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ON THE SUPPORT OF HARMONIC MEASURE FOR SETS OF CANTOR TYPE

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1. Introduction. In a recent paper [2] Kaufman and Wu have shown that the support of harmonic measure of the von Koch snowflake domain has dimension strictly smaller than the Hausdorff dimension (log 4/log 3) of the boundary curve. This result supports a very strong conjecture of Øksendal [4] to the effect that the support of any harmonic measure in the plane has dimension $\leqq 1$.

We shall here prove that the Øksendal conjecture is true for any "fractal" curve such as the snowflake. It is also true for any two-dimensional Cantor set with constant ratio in the very strong sense that the dimension for these supports is strictly <1 even when the Hausdorff dimension of the Cantor set is close to 2.

If one thinks of harmonic measure as hitting probability it may not be so surprising that the results are consequences of results on stationary processes. More precisely, they belong to the field of information theory and we shall here review what is needed (see e.g. [1]).

After this manuscript was finished I was informed that A. Manning has a similar result for invariant measures of a polynomial map. (See [3].)

Let $x=\{x_n\}_{-\infty}^{\infty}$ be a double infinite sequence of symbols x_n taken from a finite set, "alphabet". Let μ be a probability measure on the set of x's. We assume μ to be *stationary* and *ergodic* with respect to the shift T: $Tx=\{x_{n+1}\}_{-\infty}^{\infty}$. Denote by C a "*chain*" $C=(x_1, x_2, \ldots, x_n)$, of length n. The *entropy* H of μ is defined as

$$H = \lim_{n\to\infty} \frac{H_n}{n} = \sup_n \frac{H_n}{n}$$

where

$$H_n = -\sum_{C_n} \mu(C_n) \log \mu(C_n).$$

McMillan's basic theorem now describes how many C_n that actually occur.

Theorem. *Given* $\varepsilon>0$, *there is* $n_0(\varepsilon)$ *so that for* $n>n_0(\varepsilon)$, *we can find distinct chains* $C_{n1}, C_{n2}, \ldots, C_{nm}$ *of length* n *with* $m<e^{n(H+\varepsilon)}$ *so that*

$$\sum_{j=1}^{m} \mu(C_{nj}) > 1-\varepsilon.$$

It is clear that this is a description of the support of μ. Our scheme is therefore to relate harmonic measure to a stationary μ for which we can actually compute the entropy. We also obtain a connection to the geometry of the situation. Since the sup-

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port of our harmonic measure in the sense of distributions is the entire set we have to make a more careful definition.

Let w be compactly supported positive measure. We say that the support $S(\omega)$ of ω has dimension α if for every $\beta>\alpha$ and every $\varepsilon>0$ we can find discs D_ν of radii r_ν so that

$$\sum r_\nu^\beta < \varepsilon,$$

$$\omega(\boldsymbol{R}^2 \setminus \cup D_\nu) < \varepsilon,$$

while this fails for every $\beta<\alpha$.

2. Let us begin by presenting a simple proof of Øksendal's theorem.

Øksendal's theorem. *Let E be a compact set in the plane with connected complement. Then $S(\omega_E)$ has two-dimensional Lebesgue measure zero.*

Lemma. *Suppose $E\subset\{z||z|<1\}$ and assume $m_2E\geqq\pi-\delta\pi$, $\delta>0$. Let $E_0=E\cap\{z|1/2\leqq|z|\leqq1\}$. Then there is a universal constant C so that*

$$\omega(z;\, E_0) \geqq 1-C\delta \quad \text{on} \quad |z|=\frac{3}{4}.$$

Proof. Choose a radius r_1, $7/8<r_1<1$, so that

$$m_1\{\theta|r_1e^{i\theta}\in E\} \geqq 2\pi-\delta/20 \tag{1}$$

and r_2 similarly in $1/2<r_2<5/8$. Let $G(z;\,\zeta)$ be the Green's function for R: $r_2<|z|<r_1$, and define in R

$$u(z)=\frac{1}{2\pi}\int_{\substack{\partial R\\ \zeta\notin E}} \frac{\partial G(z;\,\zeta)}{\partial n_\zeta}\,|d\zeta|.$$

By the maximum principle $\omega(z;\,E_0)+u(z)\geqq1$ in R. Writing down the explicit form of G and using (1), we see that

$$u(z) \leqq (\text{Const.})\,\delta \quad \text{on} \quad |z|=3/4$$

so that

$$\omega(z;\,E_0) \geqq 1-C\delta \quad \text{on} \quad |z|=3/4.$$

For the *proof of the theorem* we may of course assume that $m_2E>0$. Let ω denote harmonic measure for the complement of E, so that $\omega(z;\,D)$, $D\subset E$, is harmonic outside E and $=1$ on D, $=0$ on $E\setminus D$. We shall prove that if z_0 is a point of density of E, then the Radon–Nikodym derivative of ω vanishes at z_0.

We take $z_0=0$. Let $B_\varepsilon=\{v||v|<\varepsilon\}$ and define

$$\varphi(\varepsilon)=\omega(B_\varepsilon)$$

and

$$\delta(\varepsilon)=1-\frac{m_2(E\cap B_\varepsilon)}{\pi\varepsilon^2}.$$

By the lemma $\omega(z;\ E\cap B_{\varepsilon/2})\leqq C\delta(\varepsilon)$ on $|z|=3\varepsilon/4$ so that for $|z|>\varepsilon$

$$\omega(z;\ E\cap B_{\varepsilon/2})\leqq\frac{C\delta(\varepsilon)}{1-C\delta(\varepsilon)}\omega(z;\ E\cap B_{\varepsilon}).$$

This yields

$$\varphi\left(\frac{\varepsilon}{2}\right)\leqq 2\delta(\varepsilon)\varphi(\varepsilon)<\frac{1}{5}\varphi(\varepsilon)$$

if $\delta(\varepsilon)<1/10$. At a point of density this holds for $\varepsilon<\varepsilon_0$ and hence

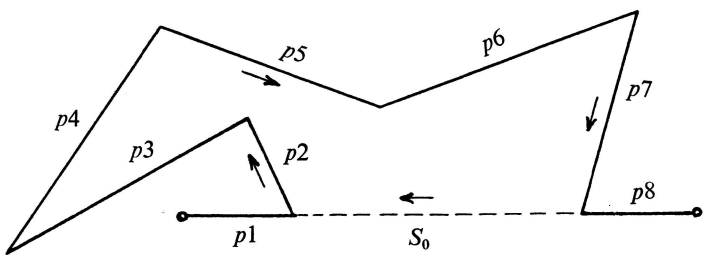
$$\varphi(\varepsilon_0 2^{-n})<5^{-n}$$

so that

$$\lim\frac{\varphi(\varepsilon)}{\varepsilon^2}=0.$$

3. Let $\boldsymbol{P}$ be a fixed simple polygon with N sides and with the following relation to a basic directed line segment S_0 of length 1.

1. The endpoints of $\boldsymbol{P}$ and S_0 coincide.
2. The first and the last side of $\boldsymbol{P}$ fall on S_0.
3. The sides 2, ..., $N-1$ of $\boldsymbol{P}$ bound with S_0 a simply-connected polygon with negative orientation.
4. If the sides of $\boldsymbol{P}$ have lengths $p(1), p(2), \ldots, p(N)$ then $p(1),\ p(N)<1/2$ and $p(2), \ldots, p(N-1)<1$.



$N=8.$

If S is a directed line segment we denote by $P(S)$ the polygon similar to P with base S instead of S_0.

We start from any fixed simple polygon, say the unilateral triangle Δ_0 bounding Ω_0 and containing the origin. For each side S of Δ_0 we construct $\Sigma(S)$. We obtain a new polygon Δ_1 with $3N$ sides. These may intersect but because of our assumption 3, Δ_1 bounds a simply-connected domain $\Omega_1\supset\Omega$ on a Riemann surface. We continue the construction and obtain at the n-th stage a polygon Δ_n with $3N^n$ sides bounding a domain Ω_n. We note that by assumption 4 the Δ_n are all located in a fixed bounded domain and the sides of Δ_n tend to zero exponentially. Ω_n tends to a domain Ω.

Theorem. *The support of harmonic measure inside Ω has dimension $\leqq 1$ on $\partial\Omega$.*

The boundary sides of Δ_n have a natural representation by a sequence $(x_0, x_1, \ldots, x_n)$ where $x_0=1, 2, 3$ and $x_i=1, 2, \ldots, N$, $i\geqq 1$, indicates on which side the construction was made at the i-th stage. We are going to disregard the special role of $i=0$ in the notation. Each domain Ω_n has harmonic measure ω^n and $\omega^n \to \omega$ (weak-*) which is harmonic measure for the limit domain Ω. The value of ω for a boundary sequence $C=x_0, x_1, \ldots, x_n$ is denoted $\omega(z;\ C)$ and for $z=0$ we write simply $\omega(C)$.

To obtain a stationary measure we now introduce the space X^* of doubly infinite sequences $X^*=\{x_n\}_{-\infty}^{\infty}$. The measure μ is defined in this space X^* as follows:

Let C be a chain in some arbitrary position in X^*. We extend C to the left by a sequence A of length $|A|=n$ obtaining after translation a new chain of length$=$ $=n+\text{length}\,(C)$ at position $(x_0, x_1, \ldots, x_{n+|C|-1})$. We shall presently show that the following limit exists and gives a definition of a probability measure μ:

$$\mu(C) = \lim_{n\to\infty} \sum_A \omega(A, C). \tag{1}$$

The definition is clearly translation invariant. Before proceeding let us drop the *-notation. All sequences will be doubly infinite but ω is naturally only used for sequences $(x_0, x_1, \ldots, x_n)$.

The existence and other basic properties of μ now follow from the following

Lemma 1.

$$\frac{\omega(A, C, B)}{\omega(A, C)} \frac{\omega(C)}{\omega(C, B)} \to 1$$

uniformly in A and B as $|C|\to\infty$.

Proof. Let us first disentangle the notation. For a chain D, $\omega(D)$ denotes the value at $z=0$ of the harmonic measure $\omega(z;\ D)$. In the lemma, B denotes a sequence of choices of sides and the lemma means that the relative harmonic measure

$$\frac{\omega(z;\ C, B)}{\omega(z;\ C)} \quad (z=0)$$

is not influenced by earlier conditionings (by A).

To prove the lemma let us map the situation to the unit disc. The length-area proof of the Carathéodory theorem applies and we conclude that the mapping is bicontinuous. Let C have length n, $C=(c_1, \ldots, c_n)$. Choose the mapping so that the midpoint of the segment $S=(c_1, \ldots, c_{n/2})$ is mapped to $\zeta=0$. C defines an arc γ on $|\zeta|=1$ and (C, B) an arc $\beta\subset\gamma$. Then $u(\zeta;\ \gamma)$ and $u(\zeta;\ \beta)$ are the corresponding harmonic measures

$$u(\zeta;\ \gamma) = \frac{1}{2\pi}\int_\gamma \frac{1-|\zeta|^2}{|\zeta-e^{it}|^2}\,dt.$$

It is clear from the geometric construction of $\Sigma(S)$ that the preimage σ of S in $|\zeta|<1$ satisfies, since the mapping is continuous,

$$|\gamma| = (\text{length}\,\gamma) = o\big(\text{dist}\,(\sigma, \gamma)\big), \quad n\to\infty,$$

uniformly in all choices of C and B. This implies that

$$\lim_{n=\infty} \frac{u(\zeta; \beta)}{u(\zeta; \gamma)} = \lim_{n=\infty} \frac{|\beta|}{|\gamma|} = q$$

uniformly for $\zeta \in \sigma$. Hence

$$(q-\varepsilon)\omega(z; C) \leqq \omega(z; C, B) \leqq (q+\varepsilon)\omega(z; C)$$

"inside" the cross-section S and in particular for $z=0$. If we add to C the sequence A but otherwize do the same construction we obtain the same limit q and putting $z=0$ we get the lemma.

Let us now prove that the limit (1) exists. We write $A=A'$, A_0 and keep A_0 fixed and long. Then

$$\omega(A, C) = \omega(A', A_0, C) = \frac{\omega(A_0, C)}{\omega(A_0)} \omega(A', A_0)(1+\varepsilon)$$

where $\varepsilon(A') \to 0$, as $|A_0| \to \infty$. Hence

$$\sum_{A'} \omega(A', A_0, C) = \omega(A_0, C)(1+\varepsilon')$$

which proves the existence of μ. Actually we have the stronger result

$$\lim_{m \to \infty} \frac{\sum_{|A|=m} \omega(A, C)}{\mu(C)} = 1 \tag{2}$$

uniformly in C.

Lemma 2. *The measure μ is a probability measure with the following properties:*

(a) *μ is invariant under T*

$$\mu(TC) = \mu(C),$$

(b) *μ is equivalent to ω, i.e.*

$$\text{Const. } \omega(C) \leqq \mu(C) \leqq \text{Const. } \omega(C),$$

(c) *μ is ergodic for T.*

Proof. We have already observed that the definition of μ makes sense. We also have $\sum_{|C|=k} \mu(C)=1$ since this holds for ω and μ extends to a probability measure on X^*. Because of (2) μ is equivalent to a fixed finite sum $\sum_A \omega(A, C)$ and this is clearly equivalent to $\omega(C)$.

To prove (c) let us consider a cylinder set defined by C and C' at distance m from each other. Let us consider $D=ACBC'$, $|B|=m$, $|A|=n$. On cross-sections β in B

$$\frac{\omega(z; ACBC'')}{\omega(z; ACBC')} \leqq \text{Const.} \frac{\mu(C'')}{\mu(C')}$$

with a uniform constant. It now follows that

$$\sum_{|B|=m} \mu(CBC') \geqq \text{Const. } \mu(C)\mu(C')$$

uniformly in m. This weak form of mixing implies ergodicity. This is certainly well-known. For convenience, a sketch of a proof is included.

Suppose E is an invariant set with $0<\mu(E)<1$ and let F be the complement of E. Let A and B be cylinder approximations to E and F and translate B by N so that A and B have disjoint positions. Then

$$\mu(A\cap B) \leqq \mu(E\cap T^{-N}F)+\varepsilon \leqq \mu(E\cap F)+\varepsilon = \varepsilon$$

but also

$$\mu(A\cap B) \geqq \text{Const. } \mu(A)\mu(B) \geqq \text{Const. } (\mu(E)-\varepsilon)(\mu(F)-\varepsilon).$$

Hence $\mu(E)$ or $\mu(F)=0$.

Let us now compute the entropy of μ. We fix some large m so that

$$1-\varepsilon \leqq \frac{\sum_{|A|=m} \omega(A, C)}{\mu(C)} \leqq 1+\varepsilon$$

for all C. Then

$$H_n = -\sum_{|C|=n} \mu(C)\log\mu(C)$$

$$\leqq (1+\varepsilon)\sum_{|C|=n}\Big(-\sum_{|A|=m}\omega(A, C)\Big)\log\Big(\sum_{|A|=m}\omega(A, C)\Big)+(1+\varepsilon)\log(1+\varepsilon)$$

$$\leqq (1+\varepsilon)\sum'_{|C|=n+m}\big(-\omega(C)\log\omega(C)\big)+(1+\varepsilon)\log(1+\varepsilon)$$

since $f(x)=-x\log x$ satisfies $f(x_1+x_2+\ldots+x_n)\leqq\sum_{\nu=1}^{n} f(x_\nu)$. Hence

$$H = \lim_{n=\infty}\frac{H_n}{n} \leqq \lim -\frac{1}{n}\sum_{\omega_n}\omega_n\log\omega_n$$

where $\omega_n=\omega(C)$ denote the harmonic measure of the boundary intervals σ_n at the n-th stage of the construction. Formally

$$\omega_n = \frac{1}{2\pi}\int_{\sigma_n}\frac{\partial g}{\partial n}\,ds$$

where g is the Green's function of the limit domain with pole at 0. Let us stop the construction at same stage $k>n$. Then the formula is valid and if we make estimates independent of k, these estimates will hold for the limit domain.

Let us consider

$$(2)\qquad E(\sigma) = \frac{1}{2\pi}\int_\sigma \frac{\partial g}{\partial n}\log\frac{\partial g}{\partial n}\,ds = \frac{1}{2\pi}\int_\sigma\left(\frac{\partial g}{\partial n}\delta\right)\left(\log\frac{\partial g}{\partial n}\delta\right)\frac{ds}{\delta}-\omega_n\log\delta$$

where δ is the length of the arc σ.

The first integral I is scale invariant and we may assume that our basic interval is (0, 1) on the real axis. Let $F(\zeta)=Z$ be the conformal mapping from $\eta>0$ to the

domain with $F(i)=0$ and $F(0)\in\sigma$. Then

$$I=\frac{1}{\pi}\int_{F^{-1}(\sigma)}-\log\left(|F'(x)|(1+x^2)\right)\frac{dx}{1+x^2}.$$

Here $F^{-1}(\sigma)=\tau$ is an interval of length $l(\leqq 1$, say) and

$$\omega=\omega_n=\frac{1}{\pi}\int_\tau\frac{dx}{1+x^2}\sim\frac{l}{\pi},$$

$$I=G(\omega)-\frac{1}{\pi}\int_\tau\frac{\log|F'(x)|\,dx}{1+x^2}=\left(G(\xi)=F(l\xi)\right)$$

$$=O(\omega)-l\frac{1}{\pi}\int_{\tau_0}\frac{\log|G'(\xi)|\,d\xi}{1+l^2\xi^2}+\omega\log\omega.$$

Since F is bounded $|G'(i)|\leqq$Const. Here τ_0 is an interval of length 1 containing $\xi=0$. We shall prove that integrals of this type are bounded.

Lemma 3. *Let $G(\zeta)$ be univalent in $\eta>0$ and smooth up to $\eta=0$ and assume $|G'(i)|\leqq 1$. Let $\psi(\zeta)\in C^2$ in $\eta\geqq 0$. Then there is universal constant C so that*

$$\int_{-1}^{1}\psi(\xi)\log|G'(\xi)|\,d\xi\leqq C\|\psi\|_{C^2}.$$

Proof. We may of course assume $\psi(\zeta)=0$, $|\zeta|>2$. Set

$$U(\xi,\eta)=\frac{1}{\pi}\left(\operatorname{arc\,tg}\frac{\xi-1}{\eta}-\operatorname{arc\,tg}\frac{\xi+1}{\eta}\right)$$

and

$$u(\xi,\eta)=\eta\psi(\zeta)\;U(\xi,\eta).$$

By Green's formula

$$\int_{-1}^{+1}\psi(\xi)\log|G'(\xi)|\,d\xi=\iint_{\eta>0}\log|G'(\zeta)|\,\Delta u(\zeta)\,d\zeta\,d\eta.$$

To obtain the desired result we now only have to use the estimates in $|\zeta|<2$:

$$\left|\log|G'(\zeta)|\right|\leqq\text{Const.}\log\frac{3}{\eta},$$

$$|\Delta u|\leqq\text{Const.}\frac{1}{|\zeta^2-1|},$$

$$\iint_{|\zeta|<2}\log\frac{3}{\eta}\frac{d\xi\,d\eta}{|\zeta^2-1|}<\infty.$$

The proof of the lemma is complete.

The result of the computation is that, from (2),

$$E(\sigma)=\omega\log\omega-\omega\log\delta+O(\omega)$$

for all $\sigma=\sigma_n$ in the n-th generation. We also observe that

$$\sum_{\sigma_n} E(\sigma_n) = \frac{1}{2\pi}\int \frac{\partial g}{\partial n}\log|G'|\,ds = \log|G'(0)|$$

where G is the conformal mapping to $|\zeta|<1$. Hence

$$H_n \leqq -\sum_{\omega_n}(\log\delta_n)\omega_n + O(1)$$

and

$$H \leqq -\overline{\lim}\frac{1}{n}\sum_{\omega_n}(\log\delta_n)\omega_n. \tag{3}$$

We now observe that

$$\delta_n = \prod_{i=1}^{n} p(x_i).$$

Since μ and ω are equivalent measures we can use the individual ergodic theorem for μ to conclude that

$$e^{-n(K-\varepsilon)} < \delta_n < e^{-n(K+\varepsilon)}$$

for all δ_n except a set of small harmonic measure. It follows from (3) that $K\geqq H$. On the other hand by the McMillan theorem only $e^{(H+\varepsilon)n}$ intervals are needed to cover essentially all harmonic measure. We conclude that the $(1+2\varepsilon)$-dimensional Hausdorff measure of the support is zero.

4. Let us now consider the case of the exterior of a two-dimensional Cantor set. In this case the n-th stage consists of 4^n squares of sides a^n, $a<1/2$ and we want to prove the following.

Theorem. *For any two-dimensional Cantor set the support of harmonic measure has dimension* <1.

The previous proof applies with two rather essential changes. Lemma 1 cannot be proved using conformal mapping and the computation of the entropy is different since $\log|\nabla g|$ has singularities.

A substitute for Lemma 1 is given in the following

Lemma 5. *Let D be a domain in the complex plane. A is a conformal annulus, $A=B\setminus C$, B, C conformal disks, $E\subset\partial D$ is such that $E\subset C_1\subset B_1\subset C_2\subset B_2\subset\ldots$ $\ldots\subset C_n\subset B_n$ where the annuli $A_\nu=B_\nu\setminus C_\nu$ are similar to A and $A_\nu\subset D$. Let u and v be non-negative harmonic functions in D so that u and $v=0$ on $\partial D\setminus E$. Then there is a constant C_0, independent of u and v and another constant C, depending on u and v but not on n so that*

$$\left|C\frac{u(z)}{v(z)}-1\right| \leqq \frac{C_0}{n^2} \quad \text{in} \quad A_n$$

and hence in the part of D exterior to B_n.

Proof. In each annulus A_ν pick a smooth curve γ_ν so that all the γ_ν are similar. We define

$$M_\nu = \sup_{\gamma_\nu} \frac{u(z)}{v(z)}, \quad m_\nu = \inf_{\gamma_\nu} \frac{u(z)}{v(z)}, \quad Q_\nu = \frac{M_\nu}{m_\nu}.$$

From the assumption it follows that $M_\nu \searrow$ and $m_\nu \nearrow$. We shall study the sequence Q_ν. Consider $\nu < n$ fixed. By multiplying u and v by constants we may assume that $M_\nu = 1$ and $\operatorname{Max}_{\gamma_\nu} u(z) = 1$. We may also assume that γ_ν has length 1. By Harnack's inequality $u(z) \leqq c_1$ on γ_ν. It also follows that $|\nabla^2 u|, |\nabla^2 v| \leqq c_2$ on γ_ν. Let z_0 be a point on γ_ν where

$$Q_\nu u(z_0) = v(z_0).$$

We conclude that

$$v \geqq (Q_\nu - 1)c_3 + u$$

on an interval I on γ_ν of length $\geqq (Q_\nu - 1)^{1/2} C_4$. Now let G_ν be the Green's function for $D_\nu = D \setminus$ (inside of γ_ν). For $z \in D_\nu$

$$u(z) = \frac{1}{2\pi} \int_{\gamma_\nu} \frac{\partial G_\nu}{\partial n}(z; \zeta) u(\zeta)\, ds_\zeta$$

$$\leqq \frac{1}{2\pi} \int_{\gamma_\nu} \frac{\partial G_\nu}{\partial n}(z; \zeta) v(\zeta)\, ds_\zeta - \frac{1}{2\pi} \int_I \frac{\partial G_\nu}{\partial n}(z; \zeta) C_3 (Q_\nu - 1)\, ds_\zeta.$$

For $z \in \gamma_{\nu+1}$, $\dfrac{\partial G_\nu}{\partial n}$ satisfies $\dfrac{\sup_{\gamma_\nu} \partial G_\nu / \partial n}{\inf_{\gamma_\nu} \partial G_\nu / \partial n} \leqq C_5$. We conclude that on $\gamma_{\nu+1}$

$$u(z) \leqq v(z)\big(1 - C_6 (Q_\nu - 1)^{3/2}\big).$$

Hence

$$M_{\nu+1} \leqq M_\nu \big(1 - C_6 (Q_\nu - 1)^{3/2}\big)$$

and so

$$Q_{\nu+1} = \frac{M_{\nu+1}}{m_{\nu+1}} \leqq \frac{M_{\nu+1}}{m_\nu} \leqq Q_\nu \big(1 - C_6 (Q_\nu - 1)^{3/2}\big).$$

This iteration is easily estimated and we find

$$0 \leqq Q_n - 1 \leqq \frac{C_0}{n^2}$$

and Lemma 5 is proved.

For the entropy we get the following formula

$$\frac{1}{2\pi} \int_{\Gamma_n} \frac{\partial g}{\partial n} \log |\nabla g|\, ds = \sum_{(n)} g(\zeta_\nu) + \text{Const.}$$

where ζ_ν are the critical points of g and (n) indicates that we should sum over those ζ_ν which are inside the n-th approximation of the Cantor region. For the final inequality

we get

$$\frac{H_n}{n} \leqq (-\log a) - \frac{1}{n} \sum_{(n)} g(\zeta_\nu).$$

The theorem therefore follows from the following.

Lemma 6. *There is a constant* $c>0$ *so that*

$$\sum_{(n)} g(\zeta_\nu) \geqq cn.$$

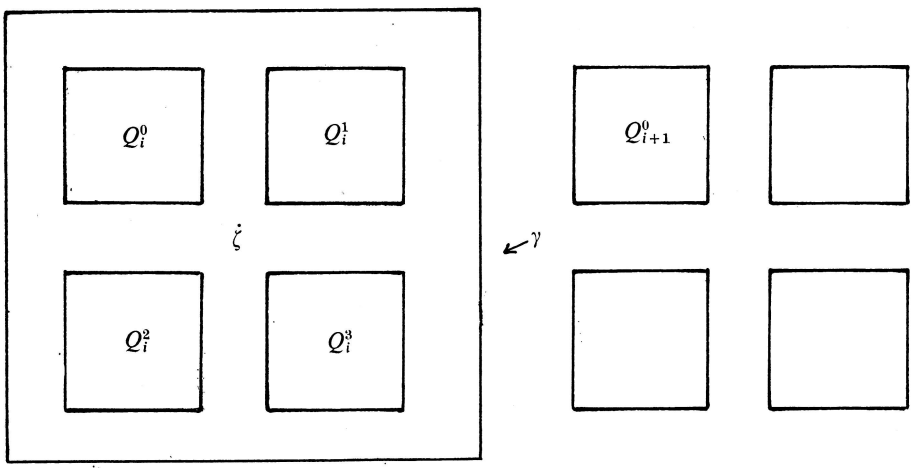
Proof. (The idea of the proof is due to Peter Jones.)

Let us consider the i-th stage in the construction of the Cantor set and a square Q_i of the corresponding generation. We shall prove that we associate to every Q_i^0 one ζ_ν so that $\nu=\nu(Q_i^0)$,

$$g(\zeta_\nu) \geqq \text{Const. } \omega(Q_i)$$

and so that the same ζ_ν is only used four times.

We consider the configuration of 4 squares Q, related to Q_i^0:



By Harnack's inequality $g(z;\ \zeta)$ is comparable to harmonic measure for all four Q_i^k, $k=0, 1, 2, 3$, on γ around the Q_i^k:s. Hence

$$g(\infty;\ \zeta) = g(\zeta;\ \infty) \asymp \omega(Q_i^k), \quad k = 0, 1, 2, 3.$$

It now follows that there is also a critical point inside γ which has the same property at a point where level lines of $g \asymp g(\zeta;\ \infty)$ meet.

We now sum and find

$$\sum_{Q_i} g(\zeta_\nu(Q_i)) \geqq c_0$$

and summation over i gives the lemma.

Added in proof. N. G. Makarov has proved that the dimension of support of harmonic measure is ≤ 1 for all simply-connected domains.

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