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Commentationes in honorem
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Appendix

INTERPOLATION WITH MULTIPLICITIES

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Let $a_1, a_2, \ldots, a_n, \ldots$ be points in the unit disc. Assign to every a_n a multiplicity p_n so that

$$\sum p_n(1-|a_n|) < \infty.$$

We can then form the Blaschke product

$$A(z) = \Pi_1^\infty \left(\frac{a_n - z}{1-z_n\bar{a}_n}\frac{\bar{a}_n}{|a_n|}\right)^{p_n} = \left(\frac{a_n - z}{1-z\bar{a}_n}\frac{\bar{a}_n}{|a_n|}\right)^{p_n} A_n(z).$$

We are interested in general interpolation with multiplicity for functions $f(z) \in H^\infty$, i.e. if the interpolation

(1) $$f(a_n) = c_n, \quad f^{(\nu)}(a_n) = 0, \quad 0 < \nu < p_n,$$

is possible with $f \in H^\infty$ for arbitrary $(c_n) \in l^\infty$.

The problem was considered by Katsnel'son [2], who proved that the condition

(2) $$|A_n(a_n)|^{p_n} \geqq \delta > 0$$

is sufficient. Simple examples — equally spaced a_n on a circle $|z|=R$ and high multiplicity at $z=0$ — show that this condition is not necessary.

A complete solution is given in the following theorem.

Theorem. *The problem* (1) *has a solution in* H^∞ *for all* $c \in l^\infty$ *if and only if*

(3) $$\left|\frac{a_n - z}{1-z\bar{a}_n}\right|^{p_n} + |A_n(z)| \geqq \delta > 0, \quad n = 1, 2, \ldots.$$

Corollary 1. *The condition* (2) *is sufficient.*

We may assume that $a_n = 0$. If (2) holds $|a_\nu| \geqq 1 - c/p_n = R$, $\nu \neq n$. Consider $|z| \leqq 1 - 2c/p_n$. Then

$$\log|A_n(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \log|A_n(Re^{i\theta})|\, d\theta$$

$$\geqq \text{const.}\; p_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|A_n(Re^{i\theta})|\, d\theta$$

$$\geqq -\text{const.}$$

by (2). Hence (3) holds for $|z| \leqq 1 - 2c/p_n$ and (3) is obvious if $|z| > 1 - 2c/p_n$.

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As pointed out to me by Svante Jansson we have the following converse.

Corollary 2. *If* a_n *are real* (2) *is necessary and sufficient.*

We must prove that in this case (3) implies (2). Assume again $a_n=0$, $p_n=p$. Choose c so that

$$x^p = \left(1-\frac{c}{p}\right)^p = \frac{1}{2}\delta.$$

$$\log\frac{\delta}{2} \leqq \log\left|A_n\left(1-\frac{c}{p}\right)\right| \leqq \sum_{a_\nu>x} p_\nu \log\frac{a_\nu-x}{1-a_\nu x}$$

$$\leqq -\text{const.}\,(1-x^2)\sum_{a_\nu>x} p_\nu \frac{1-a_\nu^2}{(1-a_\nu x)^2}$$

$$\leqq -\text{const.}\frac{1}{1-x}\sum_{a_\nu>x}(1-a_\nu)\,p_\nu.$$

This proves

$$\left(\prod_{a_\nu>0} a_\nu^{p_\nu}\right)^p \geqq \text{const.}$$

and similarly for $a_\nu<0$.

Proof of the Theorem. We consider a finite case. We replace every a_n by p_n points corresponding to the solutions of

$$(z-a_n)^{p_n} = \varepsilon, \quad \text{i.e.} \quad z = a_{nj}, \quad j = 1, \ldots, p_n,$$

where $\varepsilon>0$ is small. It is easy to see that if the interpolation problem

$$f(a_{nj}) = c_n, \quad j = 1, \ldots, p_n, \tag{4}$$

has a uniformly bounded solution as $\varepsilon\to 0$, (1) has a solution.

Assume now that (3) holds. Then Theorem 4 of [1] can be used and proves that (4) does indeed have uniformly bounded solutions, as $\varepsilon\to 0$.

Conversely, assume that (1) has a solution and choose $c_n=1$ and $c_\nu=0$, $\nu\neq n$. Call the solution $f_n(z)$. Then

$$|f_n(z)-1| \leqq (\|f_n\|+1)\left|\frac{a_n-z}{1-za_n}\right|^{p_n}$$

and

$$|f_n(z)| \leqq \|f_n\|\,|A_n(z)|.$$

This proves (3).

References

[1] Carleson, L.: Interpolations by bounded analytic functions and the corona theorem. - Ann. of Math. (2) 76, 1962, 547—559.

[2] Katsnel'son, V. È.: Conditions for a system of root vectors of certain classes of operators to be a basis. - Funkcional. Anal. i Priložen. 1, 1967, 39—51 (Russian).

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