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MÖBIUS AUTOMORPHISMS OF PLANE DOMAINS

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1. Introduction

Let D be a domain on the Riemann sphere $C\cup\{\infty\}$, and let Aut (D) be the group of biholomorphic automorphisms of D. We say that a subgroup G of Aut (D) is *discontinuous* if for each $z\in D$ the orbit $\{g(z); g\in G\}$ has no accumulation points in D. By a classical theorem Aut (D) is discontinuous if and only if the fundamental group of D is not abelian.

Let Möb (D) be the group of all biholomorphic automorphisms of D which are restrictions of Möbius transformations. If Aut (D) is a discontinuous group, then it is clear that the subgroup Möb (D) is also discontinuous. However, the converse is not true. In this paper we classify all domains D having a non-discontinuous group of Möbius automorphisms. Partial results in this direction appear in [1] and the special case Möb $(D)=$Aut (D) has also been studied by Minda [7]. The author is grateful to Professor Olli Lehto for stimulating his interest in this research.

Examples. (a) If D is the sphere $C\cup\{\infty\}$, the plane C or the punctured plane $C^*=C\setminus\{0\}$, then Möb (D) is not discontinuous. In fact, Möb (D) then contains all rotations $z\to xz$ with $|x|=1$. These rotations are in Möb (D) also if D is the unit disc $U=\{z; |z|<1\}$, the punctured unit disc $U^*=U\setminus\{0\}$ or an annulus

$$\{z; r_1<|z|<r_2\}.$$

(b) If D is a horizontal strip $\{z; y_1<\operatorname{Im} z<y_2\}$, then Möb (D) contains all translations $z\to z+b$, where b is real.

(c) Let $D=\{z\in C^*; \theta_1<\arg z-\alpha\log|z|<\theta_2\}$, where α is a real constant and $0<\theta_2-\theta_1\leqq 2\pi$. If $\alpha=0$, then Möb (D) contains all homotheties $z\to az$ with $a>0$. If $\alpha\neq 0$, then the boundary of D consists of one or two spirals, and D is invariant under the action of a one-dimensional group of loxodromic transformations $z\to e^{(1+\alpha i)t}z$, where t is a real parameter.

Suppose that h is a Möbius transformation which maps D onto a domain D'. Then the elements of Möb (D') are of the form $h\circ\varphi\circ h^{-1}$, where $\varphi\in$Möb (D). It follows that Möb (D') is discontinuous if and only if Möb (D) is discontinuous. In particular, if h maps D onto one of the domains mentioned in the above examples,

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then Möb (D) is not discontinuous. The following theorem shows that the converse is also true.

Theorem 1. *Suppose that* Möb (D) *is not discontinuous. Then there exists a Möbius transformation which maps* D *onto one of the domains described in examples* (a), (b) *and* (c).

The proof of Theorem 1 is elementary if D has at most two boundary points, for then there exists a Möbius transformation which maps D onto $C \cup \{\infty\}$, C or C^*. Thus we may assume that D has more than two boundary points. Since Möb (D) is not discontinuous by hypothesis, the fundamental group of D is abelian. It follows that D is conformally equivalent to a disc, a punctured disc or an annulus.

Theorem 1 will be proved in Sections 2 and 3. In Section 2 we consider domains conformally equivalent to a disc; the proof in this case depends on a characterization of discontinuous groups acting in the upper half plane (Theorem 2). Section 3 is devoted to the doubly connected case. Finally, in Section 4 we characterize domains D such that Möb (D) acts transitively on D.

It is important to note that Aut (D) is a topological group. A subset S of Aut (D) is *closed* if it contains the limit of every sequence of S converging to an element of Aut (D) uniformly on compact subsets of D. A well-known property of Möbius transformations [6, p. 73] implies that Möb (D) is a closed subgroup of Aut (D).

2. Simply connected domains

Let Γ be the group of all sense-preserving Möbius transformations mapping the upper half plane $H=\{z; \operatorname{Im} z>0\}$ onto itself. We shall identify Γ with the group Aut (H) of biholomorphic automorphisms of H. In particular, a subgroup of Γ is discontinuous if it is discontinuous as a subgroup of Aut (H).

If $g \in \Gamma$ is not the identity of Γ, then g is called *elliptic, parabolic* or *hyperbolic* according as g has 0, 1 or 2 fixed points on the boundary ∂H of H. It is clear that the classes of elliptic, parabolic and hyperbolic elements of Γ are invariant under inner automorphisms of Γ.

For $\zeta \in C \cup \{\infty\}$ we denote by Γ_ζ the isotropy group of ζ in Γ. Thus Γ_ζ consists of elements $g \in \Gamma$ such that $g(\zeta)=\zeta$. More generally, if A is a subset of $C \cup \{\infty\}$, we denote by Γ_A the set of elements $g \in \Gamma$ such that $gA=A$. A subgroup G of Γ is *elementary* if there exists a nonempty set A containing at most two points such that $G \subset \Gamma_A$.

The following elementary subgroups of Γ are of particular interest. An *elliptic continuum* is of the form Γ_ζ for some $\zeta \in H$; it contains all elliptic elements $g \in \Gamma$ such that $g(\zeta)=\zeta$. A subgroup of Γ is a *parabolic continuum* if it is conjugate to the subgroup of all translations $z \to z+b$ where b is real. A *hyperbolic continuum* is of the form $\Gamma_\zeta \cap \Gamma_{\zeta'}$, where ζ and ζ' are distinct points of ∂H. Equivalently, a hyperbolic continuum is a subgroup of Γ which is conjugate to the group of homotheties $z \to az$ with $a>0$. It is clear that all proper closed subgroups of an elliptic, parabolic or hyperbolic continuum are discontinuous.

We shall need the following result which is related to a theorem of Jørgensen [4, Theorem 2].

Theorem 2. *A closed subgroup of Γ is discontinuous if and only if it does not contain any elliptic, parabolic or hyperbolic continua.*

Proof. The necessity is obvious because a subgroup of Γ is never discontinuous if it contains a continuum. For the sufficiency, assume that G is a closed subgroup of Γ and that G is not discontinuous. We have to prove that G contains at least one elliptic, parabolic or hyperbolic continuum.

Suppose first that G is elementary. Then there exists a nonempty set A containing at most two points such that $G \subset \Gamma_A$.

If A contains two distinct points ζ and ζ', then Γ_A contains $\Gamma_\zeta \cap \Gamma_{\zeta'}$ as a subgroup of finite index. Since G is not discontinuous and $G \subset \Gamma_A$, it follows that $\Gamma_\zeta \cap \Gamma_{\zeta'}$ is an elliptic or hyperbolic continuum and that $G \cap (\Gamma_\zeta \cap \Gamma_{\zeta'})$ is not discontinuous. Since all proper closed subgroups of $\Gamma_\zeta \cap \Gamma_{\zeta'}$ are discontinuous, we conclude that $G \supset \Gamma_\zeta \cap \Gamma_{\zeta'}$. Hence G contains an elliptic or hyperbolic continuum.

In the remaining case G is not contained in any subgroup $\Gamma_\zeta \cap \Gamma_{\zeta'}$ with $\zeta \neq \zeta'$. It follows that $A = \{\zeta\}$, where ζ is the only common fixed point for elements of G. Hence $\zeta \in \partial H$, so that G contains no elliptic elements.

If G contains no hyperbolic elements, then it is clear that G is the parabolic continuum contained in Γ_ζ. If G contains a hyperbolic element, then G contains also parabolic elements because otherwise G would be contained in a hyperbolic continuum [1, Lemma 3.2 (b)]. Let p and h be parabolic and hyperbolic elements of G, respectively, and suppose that ζ is the attractive fixed point of h. If k is a positive integer, then $p_k = h^{-k} p h^k$ is a parabolic element of G, and a computation shows that the sequence $\{p_k\}$ converges to the identity of G as $k \to \infty$. Hence G contains infinitesimal parabolic elements of Γ_ζ. Since G is closed, we conclude that G contains the parabolic continuum contained in Γ_ζ.

Finally, suppose that G is non-elementary. In [4] it is shown that a non-elementary subgroup of Γ is discontinuous if it does not contain elliptic elements of infinite order. Since G is not discontinuous, it follows that at least one elliptic element $g \in G$ is of infinite order. The smallest closed subgroup of Γ containing g is an elliptic continuum. Since G is closed, this elliptic continuum is contained in G. The proof of Theorem 2 is now complete.

We wish to apply Theorem 2 to the proof of Theorem 1 in the case of a simply connected domain D with more than two boundary points. In the remainder of this section F denotes a fixed conformal mapping from H onto D.

There is a bicontinuous isomorphism F_* from Aut (D) onto Aut (H) such that $F_*(\varphi) = F^{-1} \circ \varphi \circ F$ for each $\varphi \in$ Aut (D). Let G be the image of Möb (D)

under F_*. Since Möb (D) is a closed subgroup of Aut (D), G is a closed subgroup of Aut (H). Furthermore G is not discontinuous, because Möb (D) is not discontinuous by hypothesis.

By Theorem 2 we have an elliptic, parabolic and hyperbolic case according as G contains an elliptic, parabolic or hyperbolic continuum. We now show in each case that the assertion of Theorem 1 follows.

Elliptic case. In this case G contains an elliptic continuum; hence there exists $\zeta\in H$ such that $G\supset\Gamma_\zeta$.

Let h be a conformal map from the unit disc U onto H such that $h(0)=\zeta$. It suffices to prove that $f=F\circ h$ is the restriction of a Möbius transformation, because then the inverse of f is a Möbius transformation mapping D onto U.

Suppose $\varrho\in\operatorname{Aut}(U)$ and $\varrho(0)=0$. Then $h\circ\varrho\circ h^{-1}$ is in Γ_ζ. Since $\Gamma_\zeta\subset G$, there is $\varphi\in$ Möb (D) such that $h\circ\varrho\circ h^{-1}=F^{-1}\circ\varphi\circ F$; hence

$$f\circ\varrho=\varphi\circ f. \tag{1}$$

As in [1, p. 22], we shall use properties of the Schwarzian derivative

$$Sf=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of f. Above all, f is the restriction of a Möbius transformation if and only if $Sf\equiv 0$. Since ϱ and φ are restrictions of Möbius transformations, it follows from (1) that

$$(Sf\circ\varrho)(\varrho')^2=Sf.$$

On the other hand, ϱ is a rotation $z\to xz$ where $x\in C$ and $|x|=1$, so that

$$Sf(xz)x^2=Sf(z)\quad(z\in U,\ |x|=1). \tag{2}$$

For a fixed $z\in U$, the left side of (2) is a holomorphic function of x in the domain $\{x\in C;\ xz\in U\}$ and assumes the constant value $Sf(z)$ on the unit circle. Thus (2) holds by analytic continuation for every x with $xz\in U$, and the substitution $x=0$ yields $Sf(z)=0$. We conclude that f agrees in U with a Möbius transformation.

Parabolic case. In this case there exists $\zeta\in\partial H$ such that G contains all parabolic elements of Γ_ζ. For any $h\in\Gamma$ with $h(\infty)=\zeta$ the composite $f=F\circ h$ is defined in H and maps H conformally onto D. As in the elliptic case it suffices to show that $Sf\equiv 0$ in H.

Let $\tau\in\Gamma$ be a translation $z\to z+b$ where b is real. Then $h\circ\tau\circ h^{-1}$ is a parabolic element of Γ_ζ; hence there exists $\varphi\in$ Möb (D) such that $h\circ\tau\circ h^{-1}=F^{-1}\circ\varphi\circ F$ or

$$f\circ\tau=\varphi\circ f.$$

Taking Schwarzian derivatives of both sides yields

$$Sf(z+b)=Sf(z)\quad(z\in H,\ b\in R).$$

Thus Sf assumes a constant value on straight lines parallel to the real axis. Since Sf is holomorphic, it follows that $Sf \equiv c$, where c is a complex constant.

For $c \neq 0$ the solutions of the differential equation $Sf \equiv c$ are of the form $f = g \circ f_0$, where g is a Möbius transformation and $f_0(z) = e^{az}$ for some $a \in C^*$. Such solutions fail to be one-to-one in the upper half plane. In the present situation, however, $f = F \circ h$ is one-to-one in H. Hence $c = 0$, and the proof of the parabolic case is complete.

Hyperbolic case. Since G contains a hyperbolic continuum, there exist distinct points $\zeta, \zeta' \in \partial H$ such that $G \supset \Gamma_\zeta \cap \Gamma_{\zeta'}$. Choose $h \in \Gamma$ so that $h(0) = \zeta$ and $h(\infty) = \zeta'$. Then $f = F \circ h$ maps H conformally onto D. We shall prove that $D = fH$ can be mapped by means of a Möbius transformation onto a horizontal strip or onto a domain of the form

$$\{z \in C^*;\ \theta_1 < \arg z - \alpha \log |z| < \theta_2\}, \tag{3}$$

where $0 < \theta_2 - \theta_1 \leqq 2\pi$.

Let $\sigma \in \Gamma$ be a stretching $z \to az$ where $a > 0$. Then $h \circ \sigma \circ h^{-1}$ is in $\Gamma_\zeta \cap \Gamma_{\zeta'}$; hence there exists $\varphi \in \text{Möb}\,(D)$ such that $h \circ \sigma \circ h^{-1} = F^{-1} \circ \varphi \circ F$ or

$$f \circ \sigma = \varphi \circ f.$$

By taking Schwarzian derivatives again it follows that

$$Sf(az)a^2 = Sf(z) \quad (z \in H,\ a > 0). \tag{4}$$

For a fixed $t \in H$, the left side of (4) is a holomorphic function of a in the domain $\{a \in C;\ az \in H\}$ and assumes the constant value $Sf(z)$ on the positive real axis. Thus (4) holds by analytic continuation for all values of a such that $az \in H$. The substitution $w = az$ yields

$$Sf(w)w^2 = Sf(z)z^2.$$

Since this holds for each $w \in H$, we conclude that f satisfies in H the differential equation

$$Sf(z)z^2 = c, \tag{5}$$

where c is a complex constant.

The solutions of (5) are of the form $f = g \circ f_0$, where g is a Möbius transformation and either $f_0(z) = \log z$ or $f_0(z) = z^\varkappa$ for some $\varkappa = \gamma + i\delta \in C^*$. If $f_0(z) = \log z$, then $f_0 H$ is a horizontal strip and g^{-1} maps D onto this strip. If $f_0(z) = z^{\gamma + i\delta}$, then $\gamma \neq 0$, because $f = F \circ h$ is one-to-one in H. In this case $f_0 H$ is a domain of the form (3), where $\alpha = \delta/\gamma$ and the interval $[\theta_1, \theta_2]$ has 0 and $\gamma\pi(1+\alpha^2)$ as its endpoints. Hence g^{-1} maps D onto a domain of the form (3). Note that $\theta_2 - \theta_1 \leqq 2\pi$, because otherwise (3) would agree with C^* which is not a simply connected domain.

The proof of the simply connected case of Theorem 1 is now complete.

3. Doubly connected domains

We have proved Theorem 1 so far for domains D which are simply connected or have only two boundary points. As pointed out in the Introduction, it remains to consider the case of a domain D which is conformally equivalent to a punctured disc or an annulus. In this section f denotes a conformal map from the domain

$$R = \{z \in C;\ r_1 < |z| < r_2\}$$

onto D. We may assume $0 \leqq r_1 < r_2 < \infty$.

Just as in the simply connected case there is a bicontinuous isomorphism f_* from Aut (D) onto Aut (R) such that $f_*(\varphi) = f^{-1} \circ \varphi \circ f$ for each $\varphi \in \text{Aut}(D)$. Since Möb ($D$) is a closed and non-discontinuous subgroup of Aut (D), f_* maps Möb (D) onto a closed and non-discontinuous subgroup G of Aut (R).

The identity component $\text{Aut}_0(R)$ of Aut (R) is a subgroup of finite index in Aut (R) and consists of rotations $z \to xz$, where $|x| = 1$. Since G is not discontinuous, it follows that $G \cap \text{Aut}_0(R)$ is dense in $\text{Aut}_0(R)$. Hence $G \supset \text{Aut}_0(R)$, because G is closed.

Suppose $\varrho \in \text{Aut}_0(R)$; since $\text{Aut}_0(R) \subset G$, there exists $\varphi \in \text{Möb}(D)$ such that $\varrho = f^{-1} \circ \varphi \circ f$. Hence $f \circ \varrho = \varphi \circ f$, and taking Schwarzian derivatives yields

$$Sf(xz)x^2 = Sf(z) \quad (z \in R,\ |x| = 1). \tag{1}$$

For a fixed $z \in R$, the map $x \to Sf(xz)x^2$ is holomorphic in the domain $\{x \in C;\ xz \in R\}$ and assumes the constant value $Sf(z)$ on the unit cirle. Thus (1) holds by analytic continuation for all values of x such that $xz \in R$. As in the hyperbolic case of the previous section we conclude that f satisfies the differential equation

$$Sf(z)z^2 = c, \tag{2}$$

where c is a complex constant.

The solutions of (2) are again of the form $f = g \circ f_0$, where g is a Möbius transformation and either $f_0(z) = \log z$ or $f_0(z) = z^\varkappa$ for some $\varkappa \in C^*$. However, $f_0(z) = \log z$ does not yield an admissible solution because $\log z$ is not single-valued in R. Moreover, $f_0(z) = z^\varkappa$ is single-valued and one-to-one in R only if $\varkappa = \pm 1$. Hence $f = g \circ f_0$ is the restriction of a Möbius transformation, and the inverse of this Möbius transformation maps D onto a punctured disc or onto an annulus. The proof of Theorem 1 is now complete.

Remark. The group Möb(D) can be identified with a closed subgroup of the Lie group Aut ($C \cup \{\infty\}$) of all biholomorphic automorphisms of the Riemann sphere. Hence Möb (D) is a Lie group. Theorem 1 could have been proved also by using the classification of Lie subgroups of Aut ($C \cup \{\infty\}$) given in [3]. However, our method has the slight advantage that we may restrict ourselves to the study of closed subgroups of the real Möbius group Γ.

4. Möbius-homogeneous domains

We say that a domain D is *Möbius-homogeneous* if for each pair of points z and $w \in D$ there exists $\varphi \in \text{Möb}(D)$ such that $\varphi(z)=w$. In this case $\text{Möb}(D)$ is not discontinuous, because it acts transitively on D.

Möbius-homogeneous domains have been studied in arbitrary dimensions in [2] and [5]. We shall apply the results of Section 2 to the characterization of Möbius-homogeneous domains in the plane.

Theorem 3. *A domain D on the Riemann sphere is Möbius-homogeneous if and only if D is a disc or a domain having at most two boundary points.*

It is clear that the condition of Theorem 3 is sufficient, because every disc and every domain having at most two boundary points is Möbius-homogeneous. To prove the necessity, we need the following information about those subgroups of Γ which act transitively on H.

Theorem 4. *Let G be a closed subgroup of Γ acting transitively on H. Then either $G=\Gamma$ or $G=\Gamma_\zeta$ for some $\zeta \in \partial H$.*

Proof. Suppose first that G is elementary. Since G acts transitively on H, G is contained in Γ_ζ for some $\zeta \in \partial H$. By conjugation, we may assume $\zeta=\infty$.

The group Γ_∞ consists of affine transformations $z \to az+b$, where a and b are real and $a>0$. Given such numbers a, b, by homogeneity there exists $g \in G$ such that $g(i)=b+ai$. Since $g \in \Gamma_\infty$, it follows that $g(z)=az+b$ for each $z \in H$. Thus G contains all elements of Γ_ζ, so that $G=\Gamma_\zeta$.

In the remaining case G is non-elementary. By the result of Jørgensen mentioned in Section 2, G then contains elliptic elements of infinite order. As in the proof of Theorem 2 we conclude that G contains an elliptic continuum Γ_ζ, where $\zeta \in H$.

Elliptic continua do not act transitively on H. Hence Γ_ζ is a proper subgroup of G. On the other hand, Γ_ζ is a maximal subgroup of Γ [1, Lemma 3.3]. Therefore $G=\Gamma$, and the proof of Theorem 4 is complete.

We proceed with the proof of Theorem 3. Suppose that D is Möbius-homogeneous and has more than two boundary points. By Theorem 1, D is either simply or doubly connected. Moreover, if D were doubly connected, D could be mapped by means of a Möbius transformation onto a punctured disc or onto an annulus. However, these domains are not Möbius-homogeneous, a contradiction. Hence D is simply connected.

Let F be a conformal map from H onto D. As in Section 2 let G be the group of all automorphisms of H of the form $F^{-1} \circ \varphi \circ F$, where $\varphi \in \text{Möb}(D)$. Then G is a closed subgroup of Γ acting transitively on H.

By Theorem 4 there exists $\zeta \in \partial H$ such that $G \supset \Gamma_\zeta$. In particular, D contains all parabolic elements of Γ_ζ. We can now repeat the argument of the parabolic case of Section 2 to conclude that F is the restriction of a Möbius transformation. Hence $D=FH$ is a disc.

References

[1] ERKAMA, T.: Group actions and extension problems for maps of balls. - Ann. Acad. Sci. Fenn. Ser. A I Math. 556, 1973, 1—31.

[2] GEHRING, F. W., and B. P. PALKA: Quasiconformally homogeneous domains. - J. Analyse Math. 30, 1976, 172—199.

[3] GREENBERG, L.: Discrete subgroups of the Lorentz group. - Math. Scand. 10, 1962, 85—107.

[4] JØRGENSEN, T.: A note on subgroups of SL(2, C). - Quart. J. Math. Oxford (2) 28, 1977, 209—211.

[5] KIMEL'FEL'D, B. N.: Homogeneous regions on the conformal sphere. - Mat. Zametki 8, 1970, 321—328 (Russian).

[6] LEHNER, J.: Discontinuous groups and automorphic functions. - Mathematical Surveys, No. VIII. American Mathematical Society, Providence, 1964.

[7] MINDA, C. D.: Conformal automorphisms of circular regions. - Amer. Math. Monthly 86, 1979, 684—686.

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