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SPACES OF MEASURES ON COMPLETELY REGULAR SPACES

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Let X be a regular topological space. If $(\mu_n)_{n\in N}$ is a sequence of Radon (i.e., inner regular by compact) measures on X such that $(\mu_n(T))_{n\in N}$ converges for every regular open set T of X (i.e., for which $\mathring{\bar{T}}=T$), then $(\mu_n(A))_{n\in N}$ converges for every Borel set A of X. This result was proved by P. Gänssler ([4] Theorem 3.1) for real measures and by S. S. Khurana ([7] Theorem 4) for group valued measures. It will be shown in this paper (Theorem 3) that, if X is completely regular, this result can be improved by assuming only that $(\mu_n(T))_{n\in N}$ converges for those regular open sets T of X for which there exists a continuous real function f on X such that

$$\{f>0\}\subset T\subset \overline{\{f>0\}}$$

(or equivalently $T=\mathring{\overline{\{f>0\}}}$); we denote the set of these sets T by $\mathfrak{T}$. If the vector lattice of continuous real functions on X is order σ-complete, then $\mathfrak{T}$ is exactly the set of closed open sets of X and so the above formulation contains the corresponding result of Z. Semadeni ([8] Theorem (i)⇒(iv)). Let $\mathscr{C}$ be the vector space of continuous bounded real functions on X endowed with the strict topology and E be a quasicomplete G-space ([2] Definition 5.9.11). We show (Theorem 12) that a continuous linear map $u:\mathscr{C}\to E$ is boundedly weakly compact (or equivalently possesses an integral representation) if and only if the sets of $\mathfrak{T}$ are sent into E by the biadjoint map of u. The special case of E equal to the vector space of continuous real functions on a metrizable topological space endowed with the topology of compact convergence is discussed in greater detail (Theorems 15 and 16).

We use the notations and the terminology of [1] *and* [2]. The expression *locally convex space* will mean Hausdorff real locally convex space. For every locally convex space E we denote its dual and bidual by E' and E'', respectively, and identify E with a subspace of E'' via the evaluation map

$$E\to E'',\quad x\mapsto\langle x,\cdot\rangle.$$

For every continuous linear map u of locally convex spaces, u' and u'' will denote the adjoint and the biadjoint of u, respectively. $\boldsymbol{N}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ denote the sets of natural numbers, rational numbers, and real numbers respectively.

Throughout this paper we denote by E a locally convex space, by Y a completely regular space, by X a subspace of Y, by $\mathscr{C}$ the vector space of continuous bounded

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real functions on X endowed with the strict topology, by $\mathfrak{K}$ the set of compact sets of X, by $\mathfrak{R}$ the σ-ring of Borel sets of X, and by $\mathcal{M}$ the band $\mathcal{M}(\mathfrak{R}, \boldsymbol{R}; \mathfrak{K})$ of $\mathcal{M}(\mathfrak{R}, \boldsymbol{R})$ ([2] Proposition 5.6.3)[1]. For every subset A of X we denote its closure and its interior in X by $\bar{A}$ and by $\mathring{A}$, respectively, and we set[2]

$$\mathfrak{T} := \{\overset{\circ}{\overline{V \cap X}} \mid V \text{ exact open set of } Y\}.$$

For every set T of $\mathfrak{T}$ there exists an exact open set V of X such that $T = \mathring{\bar{V}}$; hence T is an open regular set of X. But it may happen that $\mathfrak{T}$ is strictly contained in the set

$$\{\mathring{\bar{V}} \mid V \text{ exact open set of } X\},$$

and this will make our results more general. This is the reason for the introduction of Y.

Y will be called *σ-Stonian* if the vector lattice of continuous real functions on Y is order σ-complete. This is equivalent to the assertion that the closure of every exact open set of Y is open ([2] Lemma 5.9.15 a$\Leftrightarrow$c). If Y is σ-Stonian, then every set of $\mathfrak{T}$ is a closed open set of X.

Proposition 1. *The set $\mathfrak{T}$ is a base of X closed with respect to finite intersections such that $\overset{\circ}{\overline{\bigcup_{\iota \in I} T_\iota}} \in \mathfrak{T}$ for every countable family $(T_\iota)_{\iota \in I}$ in $\mathfrak{T}$.*

Let $x \in X$ and let U be a neighbourhood of x in X. There exists a neighbourhood V of x in Y such that $V \cap X \subset U$. Further, there exists a continuous real function f on Y equal to 0 at x and equal to 2 on $Y \setminus V$. We set

$$W := \{f < 1\}, \quad T := \overset{\circ}{\overline{W \cap X}}.$$

Then $x \in T \subset U$ and $T \in \mathfrak{T}$. Hence $\mathfrak{T}$ is a base of X.

Let $T', T'' \in \mathfrak{T}$. There exist exact open sets V', V'' of Y such that

$$T' = \overset{\circ}{\overline{V' \cap X}}, \quad T'' = \overset{\circ}{\overline{V'' \cap X}}.$$

We set $T := T' \cap T''$, $V := V' \cap V''$. Then V is an exact open set of Y and

$$\overset{\circ}{\overline{V \cap X}} = \overset{\circ}{\overline{(V' \cap X) \cap (V'' \cap X)}} \subset T.$$

Let U be a nonempty open set of X contained in T. Since $T \subset \overline{V' \cap X}$, the set $U \cap V' \cap X$ is nonempty. Since $T \subset \overline{V'' \cap X}$, the set $(U \cap V' \cap X) \cap (V'' \cap X)$ is also

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[1] Let F be a complete vector lattice; a *band of* F is a vector subspace G of F such that:

$$x \in F,\ y \in G,\ |x| \leq |y| \Rightarrow x \in G, \qquad x \in F,\ x = \bigvee_{G \ni y \leq x} y \Rightarrow x \in G.$$

[2] An open set V of Y is called *exact* if there is a continuous real function f on Y, such that

$$V = \{x \in Y \mid f(x) > 0\}.$$

nonempty. Hence $T \subset \overline{V \cap X}$ and we get

$$T = \overset{\circ}{\overline{V \cap X}} \in \mathfrak{T}.$$

This shows that $\mathfrak{T}$ is closed with respect to finite intersections.

Let $(T_\iota)_{\iota \in I}$ be a countable family in $\mathfrak{T}$. For every $\iota \in I$ there exists an exact open set V_ι of Y such that $T_\iota = \overset{\circ}{\overline{V_\iota \cap X}}$. We set $V := \bigcup_{\iota \in I} V_\iota$. Then V is an exact open set of Y and

$$V \cap X = \bigcup_{\iota \in I} (V_\iota \cap X) \subset \bigcup_{\iota \in I} T_\iota \subset \bigcup_{\iota \in I} \overline{V_\iota \cap X} \subset \overline{V \cap X},$$

$$\overset{\circ}{\overline{\bigcup_{\iota \in I} T_\iota}} = \overset{\circ}{\overline{V \cap X}} \in \mathfrak{T}. \quad \square$$

Proposition 2. *Let K be a compact set of X and F be a closed set of X such that $K \cap F = \emptyset$. Then there exist disjoint sets $T', T'' \in \mathfrak{T}$ such that $K \subset T'$, $F \subset T''$. If X is normal and equal to Y, then we may take K closed.*

There exists a continuous real function f on Y such that $f=0$ on K and $f=2$ on F. The sets $\{f<1\}$, $\{f>1\}$ are exact open sets of Y and so the sets

$$T' := \overset{\circ}{\overline{\{f<1\} \cap X}}, \quad T'' := \overset{\circ}{\overline{\{f>1\} \cap X}}$$

possess the required properties. $\square$

Theorem 3. *The identity map*

$$\mathcal{M}(\mathfrak{R}, G; \mathfrak{K})_{\mathfrak{T}} \to \mathcal{M}(\mathfrak{R}, G; \mathfrak{K})$$

is uniformly Φ_3-continuous for every Hausdorff topological additive group G. If X is normal and equal to Y then we may replace $\mathfrak{K}$ by the set of closed sets of X.

Let $\mathfrak{L}$ be the set of closed sets of X. We want to use Theorem 4.5.13 of [2] in order to show that the identity map

$$\mathcal{M}(\mathfrak{R}, G; \mathfrak{K})_{\mathfrak{T}} \to \mathcal{M}(\mathfrak{R}, G; \mathfrak{K})_{\mathfrak{L}}$$

in uniformly Φ_3-continuous. In fact, the hypotheses a), d), and e) of that theorem follow from Proposition 2 and the hypotheses b) and c) from Proposition 1. By [2] Proposition 4.5.6 the identity map

$$\mathcal{M}(\mathfrak{R}, G; \mathfrak{K})_{\mathfrak{L}} \to \mathcal{M}(\mathfrak{R}, G; \mathfrak{K})$$

is uniformly Φ_4-continuous and so by [2] Corollary 1.8.5 the identity map

$$\mathcal{M}(\mathfrak{R}, G; \mathfrak{K})_{\mathfrak{T}} \to \mathcal{M}(\mathfrak{R}, G; \mathfrak{K})$$

is uniformly Φ_3-continuous. $\square$

Remarks 1. The assertion and the proof still hold if we replace $\mathfrak{R}$ by a σ-ring of subsets of X containing $\mathfrak{T}$ and $\mathfrak{K}$ by the set of compact sets (closed sets if X is normal and equal to Y) of X belonging to $\mathfrak{R}$. This remark also holds for Corollary 4.

2. If Y is σ-Stonian, then the sets of $\mathfrak{T}$ are closed open sets of X. Hence the above formulation has the advantage of unifying the corresponding results with open regular sets ([2] Corollary 4.5.15) and with closed open sets ([2] Corollary 4.5.17).

Corollary 4. *If E is quasicomplete, then $\int \xi \, d\mu \in E$ for every $(\xi, \mu) \in \mathcal{M}^\pi \times \mathcal{M}(E)$ and the identity map*

$$\mathcal{M}(E)_{\mathfrak{T}} \to \mathcal{M}(E)_{\mathcal{M}^\pi}$$

is uniformly Φ_3-continuous.

By [1] Theorem 4.2.11, $\int \xi \, d\mu \in E$ for every $(\xi, \mu) \in \mathcal{M}^\pi \times \mathcal{M}(E)$. By Theorem 3 the identity map

$$\mathcal{M}(E)_{\mathfrak{T}} \to \mathcal{M}(E)$$

is uniformly Φ_3-continuous and the assertion follows from [2] Theorem 5.6.6. □

Proposition 5. *We have:*

a)
$$\mathscr{C} \subset \bigcap_{\mu \in \mathcal{M}} \mathscr{L}^1(\mu);$$

b) *the map*

$$\mu' : \mathscr{C} \to \mathbf{R}, \quad f \mapsto \int f \, d\mu$$

belongs to $\mathscr{C}'$ for every $\mu \in \mathcal{M}$;

c) *the map*

$$(\mathcal{M}, \mathcal{M}^\pi) \to \mathscr{C}', \quad \mu \mapsto \mu'$$

is an isomorphism of Banach spaces;

d) *the map*

$$u' : \mathcal{M} \to \mathbf{R}, \quad \mu \mapsto u(\mu')$$

belongs to $\mathcal{M}^\pi$ for every $u \in \mathscr{C}''$;

e) *the map*

$$\mathscr{C}'' \to \mathcal{M}^\pi, \quad u \mapsto u'$$

is an isomorphism of vector spaces.

The assertions follow from [5] Theorem 4.6 and Theorem 2.4 (iii) and [1] Proposition 3.4.2 b). □

Remark. We identify $\mathcal{M}$ with $\mathscr{C}'$ and $\mathcal{M}^\pi$ with $\mathscr{C}''$ via the above isomorphisms.

Theorem 6. *The identity map $(\mathscr{C}')_{\mathfrak{T}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous. If Y is σ-Stonian, then the identity map $(\mathscr{C}')_{\mathscr{C}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous.*

By Theorem 3 the identity map $\mathcal{M}_{\mathfrak{T}} \to \mathcal{M}$ is uniformly Φ_3-continuous and so, by [2] Theorem 5.6.6, the identity map $\mathcal{M}_{\mathfrak{T}} \to \mathcal{M}_{\mathcal{M}^\pi}$ is uniformly Φ_3-continuous. By the above identifications the identity map $(\mathscr{C}')_{\mathfrak{T}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous.

Assume now Y is σ-Stonian. Then every set of $\mathfrak{T}$ is a closed open set of X and therefore the identity map $(\mathscr{C}')_{\mathscr{C}} \to (\mathscr{C}')_{\mathfrak{T}}$ is uniformly continuous. Hence the identity map $(\mathscr{C}')_{\mathscr{C}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous. □

Remark. If X if σ-Stonian (or, equivalently, $\mathscr{C}$ is order σ-complete ([2] Lemma 5.9.15 a⇔b)), then (by taking $Y=X$) the identity map $(\mathscr{C}')_{\mathscr{C}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous.

Theorem 7. *Every Φ_3-set of $(\mathscr{C}')_{\mathfrak{T}}$ and every Φ_4-set of $(\mathscr{C}')_{\mathfrak{R}}$ is equicontinuous.*

Let $\mathscr{N}$ be a Φ_4-set of $(\mathscr{C}')_{\mathfrak{R}}$. Since $(\mathscr{C}')_{\mathfrak{R}} = \mathscr{M}$, we deduce by [2] Theorem 4.2.16 c, that there exists an increasing sequence $(K_n)_{n \in N}$ in $\mathfrak{K}$ such that

$$|\mu|(X \setminus K_n) < \frac{1}{4^n}$$

for every $\mu \in \mathscr{N}$ and $n \in N$. We set $K_{-1} := \emptyset$,

$$g\colon X \to \boldsymbol{R}_+,\ t \mapsto \begin{cases} \dfrac{1}{2^{n-1}} & \text{if } t \in K_n \setminus K_{n-1} \quad (n \in N), \\ 0 & \text{if } t \in X \setminus \bigcup_{n \in N} K_n, \end{cases}$$

and

$$\mathscr{U} := \{f \in \mathscr{C} \,|\, |fg| \leqq 1\}.$$

Then $\mathscr{U}$ is a 0-neighbourhood in $\mathscr{C}$. Let $\mu \in \mathscr{N}$ and $f \in \mathscr{U}$. We have

$$\left|\int f\,d\mu\right| \leqq \sum_{n \in N} \int_{K_n \setminus K_{n-1}} |f|\,d|\mu| \leqq \sum_{n \in N} \frac{2^{n-1}}{4^n} = 1.$$

Hence $\mathscr{N}$ is equicontinuous.

Assume now $\mathscr{N}$ is a Φ_3-set of $(\mathscr{C}')_{\mathfrak{T}}$. By Theorem 3 and [2] Theorem 1.8.4 a⇒h, $\mathscr{N}$ is a Φ_3-set of $(\mathscr{C}')_{\mathfrak{R}}$ and so, by the above considerations, $\mathscr{N}$ is equicontinuous. □

Corollary 8. *Every boundedly weakly compact continuous linear map $u\colon \mathscr{C} \to E$ with respect to the Mackey topology of $\mathscr{C}$ is continuous with respect to the strict topology of $\mathscr{C}$.*

Le A' be an equicontinuous set of E'. Since u is boundedly weakly compact, $u''(\mathscr{C}'') \subset E$ and so $u'(A')$ is a relatively compact set of $(\mathscr{C}')_{\mathscr{C}''}$. By Theorem 7, $u'(A')$ is equicontinuous; hence u is continuous with respect to the strict topology of $\mathscr{C}$. □

Remark. The following example[3]) will show that not every circled convex compact Φ_1-set of $(\mathscr{C}')_{\mathscr{C}}$ is equicontinuous, even if X is locally compact and normal.

[3]) This example appears in [3] Theorem 5.

In particular, the strict topology on $\mathscr{C}$ may be strictly coarser than its Mackey topology. Let ω_1 be the first uncountable ordinal number, X be the set ω_1 endowed with the usual locally compact topology

$$\{V \subset \omega_1 | \xi \in V \Rightarrow \exists \eta < \xi, \quad \{\zeta \in \omega_1 | \eta < \zeta \leqq \xi\} \subset V\},$$

and for every $\xi \in \omega_1$ let δ_ξ be the Dirac measure on X at ξ. Then X is locally compact and normal and the circled convex closed hull of

$$\{\delta_\xi - \delta_{\xi+1} | \xi \in \omega_1\}$$

is a compact Φ_1-set of $(\mathscr{C}')_{\mathscr{C}}$, which is not equicontinuous.

Corollary 9. *The set $\{x' \circ \mu | x' \in A', \ \mu \in \mathscr{N}\}$ is an equicontinuous set of $\mathscr{C}'$ for every equicontinuous set A' of E' and for every Φ_4-set $\mathscr{N}$ of $\mathscr{M}(E)$.*

We show first that the map

$$A'_E \times \mathscr{M}(E) \to \mathscr{M}, \quad (x', \mu) \mapsto x' \circ \mu$$

is continuous. Let $(x'_0, \mu_0) \in A' \times \mathscr{M}(E)$ and let $A \in \mathfrak{R}$ and $\varepsilon > 0$. There exist a 0-neighbourhood U in E such that $|x'(x)| < \varepsilon/2$ for every $(x', x) \in A' \times U$ and a neighbourhood V of x'_0 in A'_E such that

$$\left|x'(\mu_0(A)) - x'_0(\mu_0(A))\right| < \frac{\varepsilon}{2}$$

for every $x' \in V$. Further, there exists a neighbourhood $\mathscr{W}$ of μ_0 in $\mathscr{M}(E)$ such that

$$\mu(A) - \mu_0(A) \in U$$

for every $\mu \in \mathscr{W}$. We get

$$|x' \circ \mu(A) - x'_0 \circ \mu_0(A)| \leqq \left|x'(\mu(A) - \mu_0(A))\right| + \left|x'(\mu_0(A)) - x'_0(\mu_0(A))\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $(x', \mu) \in V \times \mathscr{W}$. Hence the map

$$A'_E \times \mathscr{M}(E) \to \mathscr{M}, \quad (x', \mu) \mapsto x' \circ \mu$$

is continuous.

In order to prove the assertion of the corollary we may assume E complete. Then $\mathscr{N}$ is a relatively compact set of $\mathscr{M}(E)$ ([2] Theorem 4.2.16 a)) and by the above considerations $\{x' \circ \mu | x' \in A', \ \mu \in \mathscr{N}\}$ is a relatively compact set of $\mathscr{M}$. By Theorem 7 this set is equicontinuous. □

Proposition 10. *Let E be quasicomplete and $\mathscr{L}_0(\mathscr{C}, E)$ be the vector space of boundedly weakly compact continuous linear maps of $\mathscr{C}$ into E. We denote by $\bar{\mu}$ the map*

$$\mathscr{C} \to E, \quad f \mapsto \int f d\mu$$

for every $\mu \in \mathscr{M}(E)$. Then

a) $\{\bar{\mu}|\mu\in\mathcal{N}\}$ *is an equicontinuous set of* $\mathcal{L}_0(\mathcal{C},E)$ *for every* Φ_4*-set* $\mathcal{N}$ *of* $\mathcal{M}(E)$*:*

b) *the map*

$$\mathcal{M}(E)\to\mathcal{L}_0(\mathcal{C},E),\quad \mu\mapsto\bar{\mu}$$

is an isomorphism of vector spaces;

c) *for every* $\mu\in\mathcal{M}(E)$ *the maps*

$$E'\to\mathcal{C}',\quad x'\mapsto x'\circ\mu$$

$$\mathcal{C}''\to E,\quad \xi\mapsto\int\xi\,d\mu$$

are the adjoint and the biadjoint of $\bar{\mu}$ *and the map*

$$(\mathcal{C})_{\mathcal{C}'}\to E,\quad f\mapsto\int f\,d\mu$$

is uniformly Φ_4*-continuous;*

d) *every* Φ_3*-set of* $\mathcal{L}_0(\mathcal{C},E)_{\mathfrak{T}}$ *is equicontinuous; in particular, if* Y *is* σ*-Stonian, then every* Φ_3*-set of* $\mathcal{L}_0(\mathcal{C},E)_{\mathcal{C}}$ *is equicontinuous.*

Let $\mu\in\mathcal{M}(E)$. By Proposition 5 a), $\mathcal{C}\subset\mathcal{L}^1(\mu)$ and by [1] Theorem 4.2.11, $\int f\,d\mu\in E$ for every $f\in\mathcal{C}$.

a) Let A' be an equicontinuous set of E'. By Corollary 9, $\{x'\circ\mu|x'\in A',\ \mu\in\mathcal{N}\}$ is an equicontinuous set of $\mathcal{C}'$. Since A' is arbitrary, $\{\bar{\mu}|\mu\in\mathcal{N}\}$ is an equicontinuous set of linear maps of $\mathcal{C}$ into E. By [1] Theorem 4.2.11 this set is contained in $\mathcal{L}_0(\mathcal{C},E)$.

b) It is obvious that the map

$$\mathcal{M}(E)\to\mathcal{L}_0(\mathcal{C},E),\quad \mu\mapsto\bar{\mu}$$

is injective and linear. By [1] Proposition 4.3.9 a) this map is surjective.

c) Let $x'\in E'$. Then

$$(\bar{\mu}'(x'))(f)=x'(\bar{\mu}(f))=x'\left(\int f\,d\mu\right)=\int f\,d(x'\circ\mu)=(x'\circ\mu)(f)$$

for every $f\in\mathcal{C}$ and so $\bar{\mu}'(x')=x'\circ\mu$. Hence

$$E'\to\mathcal{C}',\quad x'\mapsto x'\circ\mu$$

is the adjoint map of $\bar{\mu}$.

Let $\xi\in\mathcal{C}''=\mathcal{M}^{\pi}$. Then

$$(\bar{\mu}''(\xi))(x')=\xi(\bar{\mu}'(x'))=\xi(x'\circ\mu)=\int\xi\,d(x'\circ\mu)=\left(\int\xi\,d\mu\right)(x')$$

for every $x'\in E'$ and so $\bar{\mu}''(\xi)=\int\xi\,d\mu$. Hence ([1] Theorem 4.2.11)

$$\mathcal{C}''\to E,\quad \xi\mapsto\int\xi\,d\mu$$

is the biadjoint map of $\bar{\mu}$.

By [2] Corollary 5.8.26, $\mathcal{C}$ endowed with the order relation induced by $\boldsymbol{R}^X$ is an M-space. By [2] Corollary 5.7.7 the map

$$(\mathcal{C})_{\mathcal{C}'}\to E,\quad f\mapsto\int f\,d\mu$$

is uniformly Φ_4-continuous.

d) Let $\mathcal{N}$ be a Φ_3-set of $\mathcal{L}_0(\mathscr{C}, E)_{\mathfrak{T}}$. By b) we may consider $\mathcal{N}$ to be a Φ_3-set of $\mathcal{M}(E)_{\mathfrak{T}}$ and so, by Corollary 4 and [2] Theorem 1.8.4 a⇒h, it is a Φ_3-set of $\mathcal{M}(E)$. Let A' be an equicontinuous set of E'. By Corollary 9 there exists a 0-neighbourhood $\mathscr{U}$ in $\mathscr{C}$ such that

$$\left|x'\left(\int f d\mu\right)\right| = \left|\int f d(x' \circ \mu)\right| \leqq 1$$

for every $f \in \mathscr{U}$ and every $(x', \mu) \in A' \times \mathcal{N}$. Hence $\mathcal{N}$ is an equicontinuous set of $\mathcal{L}_0(\mathscr{C}, E)$.

If Y is σ-Stonian, then every set of $\mathfrak{T}$ is a closed open set of X, so every Φ_3-set of $\mathcal{L}_0(\mathscr{C}, E)_{\mathscr{C}}$ is a Φ_3-set of $\mathcal{L}_0(\mathscr{C}, E)_{\mathfrak{T}}$ and it is equicontinuous by the above considerations. □

Remark. The assertion b) was proved by A. Grothendieck ([6] Proposition 14) for X compact.

Proposition 11. *Let A'' be a subset of E'' such that the identity map $E'_{A''} \to E'_{E''}$ is sequentially continuous, F be a G-space and u: $E \to F$ be a continuous linear map such that $u''(A'') \subset F$. If E possesses the strong DP-property, we have:*

a) *the map $E_{E'} \to F$ defined by u is uniformly Φ_4-continuous;*

b) *if in addition E possesses the D-property and F is quasicomplete, then u is boundedly weakly compact.*

a) Let $\mathfrak{A}'$ be the set of Φ_4-sets of $E'_{E''}$ and $\mathfrak{B}'$ be the set of Φ_1-sets of F'_F. Let $B' \in \mathfrak{B}'$. Since $u''(A'') \subset F$, the map $F'_F \to E'_{A''}$ defined by u' is continuous and so $u'(B')$ is a Φ_1-set of $E'_{A''}$. The map $E_{A''} \to E'_{E''}$ being sequentially continuous, $u'(B')$ is a Φ_1-set of $E'_{E''}$ ([2] Proposition 1.5.4. a⇔c) and so it belongs to $\mathfrak{A}'$. Hence $u'(\mathfrak{B}') \subset \mathfrak{A}'$ and the map $E_{\mathfrak{A}'} \to F_{\mathfrak{B}'}$ defined by u is continuous. Since E possesses the strong DP-property, the identity map $E_{E'} \to E_{\mathfrak{A}'}$ is uniformly Φ_4-continuous. Since F is a G-space, the identity map $F_{\mathfrak{B}'} \to F$ is uniformly Φ_4-continuous. Putting together the above results we deduce by [2] Corollary 1.8.5 that the map $E_{E'} \to F$ defined by u is uniformly Φ_4-continuous.

b) Let $(x_n)_{n \in N}$ be a weak Cauchy sequence in E. By a), $(u(x_n))_{n \in N}$ is a Cauchy sequence and so a convergent sequence in F. Since E possesses the D-property, u is boundedly weakly compact. □

Theorem 12. *Let E be a G-space and u: $\mathscr{C} \to E$ be a continuous linear map such that (with the usual identifications) $u''(1_T^X) \in E$ for every $T \in \mathfrak{T}$ (this condition is automatically fulfilled if Y is σ-Stonian). We have:*

a) *the map $\mathscr{C}_{\mathscr{C}'} \to E$ defined by u is uniformly Φ_4-continuous;*

b) *if E quasicomplete, then u is boundedly weakly compact and there exists a unique $\mu \in \mathcal{M}(E)$ such that $\int \xi \, d\mu \in E$ for every $\xi \in \mathcal{M}^{\pi}$,*

$$u(f) = \int f d\mu$$

for every $f \in \mathscr{C}$,

$$u'(x') = x' \circ \mu$$

for every $x' \in E'$, *and*

$$u''(\xi) = \int \xi \, d\mu$$

for every $\xi \in \mathscr{C}''$.

By Theorem 6 the identity map $(\mathscr{C}')_{\mathfrak{T}} \to (\mathscr{C}')_{\mathscr{C}''}$ is uniformly Φ_3-continuous. By [2] Corollary 5.8.26, $\mathscr{C}$ endowed with the order relation induced by $\boldsymbol{R}^X$ is an M-space and so by [2] Corollary 5.7.9 and Theorem 5.8.9 c) it possesses the strong DP-property and the D-property. Hence by Proposition 11 the map $\mathscr{C}_{\mathscr{C}'} \to E$ defined by u is uniformly Φ_4-continuous and u is boundedly weakly compact if E is quasicomplete. The other assertions follow from Proposition 10.

If Y is σ-Stonian, then every set of $\mathfrak{T}$ is a closed open set of X and so $1_T^X \in \mathscr{C}$ and $u''(1_T^X) = u(1_T^X) \in E$ for every $T \in \mathfrak{T}$. □

Corollary 13. *If* Y *is* σ*-Stonian and* $\mathscr{C}$ *is a G-space, then every compact set of* X *is finite and* $\mathscr{C}$ *is semi-separable.*

Let K be a compact set of X and $\mathscr{F}$ be the Banach space of continuous real functions on K. By Theorem 12 a) the identity map $\mathscr{C}_{\mathscr{C}'} \to \mathscr{C}$ is uniformly Φ_4-continuous and so the map

$$\mathscr{C}_{\mathscr{C}'} \to \mathscr{F}, \quad f \mapsto f|K$$

is also uniformly Φ_4-continuous. Let $(f_n)_{n \in N}$ be a weak Cauchy sequence in $\mathscr{C}$. Then, by the above result, $(f_n|K)_{n \in N}$ is a Cauchy sequence and so a convergent sequence. By [2] Corollary 5.8.26 and Theorem 5.8.9 a) the map

$$\mathscr{C} \to \mathscr{F}, \quad f \mapsto f|K$$

is boundedly weakly compact; hence the balls of $\mathscr{F}$ are weakly compact. We deduce K is finite.

Let g be a positive real function on X such that $\{g \geqq \varepsilon\}$ is relatively compact for every $\varepsilon > 0$. Then $\{g > 0\}$ is countable. We denote by $\mathscr{H}$ the set of real functions h on $\{g > 0\}$ such that $\{h \neq 0\}$ is finite and

$$h(\{g > 0\}) \subset \boldsymbol{Q} \cap [-1, 1].$$

Then $\mathscr{H}$ is also countable and for every $h \in \mathscr{H}$ there exists an $h' \in \mathscr{C}$ such that $|h'| \leqq 1$ and $h' = h$ on $\{h \neq 0\}$. Then

$$\{\alpha h' \mid \alpha \in \boldsymbol{Q},\ h \in \mathscr{H}\}$$

is countable and for every $f \in \mathscr{C}$ and every $\varepsilon > 0$ there exist $\alpha \in \boldsymbol{Q}$ and $h \in \mathscr{H}$ such that

$$\sup_{x \in X} \left| (\alpha h'(x) - f(x))\, g(x) \right| < \varepsilon.$$

Hence $\mathscr{C}$ is semi-separable. □

Remark. Let $\mathfrak{F}$ be a filter on N finer than the section filter of N and x be a point not belonging to N. We set $X:=N\cup\{x\}$ and endow X with the topology

$$\{V\subset X\,|\,x\in V\Rightarrow V\cap N\in F\}.$$

Then X is a nondiscrete paracompact space. If $\mathfrak{F}$ is an ultrafilter, then X is σ-Stonian and its compact sets are finite. If there exist two different ultrafilters $\mathfrak{F}'$, $\mathfrak{F}''$ on N such that $\mathfrak{F}=\mathfrak{F}'\cap\mathfrak{F}''$, then the compact sets of X are finite but X is not σ-Stonian.

Proposition 14. *Let Z be a topological space such that the neighbourhood filter of every point of Z belongs to $\hat{\Phi}_1(Z)$, let $(\mu_z)_{z\in Z}$ be a family in $\mathcal{M}(E)$ such that the map*

$$Z\to E,\quad z\mapsto\mu_z(T)$$

is continuous for every $T\in\mathfrak{T}$, and let $\xi\in\mathcal{M}^\pi$ be such that $\int\xi\,d\mu_z\in E$ for every $z\in Z$. Then the map

$$Z\to E,\quad z\mapsto\int\xi\,d\mu_z$$

is continuous.

We may assume E complete. By Corollary 4 the identity map $\mathcal{M}(E)_{\mathfrak{T}}\to\mathcal{M}(E)_{\mathcal{M}^\pi}$ is uniformly Φ_3-continuous and so ([2] Proposition 1.8.3) Φ_1-continuous. By the hypothesis the map

$$Z\to\mathcal{M}(E)_{\mathfrak{T}},\quad z\mapsto\mu_z$$

is continuous, and so the map

$$Z\to\mathcal{M}(E)_{\mathcal{M}^\pi},\quad z\mapsto\mu_z$$

is Φ_1-continuous. Since the neighbourhood filter of every point of Z belongs to $\hat{\Phi}_1(Z)$, this map is continuous ([2] Proposition 1.3.6). This is exactly the assertion that the map

$$Z\to E,\quad z\mapsto\int\xi\,d\mu_z$$

is continuous for every $\xi\in\mathcal{M}^\pi$. □

Theorem 15. *Let Z be a Hausdorff topological space such that the neighbourhood filter of every point of Z possesses a countable base, $\mathcal{F}$ be the vector space of continuous maps of Z into E endowed with the topology of compact convergence and $(\mu_z)_{z\in Z}$ be a family in $\mathcal{M}(E)$ such that the map*

$$Z\to E,\quad z\mapsto\mu_z(T)$$

is continuous for every $T\in\mathfrak{T}$. If E is quasicomplete, we have:

a) *$\mathcal{F}$ is quasicomplete;*

b) *there exists a unique $\mu\in\mathcal{M}(\mathcal{F})$ such that $\int\xi\,d\mu\in\mathcal{F}$, $\int\xi\,d\mu_z\in E$ and*

$$\left(\int\xi\,d\mu\right)(z)=\int\xi\,d\mu_z$$

for every $\xi\in\mathcal{M}^\pi$ and $z\in Z$;

c) *the map*

$$\mathscr{C} \to \mathscr{F}, \quad f \mapsto \int f\,d\mu$$

is continuous and boundedly weakly compact,

$$\mathscr{M}^\pi \to \mathscr{F}, \quad \xi \mapsto \int \xi\,d\mu$$

is its biadjoint map and the map

$$\mathscr{C}_{\mathscr{C}'} \to \mathscr{F}, \quad f \mapsto \int f\,d\mu$$

is uniformly Φ_4-continuous.

a) For every $z \in Z$ let ψ_z be the map

$$\mathscr{F} \to E, \quad f \mapsto f(z)$$

and let $\mathfrak{F}$ be a Cauchy filter on $\mathscr{F}$ possessing a bounded set of $\mathscr{F}$. Then $\psi_z(\mathfrak{F})$ converges for every $z \in Z$. We set

$$f\colon Z \to E, \quad z \mapsto \lim \psi_z(\mathfrak{F}).$$

The restriction of f to every compact set of Z is continuous. Since the neighbourhood filter of every point of Z possesses a countable base, f is continuous. It is easy to see that $\mathfrak{F}$ converges to f in $\mathscr{F}$. Hence $\mathscr{F}$ is quasicompact.

b) By a) and [1] Proposition 4.2.11, $\int \xi\,d\mu \in \mathscr{F}$ and $\int \xi\,d\mu_z \in E$ for every $\xi \in \mathscr{M}^\pi$, $\mu \in \mathscr{M}(\mathscr{F})$, and $z \in Z$. By [2] Proposition 1.5.31 the neighbourhood filter of every point of Z belongs to $\Phi_1(Z)$ and so, by Proposition 14, the map

$$Z \to E, \quad z \mapsto \int \xi\,d\mu_z$$

is continuous for every $\xi \in \mathscr{M}^\pi$. We set

$$\mu(A)\colon Z \to E, \quad z \mapsto \mu_z(A)$$

for every $A \in \mathfrak{R}$ and

$$\mu\colon \mathfrak{R} \to \mathscr{F}, \quad A \mapsto \mu(A).$$

By [2] Theorem 4.6.3 b) $\mu \in \mathscr{M}(\mathfrak{R}, \mathscr{F}; \mathfrak{K})$, and by [2] Proposition 5.6.3

$$\mathscr{M}(\mathfrak{R}, \mathscr{F}; \mathfrak{K}) = \mathscr{M}(\mathscr{F}).$$

Let $z \in Z$ and $x' \in E'$. We set

$$\varphi\colon \mathscr{F} \to \mathbf{R}, \quad f \mapsto x'(f(z)).$$

The function φ is a continuous linear form and

$$\varphi \circ \mu(A) = x'((\mu(A))(z)) = x'(\mu_z(A)) = x' \circ \mu_z(A)$$

for every $A \in \mathfrak{R}$ and so $\varphi \circ \mu = x' \circ \mu_z$. Let $\xi \in \mathscr{M}^\pi$. We have

$$x'\left(\left(\int \xi\,d\mu\right)(z)\right) = \varphi\left(\int \xi\,d\mu\right) = \int \xi\,d(\varphi \circ \mu) = \int \xi\,d(x' \circ \mu_z) = x'\left(\int \xi\,d\mu_z\right).$$

Since x' is arbitrary, we deduce

$$\left(\int \xi \, d\mu\right)(z) = \int \xi \, d\mu_z .$$

The unicity of μ is trivial.

c) follows from a) and Proposition 10. □

Theorem 16. *Let E be semi-separable, Z be a locally metrizable topological space, $\mathscr{F}$ be the vector space of continuous maps of Z into E endowed with the topology of compact convergence and $u\colon \mathscr{C}\to\mathscr{F}$ a continuous map such that (with the usual identifications) $u''(1_T^X)\in\mathscr{F}$ for every $T\in\mathfrak{T}$. We have:*

a) *the map $\mathscr{C}_{\mathscr{C}'}\to\mathscr{F}$ defined by u is uniformly Φ_4-continuous;*

b) *If E is quasicomplete, then u is boundedly weakly compact and there exist uniquely a $\mu\in\mathscr{M}(\mathscr{F})$ and a family $(\mu_z)_{z\in Z}$ in $\mathscr{M}(E)$ such that*

$$u(f) = \int f \, d\mu, \quad \big(u(f)\big)(z) = \int f \, d\mu_z ,$$
$$u''(\xi) = \int \xi \, d\mu, \quad \big(u''(\xi)\big)(z) = \int \xi \, d\mu_z$$

for every $f\in\mathscr{C}$, $\xi\in\mathscr{C}''$ and $z\in Z$.

By [2] Proposition 5.9.30, $\mathscr{F}$ is a G-space and, by Theorem 15a), it is quasicomplete if E is quasicomplete. By Theorem 12 the map $\mathscr{C}_{\mathscr{C}'}\to\mathscr{F}$ defined by u is uniformly Φ_4-continuous, and if E is quasicomplete, then there exists a unique $\mu\in\mathscr{M}(\mathscr{F})$ such that

$$u(f) = \int f d\mu$$

and

$$u''(\xi) = \int \xi \, d\mu$$

for every $f\in\mathscr{C}$ and $\xi\in\mathscr{C}''$. Let $z\in Z$ and let v be the map

$$\mathscr{C} \to E, \quad f \mapsto \big(u(f)\big)(z).$$

Then v is a continuous map such that $v''(1_T^X)\in E$ for every $T\in\mathfrak{T}$. By the above considerations there exists a unique $\mu_z\in\mathscr{M}(E)$ such that

$$v(f) = \int f d\mu_z$$

for every $f\in\mathscr{C}$. We have

$$\int f d\mu_z = v(f) = \big(u(f)\big)(z) = \left(\int f d\mu\right)(z)$$

for every $f\in\mathscr{C}$ and so

$$\mu_z(A) = \big(\mu(A)\big)(z)$$

for every $A\in\mathfrak{R}$ and

$$\int \xi \, d\mu_z = \left(\int \xi \, d\mu\right)(z) = \big(u''(\xi)\big)(z)$$

for every $\xi\in\mathscr{C}''$. □

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