Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 10, 1985, 255—259

Commentationes in honorem
Olli Lehto
LX annos nato

ON THE NASH—MOSER IMPLICIT FUNCTION THEOREM

LARS HÖRMANDER

In [1] a general implicit function theorem of Moser's type was derived from the methods of Nash [2]. However, it turns out that better results and simpler proofs may be obtained by a simple modification of this approach combined with standard non-linear functional analysis. We shall present this modification here, choosing this time an abstract setting as for example in Zehnder [3].

Let E_a, $a \geqq 0$, be a decreasing family of Banach spaces with injections $E_b \hookrightarrow E_a$ of norm $\leqq 1$ when $b \geqq a$. Set $E_\infty = \bigcap E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous, and assume that we have given linear operators $S_\theta \colon E_0 \to E_\infty$ for $\theta \geqq 1$, such that with constants C bounded, when a and b are bounded,

(i) $$\|S_\theta u\|_b \leqq C\|u\|_a, \quad b \leqq a;$$

(ii) $$\|S_\theta u\|_b \leqq C\theta^{b-a}\|u\|_a, \quad a<b;$$

(iii) $$\|u-S_\theta u\|_b \leqq C\theta^{b-a}\|u\|_a, \quad a>b;$$

(iv) $$\left\|\frac{d}{d\theta} S_\theta u\right\|_b \leqq C\theta^{b-a-1}\|u\|_a.$$

Hölder spaces are classical examples (see [1, appendix]). The property (iv) is the strongest one; integration of (iv) from θ to ∞ gives (iii) and integration from 1 to θ gives (ii) (although the constants may become large as b approaches a). From (ii) and (iii) we obtain the logarithmic convexity of the norms

(v) $$\|u\|_{\lambda a+(1-\lambda)b} \leqq C\|u\|_a^\lambda \|u\|_b^{1-\lambda} \quad \text{if} \quad 0<\lambda<1.$$

In fact, if $c=\lambda a+(1-\lambda)b$ and $a<b$, we obtain from (ii) and (iii)

$$\|u\|_c \leqq \|S_\theta u\|_c + \|u-S_\theta u\|_c \leqq C(\theta^{c-a}\|u\|_a + \theta^{c-b}\|u\|_b).$$

Since $\|u\|_b \geqq \|u\|_a$ the two terms are equal for some $\theta \geqq 1$, which gives (v) with the constant $2C$.

Condition (iv) provides a convenient way of making a continuous decomposition of an arbitrary $u \in E_a$. However, we prefer to use discrete decompositions in order to have no problems with vector valued integration. We therefore choose a fixed sequence with $1=\theta_0<\theta_1<\ldots\to\infty$ such that θ_{j+1}/θ_j is bounded, set $\Delta_j=\theta_{j+1}-\theta_j$

doi:10.5186/aasfm.1985.1028

and introduce

$$R_j u = (S_{\theta_{j+1}} u - S_{\theta_j} u)/\Delta_j \quad \text{if} \quad j > 0, \quad R_0 u = S_{\theta_1} u/\Delta_0.$$

Then we have by (iii)

$$u = \sum_0^\infty \Delta_j R_j u \tag{1}$$

with convergence in E_a, if $u \in E_b$ for some $b > a$, and (iv) gives for all b

$$\|R_j u\|_b \leqq C_{a,b} \theta_j^{b-a-1} \|u\|_a. \tag{2}$$

Conversely, assume that $a_1 < a < a_2$, that $u_j \in E_{a_2}$ and that

$$\|u_j\|_b \leqq M\theta_j^{b-a-1} \quad \text{if} \quad b = a_1 \quad \text{or} \quad b = a_2. \tag{3}$$

By (v) this remains true with a constant factor on the right-hand side if $a_1 < b < a_2$ so the sum $u = \sum \Delta_j u_j$ converges in E_b if $b < a$. Let E_b' be the set of all sums $u = \sum \Delta_j u_j$ with u_j satisfying (3) and introduce as norm $\|u\|_a'$ the infimum of M over all such sum decompositions. We have then seen that $\| \ \|_a'$ is stronger than $\| \ \|_b$ if $b < a$, while (1) and (2) show that $\| \ \|_a'$ is weaker than $\| \ \|_a$. The space E_b' and, up to equivalence, its norm are independent of the choice of a_1 and a_2. In fact, assume that $u = \sum \Delta_j u_j$ with u_j satisfying (3), and let us estimate $\|R_k u\|_c$. By (3) and (iv)

$$\|R_k u_j\|_c \leqq CM\theta_k^{c-a_\nu-1} \theta_j^{a_\nu-a-1}, \quad \nu = 1, 2.$$

We multiply by Δ_j and sum for $j \leqq k$ taking $\nu = 2$ and for $j > k$ taking $\nu = 1$. This gives that

$$\|R_k u\|_c \leqq CM\theta_k^{c-a-1}.$$

Thus the decomposition (1) can be used instead, for any interval. Altogether this shows that the space E_a' and its topology are independent of the choice of the numbers a_1 and a_2; E_a' is defined by (2) for any two values of b to the left and to the right of a. (It does not depend on the smoothing operators of course.) In the particular case of Hölder spaces we have $E_a' = E_a$ unless a is an integer.

In (iii) we may replace $\|u\|_a$ by $\|u\|_a'$ if we take another constant, which may tend to ∞ as b approaches a. In fact, assume we have a decomposition $u = \sum \Delta_j u_j$ with u_j satisfying (3). Then if $b < a_1 < a < a_2$

$$u - S_\theta u = \sum \Delta_j (u_j - S_\theta u_j), \quad \|(u_j - S_\theta u_j)\|_b \leqq CM\theta^{b-a_\nu} \theta_j^{a_\nu - a - 1}.$$

We sum for $\theta_j > \theta$ with $\nu = 1$ and for $\theta_j \leqq \theta$ with $\nu = 2$ and conclude that

$$\|u - S_\theta u\|_b \leqq C_b \theta^{b-a} M$$

which proves the strengthened form of (iii).

If u_k is a bounded sequence in E_a for some fixed $a > 0$ and $u_k \to u$ in E_0, it follows from (v) that u_k is a Cauchy sequence in E_b for every fixed $b < a$ so the limit $u \in E_b$. In fact, $u \in E_a'$ for if we apply (2) to u_k and let $k \to \infty$ it follows that

$$\|R_j u\|_b \leqq C_{a,b} \theta_j^{b-a-1} \varliminf \|u_k\|_a.$$

We shall say that a sequence $u_k \in E_a$ is weakly convergent and write $u_k \rightharpoonup u$ in the preceding situation. Note that the definition of E'_a shows that every element in E'_a is the weak E_a limit of a sequence in E_∞.

To state the implicit function theorem we assume that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above; we use the same notation for the smoothing operators also. In addition we assume that the *embedding* $F_b \hookrightarrow F_a$ *is compact* when $b > a$.

Theorem. *Let* α *and* β *be fixed positive numbers,* $[a_1, a_2]$ *an interval with* $0 \leqq a_1 < \alpha < a_2$, V *a convex* E'_α *neighborhood of* 0 *and* Φ *a map from* $V \cap E_{a_2}$ *to* F_β *which is twice differentiable and satisfies, for some* $\delta > 0$,

$$\|\Phi''(u; v, w)\|_{\beta+\delta} \leqq C \sum (1 + \|u\|_{m'_j}) \|v\|_{m''_j} \|w\|_{m'''_j}, \tag{4}$$

where the sum is finite. Also assume that $\Phi'(v)$, *for* $v \in V \cap E_\infty$, *has a right inverse* $\psi(v)$ *mapping* F_∞ *to* E_{a_2}, *that* $(v, g) \to \psi(v) g$ *is continuous from* $E_\infty \cap V \times F_\infty$ *to* E_{a_2} *and that*

$$\|\psi(v) g\|_a \leqq C(\|g\|_{\beta+a-\alpha} + \|g\|_0 \|v\|_{\beta+a}), \quad a_1 \leqq a \leqq a_2. \tag{5}$$

Let a_2 *be at least as large as the subscripts on the right-hand side of* (4),

$$\max(m'_j - \alpha, 0) + \max(m''_j, a_1) + m'''_j < 2\alpha, \quad \textit{for every } j; \; \alpha - \beta < a_1. \tag{6}$$

For every $f \in F'_\beta$ *with sufficiently small norm one can then find a sequence* $u_j \in V \cap E_{a_2}$ *which has a weak limit* u *in* E'_α *such that* $\Phi(u_j)$ *converges weakly in* F'_β *to* $\Phi(0) + f$.

Proof. Let $g \in F'_\beta$ and write $g_j = R_j g$; thus

$$g = \sum \Delta_j g_j; \quad \|g_j\|_b \leqq C_b \theta_j^{b-\beta-1} \|g\|'_\beta. \tag{7}$$

We claim that if $\|g\|'_\beta$ is small enough we can define a sequence $u_j \in E_{a_2} \cap V$ with $u_0 = 0$ by the recursion formula

$$u_{j+1} = u_j + \Delta_j \dot{u}_j, \quad \dot{v}_j = \psi(v_j) g_j, \quad v_j = S_{\theta_j} u_j \tag{8}$$

and that we have the estimates

$$\|\dot{u}_j\|_a \leqq C_1 \|g\|'_\beta \theta_j^{a-\alpha-1}, \quad a_1 \leqq a \leqq a_2, \tag{9}$$

$$\|v_j\|_a \leqq C_2 \|g\|'_\beta \theta_j^{a-\alpha}, \quad \alpha < a \leqq a_2, \tag{10}$$

$$\|u_j - v_j\|_a \leqq C_3 \|g\|'_\beta \theta_j^{a-\alpha}, \quad a \leqq a_2. \tag{11}$$

Indeed, suppose that u_j is already constructed for $j \leqq k$ and that (10), (11) are proved then as well as (9) for $j < k$. We prove (9) for $j = k$ by application of (5) to v_k and g_k which gives

$$\|\psi(v_k) g_k\|_a \leqq C(\theta_k^{a-\alpha-1} \|g\|'_\beta + \theta_k^{-\beta-1} \|g\|'_\beta C_2 \|g\|'_\beta \theta_k^{\beta+a-\alpha}).$$

Here we have used the fact that $\beta+a_1>\alpha$ by (6). This gives (9) for $j=k$ if $C_1>C$ and $\|g\|_\beta$ is small, no matter what the value of C_2 is. (This point is important to avoid circularity in the choice of constants.) Since $u_{k+1}=\sum_0^k \Delta_j \dot{u}_j$ we obtain from

(12) $$\|u_{k+1}\|'_\alpha \leqq C' C_1 \|g\|'_\beta$$

and we conclude from the strengthened version of (iii) discussed above that (11) is valid for $a<\alpha$ not close to α if C_3/C_1 is large enough. When $a=a_2$ the same conclusion is obtained directly by adding (9), and the logarithmic convexity (v) then gives (11) with the same constant in the whole interval. Then we obtain (10) for $j=k+1$ by just summing (9).

We have now proved that the construction of the infinite sequences u_j, v_j, $\dot{u}_j$ is possible, for (12) and (11), with $j=k+1$, show that u_{k+1} and v_{k+1} are in V if $\|g\|'_\beta$ is sufficiently small. It follows from (9) that u_k has a weak limit u in E'_α. What remains is to examine the limit of $\Phi(u_k)$. Write

(13) $$\begin{aligned}&\Phi(u_{j+1})-\Phi(u_j)\\ &= \big(\Phi(u_j+\Delta_j\dot{u}_j)-\Phi(u_j)-\Phi'(u_j)\Delta_j\dot{u}_j\big)+\big(\Phi'(u_j)-\Phi'(v_j)\big)\Delta_j\dot{u}_j+\Delta_j g_j\\ &= \Delta_j(e'_j+e''_j+g_j).\end{aligned}$$

First we estimate

$$e''_j = \int_0^1 \Phi''\big(v_j+t(u_j-v_j);\ \dot{u}_j,\ u_j-v_j\big)\,dt$$

by means of (4); our purpose is of course to show that e'_j, e''_j are so small that $\Phi(u_k)-\Phi(u_0)$ will have a limit close to $\sum \Delta_j g_j=g$. If we combine (5) with (9), (10), (11) and recall (6), we obtain, if $\varepsilon>0$ is so small that (6) remains valid if ε is added in the left-hand side,

(14) $$\|e''_j\|_{\beta+\delta} \leqq C\theta_j^{-1-\varepsilon}\|g\|'^2_\beta.$$

For any $N>0$ we can choose θ_j so that $\Delta_j=O(\theta_j^{-N})$. For large N we obtain in the same way from Taylor's formula

(15) $$\|e'_j\|_{\beta+\delta} \leqq C\theta_j^{-1-\varepsilon}\|g\|'^2_\beta.$$

It follows that $\Phi(u_k)$ converges weakly to $\Phi(0)+T(g)+g$ where

(16) $$T(g) = \sum \Delta_j(e'_j+e''_j).$$

This sum is uniformly convergent in $F_{\beta+\delta}$ norm when $\|g\|'_\beta$ is small enough. Hence $T(g)$ is a continuous map from a neighborhood of 0 in F'_β to a compact subset of F'_β, and

$$\|Tg\|'_\beta \leqq C\|g\|'^2_\beta.$$

By the Leray–Schauder theorem it follows that $g+T(g)$ takes on all values in a neighborhood of 0, and this proves the theorem.

References

[1] HÖRMANDER, L.: The boundary problems of physical geodesy. - Arch. Rational Mech. Anal. 62, 1976, 1—52.

[2] NASH, J.: The imbedding problem for Riemannian manifolds. - Ann. of Math. (2)63, 1956, 20—63.

[3] ZEHNDER, E.: Generalized implicit function theorems with applications to some small divisor problems, I. - Comm. Pure Appl. Math. 28, 1975, 91—140.

University of Lund
Institute of Mathematics
S-220 07 Lund
Sweden

Received 30 March 1984