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ON A CONJECTURE BY M. OZAWA CONCERNING FACTORIZATION OF ENTIRE FUNCTIONS

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0. In a series of four papers [2, 3] M. Ozawa considered entire functions $F(z)$ possessing, for infinitely many k, a factorization

(1) $$F(z) = P_k \circ g_k(z),$$

where P_k is a polynomial of degree k and g_k is an entire function. He proved

Theorem A. *If* (1) *holds for* $k=2^j$ $(j=1, 2, \ldots)$ *and for either* $k=3$ *or* $k=5$, *then either*

(2) $$F(z) = ae^{H(z)}+b \quad (a, b\in \mathbf{C}, H(z) \text{ entire})$$

or

(3) $$F(z) = a\cos\left((H(z))^{1/2}\right)+b.$$

Indeed for a function of the form (2)

$$F(z) = (au^n+b)\circ e^{H(z)/n}$$

shows that (1) holds for $k=1, 2, 3, \ldots$. And if the polynomial T_n is defined by

(4) $$T_n(\cos\theta) = \cos n\theta,$$

then a function of the form (3) satisfies

$$F(z) = aT_n\left(\cos\frac{H(z)^{1/2}}{n}\right)+b \quad (n = 1, 2, \ldots)$$

and again (1) is true for $k=1, 2, \ldots$.
Ozawa also proved

Theorem B. *If* (1) *holds for* $k=3^j$, $k=2$ *and* $k=4$, *then* $F(z)$ *must be either of the form* (2) *or of the form* (3).

These results led Ozawa to the

Conjecture. If (1) holds for $k=q\geqq 2$ and $k=n_j\geqq 2$, where n_j divides n_{j+1} and $(q, n_j)=1$ $(j=1, 2, \ldots)$, then the conclusion of Theorem A holds.

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We shall show that this conjecture is not generally correct by proving

Theorem 1. *There are entire functions* $F(z)$ *which are not of the form* (2) *or* (3) *and which satisfy* (1) *for* $k=n\geqq 2$, $k=q\geqq 2$, $(n, q)=1$, *and for* $k=(1+nq)^l$ $(l=1, 2, \ldots)$.

However, if the sequence n_j does not increase too rapidly, then the conjecture is correct.

We shall prove

Theorem 2. *If the entire function* $F(z)$ *satisfies* (1) *for* $k=n_1$, $k=q$ $(2\leqq n_1<q, (n_1, q)=1)$ *and for* $k=n_j$ $(j=2, 3, \ldots)$ *where* $n_{j+1}\leqq n_j q$, $(n_j, q)=1$, *then* $F(z)$ *is either of the form* (2) *or of the form* (3).

1. *Proof of Theorem* 1. Let

$$Q_k(z) = z+z^{nq+1}/c_k \quad (c_k > 0).$$

Let

$$h_{v,m}(z) = Q_{v+1}\circ\ldots\circ Q_m(z) \quad (m > v \geqq 0).$$

We prove first that the constants c_k can be chosen so that

$$h_v(z) = \lim_{m\to\infty} h_{vm}(z)$$

is an entire function for every $v\geqq 1$.

As the first step in the construction we choose

$$c_1 = 1, \; Q_1(z) = z+z^{nq+1}.$$

Suppose $Q_1\ldots Q_k$ have already been chosen. Put

$$A_k = \max_{0\leqq v\leqq k-1} \sup_{\substack{|z|\leqq k \\ |z'|\leqq k+1}} \left|\frac{h_{v,k}(z')-h_{v,k}(z)}{z'-z}\right|,$$

with the obvious definition of the right hand side for $z=z'$.

Let $c_{k+1}=[2^k k^{nq+1}\max(1, A_k)]$. Then, in $|z|\leqq k$,

$$|Q_{k+1}(z)| < |z|+\frac{k^{nq+1}}{2^k k^{nq+1}} < |z|+1.$$

For $v \leqq k-1$, $|z|\leqq k$, by the definition of A_k

$$|h_{v,k+1}(z)-h_{v,k}(z)| \leqq A_k|Q_{k+1}(z)-z| \leqq \frac{k^{nq+1}}{2^k k^{nq+1}} = 2^{-k}.$$

Therefore

$$\lim_{k\to\infty} h_{v,k}(z) = h_{v,v+1}(z)+\sum_{k=v+1}^{\infty}\{h_{v,k+1}(z)-h_{v,k}(z)\}$$

is uniformly convergent in $|z|<m$ for every $m>0$, i.e.

$$h_v(z) = \lim_{k\to\infty} h_{v,k}(z)$$

is an entire function. Obviously

$$h_0(z) = R_v \circ h_v(z) \quad (v \geqq 1)$$

where

$$R_v = Q_1 \circ Q_2 \ldots \circ Q_v$$

is a polynomial of degree $(1+nq)^v$.

The functions $h_v(z)$ are not constants, because it follows from the definition that

$$h_v(1) \geqq 1, \; h_v(0) = 0 \quad (v = 0, 1, 2\ldots).$$

By induction on v it is easily seen that

$$R_v(u) = uk_v(u^{nq}),$$

where k_v is a polynomial of degree

$$\deg k_v = \frac{1}{nq}((1+nq)^v - 1).$$

Choose

$$F(z) = (h_0(z))^{nq}, \; g_v(z) = (h_v(z))^{nq}.$$

Then

$$F(z) = u^n \circ (h_0(z))^q = u^q \circ (h_0(z))^n = (R_v \circ h_v(z))^{nq} = (h_v(z) k_v(h_v(z)^{nq}))^{nq}$$
$$= u(k_v(u))^{nq} \circ g_v(z) = P_v \circ g_v(z),$$

where P_v is a polynomial of degree

$$1 + nq \cdot \deg k_v = (1+nq)^v.$$

The function $F(z)$ has all the required factorizations.

We must still show that $F(z)$ is not of the form (2) or (3). The equation

$$q(u) = u(k_1(u))^{nq} = u(1+u)^{nq} = \alpha$$

has at least 2 distinct roots for every value of α, since $q'(u)$ is not a perfect power. The entire function $g_1(z)$ omits at most one value. Therefore $F(z) = u(k_1(u))^{nq} \circ g_1(z)$ assumes every value and so $F(z)$ is not of the form (2).

Suppose

(5) $$F(z) = (h_0(z))^{nq} = a \cos \sqrt{H(z)} + b.$$

Choose γ so that $a \cos \gamma + b = 0$. If z_1 is a root of multiplicity l of

(6) $$H(z) = (\gamma + 2k\pi)^2 \neq 0 \quad (k \in \boldsymbol{Z}),$$

then the power series of $\sqrt{H(z)}$ in powers of $z - z_1$ is

$$\sqrt{H(z)} = \pm(\gamma + 2k\pi) + \alpha(z-z_1)^l + \ldots \quad (\alpha \neq 0).$$

By Taylor's theorem

$$a \cos \sqrt{H(z)}+b = a \cos \gamma+b+\beta(z-z_1)^l+\ldots = \beta(z-z_1)^l+\ldots$$

where $\beta \neq 0$, if $\gamma \not\equiv 0 \pmod{\pi}$. If $\gamma \equiv 0 \pmod{\pi}$, then

$$a \cos \sqrt{H(z)}+b = \beta(z-z_1)^{2l}+\ldots \quad \beta \neq 0.$$

By (5) every zero of $a \cos \sqrt{H(z)}+b$ must have a multiplicity divisible by $nq \geqq 6$. Hence every root of (6) has multiplicity $l \geqq 3$, the value $(\gamma+2k\pi)^2$ of $H(z)$ is completely ramified. But a well known theorem of Nevanlinna theory [1, p. 44] asserts that an entire function has at most 2 completely ramified values. This contradicts (5) and the proof of Theorem 1 is completed.

2. *Some preliminary results.* Our proof, like Ozawa's work, is based on

Picard's theorem. (cf. [4].) *Let* $R(u, v)$ *be an irreducible polynomial in* $\boldsymbol{C}[u, v]$. *If there are non-constant entire functions* $f(z)$, $g(z)$ *such that*

$$R(f(z), g(z)) = 0 \quad (\forall z \in \boldsymbol{C}), \tag{7}$$

then the Riemann surface defined by

$$R(u, v) = 0 \tag{8}$$

is of genus zero.

(The Theorem is usually stated in the form: If (7) holds for meromorphic functions f, g, then $R(u, v)=0$ defines a Riemann surface of genus $\leqq 1$. But Riemann surfaces of genus one can only be uniformized by elliptic functions, not by entire functions.)

A Riemann surface X of genus 0 is conformally equivalent to the Riemann sphere, i.e., its points can be put into 1—1 correspondence with a parameter s ranging over the Riemann sphere so that any holomorphic function on X can be written as a holomorphic function of s defined on the Riemann sphere, that is to say a rational function of s. In particular the points of the surface (8) are in 1—1 correspondence with the points s of the Riemann sphere by

$$u = U(s), \quad v = V(s) \quad (U, V \text{ rational}). \tag{9}$$

Note that the parameter s can be replaced by any fractional linear transform T of s, if U and V are changed into $U \circ T^{-1}$, $V \circ T^{-1}$.

Lemma 1. *If* $f(z)$ *and* $g(z)$ *are non-constant entire functions and if* P_m *and* P_n *are polynomials of relatively prime degrees* m, n *respectively, then the identity*

$$P_m \circ f(z) = P_n \circ g(z) \quad (\forall z \in \boldsymbol{C}) \tag{10}$$

implies the existence of an entire function $s(z)$ *and of polynomials* U *(of degree* n*) and* V *(of degree* m*) such that*

$$f(z) = U(s(z)), \quad g(z) = V(s(z)).$$

Proof of Lemma 1. Factorize

$$P_m(u)-P_n(v)$$

into irreducible factors in $C[u, v]$. If (10) holds, then one of these irreducible factors, $R(u, v)$, say, will satisfy

$$R(f(z), g(z)) = 0.$$

By Picard's theorem this means that there is a conformal map $s=\psi(p)$ of the points p of the Riemann surface X of

$$R(u, v) = 0$$

onto the points s of the Riemann sphere. Without loss of generality we may assume that $s=\infty$ corresponds to a point with $u=\infty$.

Except at a finite number of branch points of R we may use u as local uniformizing parameter, so that s is a holomorphic function $\sigma(u)$ of u near all points of R except the branch points. Therefore the map

$$z \mapsto (f(z), g(z)) = p \mapsto s = \psi(p) = \sigma \circ f(z) = s(z)$$

is holomorphic near all z except perhaps those for which $(f(z), g(z))=(u_1, v_1)=p_1$ is a branch point of X. These values of z form a discrete set E. If $z\to z_1\in E$, $s(z)\to s(p_1)$. Therefore z_1 is a removable singularity of $s(z)$, $s(z)$ is entire.

By (9), on X

$$u = U(s), \quad v = V(s)$$

and so

$$f(z) = U(s(z)), \quad g(z) = V(s(z)),$$

U, V rational functions.

Suppose $R(u, v)$ is of degree m_1 in u, n_1 in v. Then for a given value of v, u has in general m_1 possible values, i.e., there are m_1 values of s for given v, i.e., V is of degree m_1. Similarly U is of degree n_1.

Since $f(z)$ and $g(z)$ are entire, U and V cannot have poles at any value taken on by $s(z)$. Since $s(z)$ omits at most one finite value, U and V can have poles at one finite value s_0 at most and then $s(z)\neq s_0$ $(z\in C)$. Without loss of generality we may suppose $s_0=0$ (otherwise replace s by $s-s_0$). Combining all the information on U and V we find that we must have

(11) $$U(s) = \sum_{-\nu}^{n_1-\nu} a_k s^k, \quad V(s) = \sum_{-\mu}^{m_1-\mu} b_k s^k,$$

$$0 \leqq \nu \leqq n_1 \leqq n, \qquad 0 \leqq \mu \leqq m_1 \leqq m.$$

By (10),

(12) $$P_m \circ U \circ s(z) = P_n \circ V \circ s(z) \quad (z\in C).$$

Since $s(z)$ takes on infinitely many distinct values, this implies

(12′) $$P_m \circ U(s) = P_n \circ V(s) \quad (s\in C).$$

For large values of s, by (11) and (12′)

$$P_m \circ U(s) \sim \text{const. } s^{(n_1-\nu)m}, \quad P_n \circ V(s) \sim \text{const. } s^{(m_1-\mu)n}.$$

Therefore

$$(n_1-\nu)m = (m_1-\mu)n.$$

Since $0 \leqq n_1-\nu \leqq n$, $0 \leqq m_1-\mu \leqq m$, $(m, n)=1$, we must have

either $n_1-\nu=n$, $m_1-\mu=m$; $n_1=n$, $\nu=0$, $m_1=m$, $\mu=0$

or $n_1=\nu$, $m_1=\mu$.

That is to say that either U and V are polynomials of degree n and m respectively or they are polynomials in $1/s$. In this case $s(z)\neq 0$ $(z\in C)$. Put $1/s(z)=t(z)$. Arguing with the polynomials $U_1(t)=U(s)$, $V_1(t)=V(s)$ we obtain again that (12′) implies $n_1=n$, $m_1=m$ and the proof of the Lemma is completed.

Lemma 1 reduces the study of the identity (10) to the investigation of polynomials P_m, P_n, U_n, V_m satisfying

$$P_m \circ U_n = P_n \circ V_m \quad (n, m) = 1. \tag{13}$$

This relation was the subject of a beautiful and deep investigation by J. F. Ritt in his paper [5]. The results of Ritt are summarized by

Lemma 2. *The equation* (13) *can only hold under the following circumstances*:
(A) *There are first degree polynomials* $\lambda, \varkappa, \nu, \mu$ *such that*

$$\lambda \circ P_m \circ \varkappa = T_m, \quad \lambda \circ P_n \circ \nu = T_n,$$
$$\varkappa^{-1} \circ U_n \circ \mu = T_n, \quad \nu^{-1} \circ V_m \circ \mu = T_m,$$
$$\lambda \circ P_m \circ U_n \circ \mu = \lambda \circ P_n \circ V_m \circ \mu = T_{nm},$$

where the polynomials T *are defined by* (4).

(B) *Suppose* $m>n$. *There are first degree polynomials* $\lambda, \varkappa, \nu, \mu$ *and a polynomial* $h(u)$ *of degree* $<m/n$ *such that*

$$\lambda \circ P_m \circ \varkappa(u) = u^r h^n(u) \quad (r+n \deg h = m),$$
$$\varkappa^{-1} \circ U_n \circ \mu(s) = s^n,$$
$$\lambda \circ P_n \circ \nu(u) = u^n,$$
$$\nu^{-1} \circ V_m \circ \mu(s) = s^r h(s)^n,$$
$$\lambda \circ P_m \circ U_n \circ \mu(s) = \lambda \circ P_n \circ V_m \circ \mu(s) = \big(s^r h(s^n)\big)^n. \tag{14}$$

Lemma 3. *Unless the polynomial* Q *of degree* p *is of the form*

$$Q(u) = A(u-\alpha)^p + B,$$

there are only a finite number of pairs of first degree polynomials ν, μ *such that*

$$\nu \circ Q = Q \circ \mu.$$

Proof. Let $\nu(t)=at+b$, $\mu(t)=ct+d$. If

$$\nu\circ Q = aQ(u)+b = Q\circ\mu = Q(cu+d),$$

then

$$aQ'(u) = cQ'(cu+d).$$

Therefore the set S of zeros of Q' is invariant under the map $u\mapsto cu+d$. Since no translation leaves a finite set invariant, $c\neq 1$, unless $\nu(u)=\mu(u)=u$. If $c\neq 1$ we can write

$$\mu(t) = c(t-\alpha)+\alpha, \quad \alpha = d/(1-c).$$

The invariance of S now requires $|c|=1$, unless $S=\{\alpha\}$. (Consider a value of $t\in S$ for which $|t-\alpha|$ is maximal.)

If $S\neq\{\alpha\}$, then c must be a root of unity, $c^N=1$, $c^k\neq 1$ $(0<k<N)$. S consists of corners of some regular N-gons with center α and possibly also $u=\alpha$. Hence

$$Q'(u) = C(u-\alpha)^s \prod_{j=1}^{M}\{(u-\alpha)^N-b_j\}$$

and, by integration,

$$Q(u) = (u-\alpha)^{s+1}h[(u-\alpha)^N]+B$$

where h is a polynomial.

$$Q\circ\mu=c^{s+1}Q(u)+B(1-c^{s+1}) = \nu\circ Q,$$

$$\nu(t) = c^{s+1}t+B(1-c^{s+1}), \quad (c^N=1).$$

There are only a finite number of possibilities for μ and ν in this case.

Finally, if $S=\{\alpha\}$, then

$$Q'(u) = C(u-\alpha)^{p-1}, \quad Q(u) = A(u-\alpha)^p+B.$$

In this case any pair μ, ν with

$$\mu(t) = c(t-\alpha)+\alpha, \quad \nu(t) = c^p t+B(1-c^p) \quad (c\neq 1)$$

is possible.

3. *Proof of Theorem* 2. Without loss of generality we may suppose that (1) holds for $k=n_1, q, n_2, n_3, \ldots$ where

(15) $$(n_j, q) = 1;\ 1<n_1<q<n_2<n_3\ldots;\ n_{j+1}\leqq n_j q.$$

Using Lemma 1 with $m=n_j$ $(j\geqq 2)$, $n=q$ we see that there are polynomials U (of degree q), V (of degree m) and an entire function $s_m(z)$ such that

(16) $$P_m\circ U\circ s_m(z) = P_q\circ V\circ s_m(z) = F(z).$$

Now Lemma 2 shows that P_m, U, P_q, V must be given either by the formulae (A) of Lemma 2 or by the formulae (B).

We show next that if (B) holds for a pair $m=n_j$, $n=q$, then (B) holds also for any other pair (n_k, q) $(k\geqq 1)$.

Suppose we were in case (A) for (n_k, q). Then we can find first degree polynomials ϱ, σ such that

$$\varrho \circ P_q \circ \sigma(u) = T_q(u).$$

On the other hand (B) for m and $n=q<m$ shows that there are first degree polynomials $\varkappa$, λ such that

$$\lambda \circ P_q \circ \varkappa(u) = v^q \circ u.$$

Hence

$$\varrho \circ \lambda^{-1} \circ v^q \circ \varkappa^{-1} \circ \sigma(u) = T_q(u),$$

or, writing out the first degree polynomials

$$A(Bu+C)^q+D = T_q(u).$$

But T_q does not have any values of multiplicity >2. This leads to a contradiction, since $q \geqq 3$.

Theorem 2 will therefore be a consequence of the two statements:

(C) If there is an infinite sequence $M=\{m_k\}_{k=1}^{\infty}$ and a $q \geqq 3$ prime to all m_k such that (16) and (A) of Lemma 2 hold, then $F(z)$ is of the form (3).

(D) If, for a sequence $m=n_j$, where the n_j satisfy (15), (16) and (B) of Lemma 2 hold, then $F(z)$ is of the form (2).

Proof of (C). By (A) there are first degree polynomials $\lambda=\lambda_m$, $\nu=\nu_m$ such that

$$\lambda \circ P_q \circ \nu = T_q.$$

If $\tilde{\lambda}$ and $\tilde{\nu}$ are the first degree polynomials corresponding to another value $\tilde{m} \in M$, then

$$\lambda \circ \tilde{\lambda}^{-1} \circ T_q \circ \tilde{\nu}^{-1} \circ \nu = T_q.$$

For $q \geqq 3$ $T_q(u)$ is not of the form $A(u-\alpha)^p+B$. By Lemma 3 there are only a finite number of possible values of the pair $(\lambda \circ \tilde{\lambda}^{-1},\ \tilde{\nu}^{-1} \circ \nu)$. Keeping m fixed and replacing M by a subsequence, if necessary, we may assume that λ does not depend on the choice of m. Formulae (16) and (A) now show that there is a first degree polynomial λ such that

$$\lambda \circ F(z) = T_{qm}(S_m(z)) \quad (m \in M),$$

where $S_m(z)$ is an entire function (S_m is the composition of $s_m(z)$ in (16) with a first degree polynomial). Put

$$S_m(z) = \cos \varphi(z), \tag{17}$$

so that

$$\lambda \circ F(z) = \cos qm\, \varphi(z). \tag{18}$$

The expression $\varphi(z)$ is not uniquely determined, but in a disk U of the z-plane which contains no roots of $S_m(z)=\pm 1$ we can define $\varphi(z)$ as a holomorphic function equal to a branch of arc cos $S_m(z)$. All possible values of $\varphi(z)$ in U are obtained from one, $\varphi_0(z)$, say, by the formula

$$\varphi(z) = \pm\varphi_0(z) \pmod{2\pi} \quad (z \in U).$$

Replacing m by another member $\tilde{m}$ of the sequence M we can similarly define $\psi(z)$ by

(19) $$S_{\tilde{m}}(z) = \cos \psi(z),$$

(20) $$\lambda \circ F(z) = \cos q\tilde{m}\, \psi(z).$$

Again ψ is not uniquely defined, but in a disk in which $S_{\tilde{m}}(z) \neq \pm 1$ ψ can be chosen as a holomorphic function. We may assume without loss of generality that this disk is identical with U.

Again all possible determination of ψ can be derived from one of them, ψ_0, by the formula

(21) $$\psi(z) = \pm\psi_0(z) \pmod{2\pi} \quad (z \in U).$$

By (18) and (20),

$$q\tilde{m}\, \psi_0(z) \equiv \pm q m\, \varphi_0(z) \pmod{2\pi}.$$

Changing ψ_0 into $-\psi_0$, if necessary, we may suppose

$$q\tilde{m}\, \psi_0(z) = qm\, \varphi_0 + 2h\pi,$$

$$\psi_0(z) = (m/\tilde{m})\varphi_0(z) + 2h\pi/q\tilde{m}.$$

Changing ψ_0 by adding a suitable multiple of 2π we have

(22) $$\psi_0(z) = (m/\tilde{m})\varphi_0(z) + c,$$

where c is a real number satisfying

(23) $$-\pi < c \leqq \pi.$$

Next we observe that the functions φ_0 and ψ_0 can be analytically continued along any path C which avoids the roots of $S_m(z) = \pm 1$, $S_{\tilde{m}}(z) = \pm 1$. If C is chosen as a closed path from a point in U to U, then the results of the continuation φ_C, ψ_C still satisfy (17), (19) and (22) with φ_0 and ψ_0 replaced by φ_C and ψ_C. Let

$$\varphi_C(z) = \varepsilon\varphi_0(z) + 2l\pi \quad (\varepsilon = \pm 1).$$

Then

(24) $$\psi_C(z) = (m/\tilde{m})\varphi_C(z) + c = \varepsilon(m/\tilde{m})\varphi_0(z) + 2ml\pi/\tilde{m} + c.$$

Also

(25) $$\psi_C(z) = \eta\psi_0(z) + 2k\pi = \eta(m/\tilde{m})\varphi_0(z) + \eta c + 2k\pi \quad (\eta = \pm 1).$$

Note that k depends on $\tilde{m}$ and C, l on m and C, and c on m and $\tilde{m}$ only.

Comparing (24) and (25) we see that $\varepsilon=\eta$, since $\varphi_0(z)$ is not constant. Therefore

$$c+2lm\pi/\tilde{m} = \eta c+2k\pi.$$

If $\eta=1$ this reduces to

$$lm/\tilde{m} = k$$

and for sufficiently large $\tilde{m}$ we must have $k=0$ and therefore also $l=0$. If $\eta=-1$, then

$$2k\pi = 2c+2lm\pi/\tilde{m}. \tag{26}$$

For large $\tilde{m}$ (26) and (23) imply

$$k=-1 \quad \text{or} \quad 0 \quad \text{or} \quad 1.$$

And since c is independent of the path C, so k must be independent of C, provided $\eta=-1$. We see that ψ_C is only capable of assuming four values:

(i) If $k=0$, $\psi_C=\psi_0$ $(\eta=1)$ or $\psi_C=-\psi_0$ $(\eta=-1)$.

(ii) If $k=1$, $\psi_C=2\pi-\psi_0$.

(iii) If $k=-1$, $\psi_C=-2\pi-\psi_0$.

In the three cases respectively the functions

$$\psi^2, \ (\psi-\pi)^2, \ (\psi+\pi)^2 \tag{27}$$

are single-valued functions of z, holomorphic at all points where $S_{\tilde{m}}(z)\neq\pm 1$. As z approaches a root of this equation ψ, which is locally defined as a branch of $\arccos S_{\tilde{m}}(z)$, approaches a finite limit. The roots of $S_{\tilde{m}}(z)=\pm 1$ are therefore removable singularities of one of the functions (27). In case (i) we have

$$\psi^2(z) = H(z)/(q\tilde{m})^2 = \text{entire function}$$

and $\lambda\circ F(z)=\cos\sqrt{H(z)}$ is of the form (3). In the other cases

$$\psi\pm\pi = \sqrt{H(z)}/q\tilde{m},$$

$$\lambda\circ F(z) = \cos\left(\mp q\tilde{m}\pi+\sqrt{H(z)}\right) = \pm\cos\sqrt{H(z)}$$

and again $F(z)$ is of the form (3).

Proof of (D). By (14) and (16) with $m=n_k$, $n=q$, there is a first degree polynomial λ_n and an entire function $S_k(z)=A_k s_{n_k}(z)+B_k$ $(A, B\in C)$ such that

$$\lambda_k F(z) = [S_k^{r_k} h_k(S_k^q)]^q \quad (k=2, 3, \ldots), \tag{28}$$

where h_k is a polynomial, $h_k(0)\neq 0$, and

$$r_k + q \deg h_k = n_k.$$

By replacing S_k by cS_k, if necessary, we may assume

$$h_k(0) = 1.$$

Also, using (14) for the indices n_1 and q,

$$\lambda_0 \circ F(z) = [S^r h(S^{n_1})]^{n_1}, \tag{29}$$

$$r + n_1 \deg h = q.$$

If α is the root of $\lambda_0(t)=0$, β the root of $\lambda_k(t)=0$; then every root of $F(z)=\alpha$, has multiplicity $\geqq n_1$, every root of $F(z)=\beta$ has multiplicity $\geqq q$, by (28) and (29). By Nevanlinna's 2nd fundamental theorem [1, p. 43], if $\alpha\neq\beta$

$$T(r, F) \leqq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F-\alpha}\right) + \bar{N}\left(r, \frac{1}{F-\beta}\right) + o(T(r, F)), \tag{30}$$

as $r\to\infty$ through a suitable sequence of values. But

$$\bar{N}(r, F) = 0, \quad \bar{N}\left(r, \frac{1}{F-\alpha}\right) \leqq \frac{1}{n_1} N\left(r, \frac{1}{F-\alpha}\right) \leqq \frac{1}{n_1} T(r, F)(1+o(1)),$$

$$\bar{N}\left(r, \frac{1}{F-\beta}\right) \leqq \frac{1}{q} N\left(r, \frac{1}{F-\beta}\right) \leqq \frac{1}{q} T(r, F)(1+o(1)).$$

Therefore (30) leads to the contradiction

$$T(r, F) \leqq \left(\frac{1}{q} + \frac{1}{n_1} + o(1)\right) T(r, F).$$

This contradiction can only be avoided, if $\alpha=\beta$, i.e., if $\lambda_k(t)=c\lambda_0(t)$. Replacing $S_k(z)$ by $bS_k(z)$ with a suitable b, we may suppose that

$$\lambda_k = \lambda_0 \quad (k = 2, 3, \ldots,),$$

$$F_0(z) = \lambda_0 \circ F(z) = (S_k^{r_k} h_k(S_k^q))^q = (S^r h(S^{n_1}))^{n_1}. \tag{31}$$

Since $(n_1, q)=1$, (31) implies that $F_0(z)=0$ has only roots whose multiplicity is divisible by $n_1 q$;

$$F_0(z) = (G(z))^{n_1 q},$$

$G(z)$ entire. By (29) we can choose G so that

$$G^q = S^r h(S^{n_1}).$$

Suppose

$$h(t) = \prod_\gamma (t-\gamma^{n_1})^{\mu(\gamma)}.$$

Then $h(t^{n_1})=\prod_\gamma \prod_{j=1}^{n_1}(t-\varrho^j\gamma)^{\mu(\gamma)}$, where ϱ is a primitive n_1-th root of unity. If z_1 is a root of $S(z)=\varrho^j\gamma$ of multiplicity v, then $q|v\cdot\mu(\gamma)$. If $q\nmid\mu(\gamma)$, then $v\geqq 2$, i.e., the value $\varrho^j\gamma$ of S is completely ramified. Also $r+n_1 \deg h=q$. Therefore $0<r<q$ and each root of $S=0$ has multiplicity $\geqq 2$, by the preceding argument.

We have at least 3 ramified values of S: $0, \gamma, \varrho\gamma$, if $\deg h>0$ and not all roots of h have multiplicity divisible by q. Since an entire function has at most 2 ramified values we have either

(32) $$S^r h(S^{n_1}) = S^q$$

or $q|\mu(\gamma)$ for all γ,

$$S^r h(S^{n_1}) = S^{r_1}(k(s^n))^q.$$

But

$$q = r+n_1 \deg h_1 = r+n_1 q \deg k;$$

this is impossible, if $\deg k>0$, because $n_1>1$. Hence (32) holds and, by (29),

$$F_0(z) = S^{n_1 q}.$$

By (31) with k=2

$$(S_2^{r_2} h_2(S_2^q))^q = S^{n_1 q}$$

and we may suppose

$$S_2^{r_2} h_2(S_2^q) = S_1^{n_1}, \quad r_2+q \deg h_2 = n_2.$$

By repeating the reasoning above with S_2 in place of S_1, S_1 in place of G and n_1 and q interchanged we find

$$h_2(S_2^q) = (k_2(S_2^q))^{n_1},$$

$$r_2+n_1 q \deg k_2 = n_2.$$

Since $(n_2, q)=1$, $r_2>0$, and since $n_2\leqq n_1 q$, we must have

$$S_2^{r_2} h_2(S_2^q) = S_2^{n_2}.$$

Therefore

$$F_0(z) = (S_3^{r_3} h_3(S_3^q))^q = S_2^{n_2 q}$$

which leads by the same reasoning to

$$F_0(z) = S_3^{n_3 q} = S_4^{n_4 q}\dots.$$

If F_0 has a root we arrive at a contradiction as soon as $n_k q$ is greater than the multiplicity of the root. Therefore

$$F_0(z) = \lambda_1 \circ F(z) = e^{H(z)},$$

H entire. $F(z)$ is of the form (2).

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