ the sequential Feynman integral

$$(1.4)\qquad (\Gamma_p F)(\vec{y}) \equiv \int^{sf_{1/p}} F(\vec{x}+\vec{y})\,d\vec{x}$$

exists, then $\Gamma_p F$ is called the sequential Fourier – Feynman transform of F. If $p=0$ we define Γ_0 to be the identity transformation, $\Gamma_0 F\equiv F$.

Theorem 1. *If* $F\in\hat{S}$ *and* p *is real, then the sequential Fourier – Feynman transform of* F *exists and* $\Gamma_p F\in\hat{S}$. *Moreover for* $\vec{y}\in D^{\nu}$ *and* F *given by equation* (1.1),

(1.6)
$$(\Gamma_p F)(\vec{y}) = \int_{L_2^{\nu}} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dy_j(t)}{dt}\,dt\right\}\exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}d\mu(\vec{v}).$$

Proof of Theorem 1. By the Lemma, $F((\cdot)+\vec{y})\in\hat{S}$, and hence by Corollary 2 of Theorem 3.1 of [3] when $p\neq 0$, the right hand member of (1.4) exists. Also in terms of the measure σ given in equation (1.3), we have from (1.2) and Corollary 2 of Theorem 3.1 of [3] that

$$\int^{sf_{1/p}} F(\vec{x}+\vec{y})\,d\vec{x} = \int^{sf_{1/p}}\left[\int_{L_2^{\nu}} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dx_j(t)}{dt}\,dt\right\}d\sigma(\vec{v})\right]d\vec{x}$$

$$= \int_{L_2^{\nu}} \exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}d\sigma(\vec{v}).$$

Equation (1.6) follows by substituting for σ using equation (1.3). Now let the measure τ be defined on the Borel subsets E of L_2^{ν} by

$$(1.7)\qquad \tau(E) \equiv \int_E \exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}d\mu(\vec{v}).$$

Clearly $\tau\in\mathcal{M}$ and equation (1.6) can be written

$$(1.8)\qquad (\Gamma_p F)(\vec{y}) = \int_{L_2^{\nu}} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dy_j(t)}{dt}\,dt\right\}d\tau(\vec{v}).$$

Hence $\Gamma_p F\in\hat{S}$ and the theorem is proved for the case $p\neq 0$. When $p=0$, $\Gamma_0 F$ is the identity transformation and the theorem follows from (1.1).

Corollary *to Theorem* 1. *In addition to the hypotheses of Theorem* 1, *assume that* Φ *is a bounded measurable functional defined on* L_2^ν, *and let*

$$H(\vec{x}) \equiv \int_{L_2^\nu} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dx_j(t)}{dt}\,dt\right\}\Phi(\vec{v})\,d\mu(\vec{v}). \tag{1.9}$$

Then the functional $H\in\hat{S}$ *and*

$$(\Gamma_p H)(\vec{y}) \tag{1.10}$$

$$= \int_{L_2^\nu} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dy_j(t)}{dt}\,dt\right\}\exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}\Phi(\vec{v})\,d\mu(\vec{v}).$$

Proof. Let a measure σ be defined on each Borel set E of L_2^ν by

$$\sigma(E) \equiv \int_E \Phi(\vec{v})\,d\mu(\vec{v}).$$

Clearly $\sigma\in\mathcal{M}$ and for $\vec{x}\in D^\nu$

$$H(\vec{x}) = \int_{L_2^\nu} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dx_j(t)}{dt}\,dt\right\}d\sigma(\vec{v})$$

so that $H\in\hat{S}$. Applying the theorem to H and replacing $d\sigma(\vec{v})$ by $\Phi(\vec{v})\,d\mu(\vec{v})$, we obtain (1.10) and the Corollary is proved.

Theorem 2. *The set of sequential Fourier—Feynman transforms* Γ_p *for real* p *forms an abelian group of isometries of the Banach algebra* $\hat{S}$, *with multiplication rule*

$$\Gamma_q\Gamma_p = \Gamma_{p+q} \quad \textit{for} \quad p, q \quad \textit{real} \tag{1.11}$$

and identity

$$\Gamma_0 = I \tag{1.12}$$

and inverses

$$(\Gamma_p)^{-1} = \Gamma_{-p} \quad \textit{for} \quad p \quad \textit{real.} \tag{1.13}$$

Proof of Theorem 2. By equation (1.6),

$$(\Gamma_p F)(\vec{y}) = \int_{L_2^\nu} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dy_j(t)}{dt}\,dt\right\}\exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}d\mu(\vec{v})$$

and by equation (1.10)

$$(\Gamma_q\Gamma_p F)(\vec{y}) = \int_{L_2^\nu} \exp\left\{i\sum_{j=1}^{\nu}\int_a^b v_j(t)\frac{dy_j(t)}{dt}\,dt\right\}\exp\left\{\frac{p}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}$$

$$\cdot\exp\left\{\frac{q}{2i}\sum_{j=1}^{\nu}\int_a^b [v_j(t)]^2\,dt\right\}d\mu(\vec{v}) = (\Gamma_{p+q}F)(\vec{y}).$$

Equation (1.12) is given by the definition of the sequential Fourier—Feynman transform and equation (1.13) follows from equation (1.11) by setting $q=-p$. Finally we establish the isometric property of Γ_p. If $F \in \hat{S}$ and $G = \Gamma_p F$, then by equations (1.8), (1.7) and (1.1)

$$\|G\| = \|\Gamma_p F\| = \|\tau\| \leqq \|\mu\| = \|F\|.$$

Also by (1.13)

$$F = \Gamma_{-p} G$$

and

$$\|F\| \leqq \|G\|,$$

and the theorem is proved.

In conclusion, for the case $v=1$, $p=-1$, the definition of the sequential Fourier – Feynman transform given in this paper is similar in form to the definition of the analytic Fourier – Feynman transform T given by Brue [3]. Brue defines the transform, TF, of a functional F, in terms of the analytic Feynman integral of F; namely he lets

$$(TF)(y) \equiv \int_C^{anf_{-1}} F(x+y)\, dx$$

whenever the right hand side exists for all $y \in C$. He then proceeds to establish the existence of the transform T and its inverse T^* for several large classes of functionals. There is also a formal similarity to the definitions given in [2] and [4], but they involve more complicated forms of the analytic Feynman integral.

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