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LOCAL VALUE DISTRIBUTION OF FUNCTIONS BOUNDED IN A HALF-PLANE

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1. Introduction

Suppose that $f(z)$ is meromorphic in an angle, which we may for definiteness take to be the right half-plane

$$P:\ -\frac{\pi}{2} < \arg z < \frac{\pi}{2},$$

and smooth at the origin, so that roots of $f=a$ do not accummulate there. In a recent paper [2] the notion of the inner order $k_i(P,f)$ of $f(z)$ in P was introduced and so was the inner order $k_i(a, P, f)$ of the roots of the equation $f(z)=a$. One then obtains the following result, using ideas of Valiron [7].

Theorem A. *We have* $0 \leqq k_i(P) \leqq \infty$.

(i) *If* $1 \leqq k_i(P) \leqq \infty$,

then $k_i(a, P)=k_i(P)$ *except for at most two values* a *for which* $k_i(a, P)<k_i(P)$.

(ii) *If* $0 \leqq k_i(P) < 1$,

then we have at most two values a *for which* $k_i(a, P)<k_i(P)$ *and a certain small exceptional set* V *of values* a *for which*

$$k_i(P) < k_i(a, P) \leqq 1.$$

It turns out that the nature of the set V can be precisely described in terms of a set function due to Hyllengren [3, 4]. The positive theorems were obtained in [2] and examples are given in [1] for any value of the order $\varrho=k_i(P)$ satisfying $0\leqq\varrho\leqq 1$.

If $f(z)$ is regular and bounded in P then $f(z)$ does not assume large values a and so $k_i(a, P)=0$ for such a. The small set V in (ii) can certainly not contain all values in a disk and so we deduce that $k_i(P)=0$ for bounded functions. Thus we obtain results concerning the nature of V for bounded functions by applying Theorem A with $k_i(P)=0$. In this paper we show, using a result on interpolation due to Katsnel'son [5] and Carleson (see appendix) that the results obtained in this way are sharp. The technique used by Drasin and the author [1] to obtain functions of order zero does not appear able to yield bounded functions although it does yield func-

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tions which are regular in P and grow at most like $|z|$ as $z \to \infty$ in P. In order to obtain the result of Theorem 1 I needed an interpolation theorem such as Lemma 1. I wrote to Carleson about this and the theorem proved by him in the appendix was the result.

2. Statement of results

Let V be a plane set and write

$$e(x) = \exp\{-\exp\exp x\}. \tag{2.1}$$

Suppose that for some sequence a_n of complex numbers and some positive c, every point of V lies in infinitely many of the disks

$$|a - a_n| < e(cn). \tag{2.2}$$

Then we say following Hyllengren [3, 4] that the sequence $e(cn)$ majorises V. The span $s(V)$ of V is defined to be the greatest lower bound of all numbers c^{-1} for which $e(cn)$ majorises V. If $e(cn)$ does not majorise V for any c, we say that $s(V) = \infty$. If $V = \cup V_n$, where $s(V_n) < \infty$ for each n, we say that V has at most countably infinite span.

With the above definition the results on V in [2] can be stated as follows.

Theorem B. *If $\varrho = k_i(P) = 0$ and in particular if $f(z)$ is regular and bounded in P, suppose that $0 < \varrho' < 1$. Then if $V(\varrho')$ is the set of all a for which $k_i(a, P) \geqq \varrho'$, we have*

$$s\{V(\varrho')\} \leqq \left\{\log \frac{1+\varrho'}{1-\varrho'}\right\}^{-1}. \tag{2.3}$$

Corollary. *If $V(0)$ is the set of a for which $k_i(a, P) > 0$ then $V(0)$ has at most countably infinite span. If $V(1)$ is the set for which $k_i(a, P) = 1$, then*

$$s\{V(1)\} = 0.$$

In this paper we obtain a converse result, at least if strict inequality holds in (2.3).

Theorem 1. *Suppose that $0 < \varrho' < 1$ and that $V' = V(\varrho')$ is any bounded plane set such that*

$$s(V') < \left\{\log \frac{1+\varrho'}{1-\varrho'}\right\}^{-1}.$$

Then there exists $f(z)$ regular and bounded in P and such that

$$k_i(a, P) \geqq \varrho'$$

for every $a \in V'$.

To define $k_i(a, P)$ we let $n_\varepsilon(r, a)$ be the number of roots of $f(z)=a$ in

$$|z| < r, \ |\arg z| < \frac{\pi}{2} - \varepsilon,$$

and write

(2.4) $$k(a, \varepsilon) = \overline{\lim_{r\to\infty}} \frac{\log n_\varepsilon(r, a)}{\log r}.$$

Then

(2.5) $$k_i(a, P) = \lim_{\varepsilon\to 0} k(a, \varepsilon).$$

3. The fundamental interpolation lemma

We shall need to use the following result which is a very special case of Katsnel'son's sufficient condition. A more complete result is proved in the appendix.

Lemma 1. *Suppose that* p_n *is a sequence of positive integers,* a_n *is a sequence of complex numbers, such that* $|a_n| \leqq 1$, *and* r_n *is a sequence of positive numbers such that, for some positive constant* K,

(3.1) $$\frac{r_{n+1}}{r_n} \geqq 1 + Kp_n p_{n+1}, \quad n = 1, 2, \ldots.$$

Then there exists a function $f(z)$ *regular in* P *and bounded there by a constant* K_1 *depending only on* K, *such that*

(3.2) $$f(r_n) = a_n, \ f^{(p)}(r_n) = 0, \quad 0 < p \leqq p_n - 1.$$

We write

(3.3) $$B_n(z) = \Pi_{m\neq n}\left(\frac{r_m - z}{r_m + z}\right)^{p_m}.$$

Katsnel'son [5] shows that, if

(3.4) $$|B_n(r_n)| \geqq \delta > 0, \quad n = 1, 2, \ldots,$$

then the interpolation problem (3.2) can be solved by a function $f(z)$ regular in P and bounded by a constant depending on δ only. Thus to prove Lemma 1 we need only show that (3.1) implies (3.4). We proceed to prove this result. We denote by $K_2, K_3, \ldots$ positive constants depending on K_1 only.

We note first that (3.1) implies

(3.5) $$\frac{r_n}{r_m} > K_2(1+K)^{n-m} p_m p_n, \quad m < n.$$

If $n = m+1$ this follows from (3.1) with $K_2 = K/(1+K)$. Next, if $n - m \geqq 2$, (3.1) shows that

$$\frac{r_{\nu+1}}{r_\nu} \geqq 1+K, \quad n+1 \leqq \nu \leqq m-1.$$

Thus

$$\frac{r_{m+1}}{r_m} \geqq Kp_m, \quad \frac{r_n}{r_{n-1}} \geqq Kp_n$$

and

$$\frac{r_{n-1}}{r_{m+1}} \geqq (1+K)^{n-m-2}.$$

On multiplying these inequalities we obtain (3.5) with

$$K_2 = \left(\frac{K}{1+K}\right)^2.$$

We now form the Blaschke products (3.3). We deduce from (3.5) that these products converge and in fact (3.5) yields

$$-\log B_n(r_n) < K_3\left\{\sum_{m<n} p_m \frac{r_m}{r_n} + \sum_{m>n} p_m \frac{r_n}{r_m}\right\}$$

$$\leqq \sum_{m<n} \frac{K_3}{K_2}(1+K)^{m-n} + \sum_{m>n} \frac{K_3}{K_2}(1+K)^{n-m} = K_4.$$

This proves (3.4) and thus Lemma 1 is proved.

We also need a form of the Milloux-Schmidt inequality.

Lemma 2. *Suppose that* ε, η, r *lie between* 0 *and* 1, *that* $F(z)$ *is regular in* $|z|<1$ *and satisfies* $|F(z)|<1$ *there and further that*

(3.6) $$\inf_{|z|=\varrho} |F(z)| \leqq \eta, \quad 0 < \varrho < \varepsilon.$$

Then for $\varepsilon/2<r<1$, *we have*

(3.7) $$\log|F(z)| < \frac{\log\eta \log r}{6\log(\varepsilon/2)}, \quad |z| = r.$$

We consider first the case $r=\varepsilon/2$. Then since $|f(z)|<1$ in $|z|<\varepsilon$ and (3.6) holds, the classical Milloux-Schmidt inequality [6, Theorem 1, p. 107] yields for $|z|=r$

$$\log|F(z)| \leqq (\log\eta)\left(1-\frac{4}{\pi}\tan^{-1}\sqrt{1/2}\right) < \frac{1}{6}\log\eta.$$

Now Hadamard's convexity theorem shows that for $\varepsilon/2\leqq r<1$, we have

$$\log|F(z)| \leqq \frac{\log(1/r)}{\log(2/\varepsilon)}\frac{1}{6}\log\eta,$$

which is (3.7).

It is convenient to express this result a little differently.

Lemma 3. *Suppose that* $F(z)$ *is regular in* $|z|<1$ *and satisfies* $|F(z)|<M$ *there and further* $|F(0)|\leqq 1$ *and* $|F(r)|=2$ *where* $1/2<r<1$. *Then if* $0<\varepsilon<1/2$, *there exists* ϱ, *such that* $0<\varrho<\varepsilon$ *and*

$$|F(z)-F(0)| \geqq d, \ |z| = \varrho \tag{3.8}$$

where

$$\log \frac{1}{d} = \frac{6 \log (2/\varepsilon) \log (M+1)}{\log (1/r)}. \tag{3.9}$$

In particular if $F(z)-F(0)$ *has a zero of order* p *at the origin then the equation* $F(z)=a$ *has at least* p *roots in* $|z|<\varepsilon$, *if* $|a-F(0)|<d$.

We consider

$$G(z) = \frac{F(z)-F(0)}{M+1},$$

so that $|G(z)|<1$ for $|z|<1$. Suppose that (3.8) is false for $0<\varrho<\varepsilon$. Then we can apply Lemma 2 to $G(z)$ with

$$\eta = \frac{d}{M+1}.$$

We obtain

$$\log \left(\frac{1}{M+1}\right) \leqq \log |G(r)| \leqq \frac{\log (d/(M+1)) \log r}{6 (\log (\varepsilon/2))},$$

i.e.

$$\log \left(\frac{M+1}{d}\right) \leqq \frac{6 \log (2/\varepsilon)}{\log (1/r)} \log (M+1),$$

so that

$$\log \frac{1}{d} < \frac{6 (\log (2/\varepsilon)) \log (M+1)}{\log (1/r)}.$$

Thus when (3.9) holds, (3.8) must be true for some ϱ. The last part of Lemma 3 follows at once from Rouché's theorem.

4. A general example

We can prove

Theorem 2. *Suppose that* p_n, r_n *satisfy the hypotheses of Lemma* 1 *and that* a_n *is an arbitrary sequence of complex numbers satisfying* $|a_n|\leqq 1$. *Suppose further that* $0<\varepsilon<1/2$. *Then there exists* $f(z)$ *bounded and regular in* P *and such that for*

$|a-a_n|<d_n$, the equation $f(z)=a$ has at least p_n roots in $|z-r_n|<\varepsilon r_n$ where

$$d_n = \exp(-K_5 p_n)$$

and K_5 is a constant depending on ε and K only.

We write $C=(1+K)^{1/2}-1$ and define r_n' by

$$\frac{r_n'}{r_n} = 1+Cp_n.$$

We note that

$$(1+Cp_n)(1+Cp_{n+1}) = 1+C(p_n+p_{n+1})+C^2 p_n p_{n+1}$$

$$\leqq 1+p_n p_{n+1}(2C+C^2) = 1+Kp_n p_{n+1}.$$

Thus

$$r_{n+1}/r_n' \geqq 1+Cp_{n+1}.$$

Hence the sequence $r_1, r_1', r_2, r_2', \ldots$ and the associated sequence $p_1, 1, p_2, 1, \ldots$ satisfies (3.1) with C instead of K, and so we can find $f(z)$ satisfying the conditions of Lemma 1, and in addition

$$f(r_n') = 2, \quad 1 \leqq n < +\infty.$$

We now assume that $a_n=f(r_n)$ is a preassigned sequence such that $|a_n|\leqq 1$, and that

$$|f(z)| < M.$$

Consider

$$F(\zeta) = f\left(r_n \frac{1+\zeta}{1-\zeta}\right).$$

Then $F(\zeta)$ is regular in $|\zeta|<1$, $|F(\zeta)|<M$ there and $F(\zeta)-a_n$ has a zero order at least p_n at the origin. Also

$$F(r) = 2$$

where r is given by

$$r_n \frac{1+r}{1-r} = r_n', \quad \text{i.e.} \quad r = \frac{r_n'-r_n}{r_n'+r_n} = \frac{Cp_n}{2+Cp_n}.$$

It now follows from Lemma 3 that $F(\zeta)$ assumes at least p_n times in $|\zeta|<\varepsilon$ every value a, such that $|a-a_n|<d_n$ where

$$\log\frac{1}{d_n} = \frac{6\log(2/\varepsilon)\log(M+1)}{\log((2+Cp_n)/(Cp_n))}.$$

Thus

$$\log\frac{1}{d_n} \leqq K_5 p_n, \quad d_n \geqq \exp(-K_5 p_n),$$

where K_5 depends only on K_1, M and ε and so on K_1 and ε. Also if

$$z = r_n \frac{1+\zeta}{1-\zeta},$$

we have

$$|z-r_n| = r_n \left|\frac{2\zeta}{1-\zeta}\right| < 4\varepsilon r_n, \quad \text{if} \quad |\zeta| < \varepsilon < \frac{1}{2}.$$

Thus the function $f(z)$ assumes every value a such that $|a-a_n|<d_n$ at least p_n times in $|z-r_n|<4\varepsilon r_n$ and replacing 4ε by ε we deduce Theorem 2.

5. Proof of Theorem 1

We now choose $\alpha=\varrho'$, so that $0<\alpha<1$, set $c=\log((1+\alpha)/(1-\alpha))$, and

(5.1) $$r_n = \exp\exp(cn), \quad p_n = [2r_n^{\alpha}]+1, \quad n \geqq 1,$$

where $[x]$ denotes the integral part of x. Then

$$r_{n+1} = r_n^{(1+\alpha)/(1-\alpha)}$$

and so

$$r_{n+1}/r_n = r_{n+1}^{\alpha} r_n^{\alpha} \geqq \max\left\{r_1^{2\alpha}, \frac{p_n p_{n+1}}{9}\right\}.$$

Thus the conditions of Theorem 2 are satisfied, and taking $\varepsilon\leqq 1/2$ we see that $f(z)$ assumes the value a at least p_n times in $|z-r_n|<r_n \tan\varepsilon$ provided that

(5.2) $$|a-a_n| < d_n.$$

If a lies in infinitely many of the disks (5.2), then we see that the equation $f(z)=a$ has more than $(2r_n)^{\alpha}$ roots in $|\arg z|<\varepsilon$, $|z|<2r_n$ for infinitely many n. This implies by (2.4) and (2.5) that $k_i(a, P, f)\geqq\alpha$. Also the set of a in question includes all a lying in infinitely many of the disks (5.2). For a_n we can choose any sequence such that $|a_n|<1$, and so any bounded sequence, for if $|a_n|<M$, where $M>1$, we consider f/M, a_n/M instead of f, a_n. For d_n we have from Theorem 2 and (5.1)

$$d_n > \exp\{-3K_5 \exp(\alpha \exp(cn))\} > \exp\{-\exp\exp(cn)\}$$

for large n, since $\alpha<1$. Thus for our exceptional set we can choose any bounded set V' of span less than $c^{-1}=\{\log((1+\varrho')/(1-\varrho'))\}^{-1}$.

In conclusion we note that by using Theorem 2 and the technique employed in [1, Section 10] we can also deal with the limiting cases $\varrho'=0$ and $\varrho'=1$. In this way we can construct a regular bounded function in P, which assumes all values a of a preassigned set of countably infinite span $V(0)$ with positive order $\varrho'(a)$ and all values of a preassigned set $V(1)$ of zero span with order 1.

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