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COMPARISON OF HYPERBOLIC AND EXTREMAL LENGTHS

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Let S be a hyperbolic Riemann surface of finite type (that is, $S=U/G$, where U is the upper half-plane and G is a finitely generated, torsion free Fuchsian group), and let w be a hyperbolic simple loop on S (that is, w is a simple loop on S, and w is represented by a hyperbolic element A in G). There are two natural notions of length for such a loop: first, there is the hyperbolic length l of the shortest geodesic freely homotopic to w on S, and second, there is the extremal length m of the family of loops freely homotopic to w on S. The purpose of this note is to give some comparisons between these two notions of length.

When we need to emphasize the dependence of say l on w, or A, or S, we will write $l(w)$, or $l(A)$, or $l(w, S)$.

The proofs all take place in the context of a non-elementary finitely generated Fuchsian group; the group may have torsion. All the results are easily seen to be equally valid for elementary Fuchsian groups.

In general, we say that a set $X\subset U$ is *precisely invariant* under the element $A\in G$ if $A(X)=X$, and $B(X)\cap X=\emptyset$ for all B in G which are not powers of A.

We say that the hyperbolic element $A\in G$ is *strictly simple* if the axis L_A of A is precisely invariant under A in G. In particular, if A is strictly simple, then *there are no fixed points of elliptic elements of G lying on L_A*.

For any hyperbolic element A of G, we define $l=l(A)$ to be the geodesic length of A; that is, A is conjugate in PSL(2, R) to a unique element of the form $z\to e^l z$, $l>0$; equivalently, $|\mathrm{tr}(A)|=2\cosh(l/2)$. If G is torsion free, this definition agrees with that of the first paragraph.

We let w be the projection of L_A on U/G, so that w is a geodesic. Let U' be U with all fixed points of elliptic elements of G deleted, and let $S'=U'/G$. Then $m=m(A)$ is the extremal length of the family of loops freely homotopic to w on S'.

We normalize G so that $A(z)=e^l z$, $l>0$. We denote the projection from U to S, or from U' to S', by $p: U\to S$.

A *topological collar* about w is a subsurface S_0 of S', containing w, where S_0 is topologically an (open) annulus.

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A *topological collar* about L_A is a set X, containing L_A, which is precisely invariant under A in G. A topological collar about L_A of the form

$$\{\pi/2-\theta_1 < \arg z < \pi/2+\theta_2\}, \quad 0 \leqq \theta_1, \theta_2 \leqq \pi/2,$$

is a *collar* about L_A of angle width $\theta=\theta_1+\theta_2$.

Proposition 1. *If there is a collar about L_A of angle width θ, then*

$$m\theta \leqq l; \tag{1}$$

in any case,

$$l \leqq m\pi. \tag{2}$$

Proof. Let T be the collar about L_A of angle width θ. Then $f(z)=\log(-iz)$, $f(i)=0$, maps T onto a strip V of height θ, where V is invariant under $H=\{z\to z+lZ\}$. The extremal length $m(w, p(T))$ is the extremal length of the family of curves connecting a point z to $z+l$ in V/H; it is well known that the extremal length of this family is l/θ [2, p. 12]. We now obtain inequality (1) from $m(w, S)\leqq m(w, p(T))=l/\theta$.

Inequality (2) was proved in [7], but the statement there has the constant 2π rather than π. For the convenience of the reader, we reprove it. Let T be any topological collar about L_A. Then using the same function $f(z)=\log(-iz)$, $f(T)$ is a topological strip invariant under H. We can estimate $m(w, p(T))$ by using the Euclidean metric in $f(T)$. We observe as above that the length of any curve is at least l, and since any vertical line intersects $f(T)$ in a set of measure at most π, the area of $f(T)/H$ is at most πl. Hence, $m(w, p(T))\geqq l^2/\pi l=l/\pi$. It was shown by Jenkins [5] that the infimum of $m(w', S_0')$, where the infimum is taken over all topological collars S_0' about loops w', freely homotopic to w on S', is in fact a minimum, and this minimum value is $m(w, S')$. Inequality (2) now follows.

The loop w is called a *boundary loop* if w divides S into two subsurfaces and one of them is topologically an annulus. It is immediate that every boundary loop has a collar of angle width at least $\pi/2$.

Corollary 1. *If w is a boundary loop, then $m\pi/2\leqq l\leqq m\pi$.*

Our next proposition is a version of the collar lemma; other versions appear in Keen [6], Matelski [8], Buser [3], Randol [9], Abikoff [1], and Halpern [4].

Proposition 2. *If $A(z)=e^l z$ is strictly simple, then L_A has a collar of angle width θ, where $\sin\theta/2=e^{-l/2}$. Further, if $B\in G$ is also strictly simple, where $p(L_A)\cap p(L_B)=\emptyset$, then these collars about L_A and L_B are disjoint.*

Proof. Let L_B be a hyperbolic axis in G, where B represents a strictly simple loop v in U/G, and either $v=w$, or v is disjoint from w; we are primarily interested in the former case where $B=C\circ A\circ C^{-1}$, for some C in G. Let x and y be the endpoints of L_B; we can assume without loss of generality that $0<x<y$. Let L be the hyperbolic line with endpoints x and $e^l x = A(x)$. Since no translate of L_B can cross L_B, $y \leqq e^l x$, and so $d(L_A, L) \leqq d(L_A, L_B)$, where $d(\cdot, \cdot)$ denotes hyperbolic distance. Let M be the ray through the origin which is tangent to L. We write $M = \{\arg z = \varphi\}$, and we observe that

(3) $$\sin \varphi = (e^l x - x)/(e^l x + x) = \tanh l/2.$$

We note that $d(L_A, L) = d(L_A, M)$, and we choose M' to be that ray through the origin so that $d(M', L_A) = d(M', M)$. We write $M' = \{\arg z = \pi/2 - \theta_1\}$, and we observe that we have chosen M' so that L_A has a collar of angle width $2\theta_1 = \theta$.

An easy computation shows that

$$d(L_A, M) = \log(\csc \varphi + \cot \varphi),$$

and

$$d(L_A, M') = \log\big(\csc(\pi/2 - \theta_1) + \cot(\pi/2 - \theta_1)\big).$$

Hence

(4) $$(1 + \sin \theta_1)/\cos \theta_1 = ((1 + \cos \varphi)/\sin \varphi)^{1/2}.$$

We set the right hand side of (4) equal to Q; note that $\sin \theta_1 > 0$, and solve (4) for $\sin \theta_1$. We obtain

(5) $$\sin \theta_1 = \frac{Q^2 - 1}{Q^2 + 1}.$$

We combine (3), (4), and (5) to obtain

$$\sin \theta_1 = \frac{1 + \cosh l/2 - \sinh l/2}{1 + \cosh l/2 + \sinh l/2} = e^{-l/2}.$$

Once we have chosen A in G, we can of course consider l and m as functions on the Teichmuller space $T(G)$. The remainder of our note takes place in this setting.

Corollary 2. *The lengths l and m go to zero together, and* $\lim_{l \to 0} l/m = \pi$.

Corollary 3. $m \leqq (1/2) l e^{l/2}$.

Proof. We know from Proposition 2 that L_A has a collar of angle width θ, where $\sin \theta/2 = e^{-l/2}$. Thus $\theta/2 \geqq \sin \theta/2 = e^{-l/2}$; hence by Proposition 1, $2me^{-l/2} \leqq l$, or

(6) $$m \leqq (1/2) l e^{l/2}.$$

We conclude this note with an example showing that the estimate (6) is not very far from being sharp.

For each α, $0 < \alpha < \pi/2$, we write down the Fuchsian group G_α, generated by

$$A_\alpha = \begin{bmatrix} \csc \alpha & \cot \alpha \\ \cot \alpha & \csc \alpha \end{bmatrix}, \qquad B_\alpha = \begin{bmatrix} \sec \alpha & i \tan \alpha \\ -i \tan \alpha & \sec \alpha \end{bmatrix}.$$

Observe that A_α has its fixed points at ± 1, while B_α has its fixed points at $\pm i$.

A fundamental domain D_α for G_α can be obtained by drawing the four hyperbolic lines with endpoints $e^{i\alpha}$ and $-e^{-i\alpha}$, $-e^{-i\alpha}$ and $-e^{i\alpha}$, $-e^{i\alpha}$ and $e^{-i\alpha}$, and $e^{-i\alpha}$ and $e^{i\alpha}$; see Figure 1. We see at once that G has signature (1, 1; ∞).

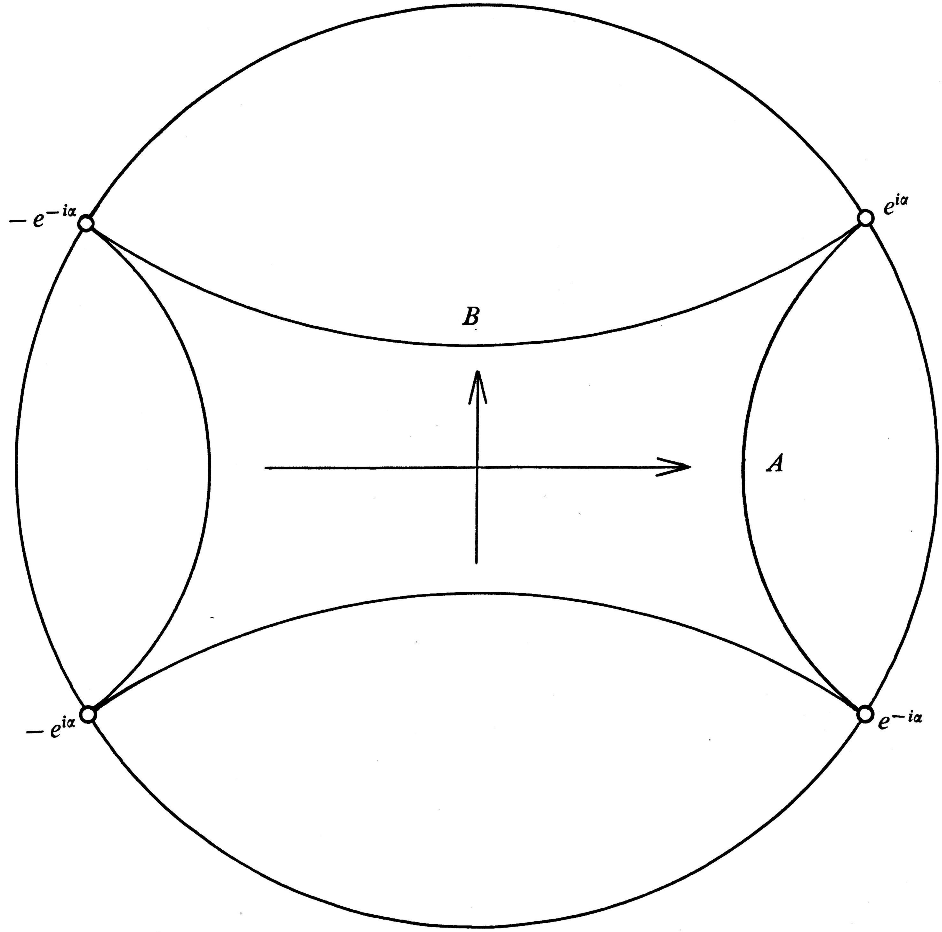

Figure 1

The reflection $j(z)=\bar{z}$ commutes with $A=A_\alpha$, and conjugates $B=B_\alpha$ into B^{-1}. Hence the elliptic modulus of the torus U/G with the generators A and B is pure imaginary. Since the sides of D_α are fixed point sets of reflections in the group generated by G_α, j, and the reflection $z \to -\bar{z}$, the covering map φ, from U onto the plane punctured at the lattice points, maps D_α onto a rectangle, and conjugates A and

B into translations in the plane. We conclude that the extremal length of the family of paths joining a point z on the boundary of D_α to $A(z)$ (or z to $B(z)$), is equal to the extremal length of the family of paths joining opposite sides of the rectangle, $\varphi(D_\alpha)$. In particular $m(A)=1/m(B)$.

Let θ_B be the angle width of the largest collar about L_B. Then,

$$m(A) = 1/m(B) \geqq \theta_B/l(B) = 2l(B)^{-1}\arcsin(e^{-l(B)/2}) \geqq 2l(B)^{-1}e^{-l(B)/2}.$$

An easy computation shows that $l(B)=2\log(\sec\alpha+\tan\alpha)$, and so

$$m(A) \geqq 1/(\sec\alpha+\tan\alpha)\log(\sec\alpha+\tan\alpha)$$
$$\geqq 1/(\sec\alpha+\tan\alpha)(\sec\alpha+\tan\alpha-1).$$

We rewrite the right hand side of the above as

$$R = \cos^2\alpha/(1+\sin\alpha)(1+\sin\alpha-\cos\alpha).$$

We note that $l(A)=2\log(\csc\alpha+\cot\alpha)$, and we observe that

$$\lim_{\alpha\to 0} \mathrm{Re}^{-l(A)/2} = \lim_{\alpha\to 0} R/(\csc\alpha+\cot\alpha) = 1/2.$$

We conclude that for every $\varepsilon>0$, we can find an α so that

$$m(A) \geqq R \geqq (1/2-\varepsilon)e^{l(A)/2}.$$

Remark 1. The requirement that G be finitely generated was used only in Jenkins' theorem. Proposition 2 is valid for an arbitrary Fuchsian group. Proposition 1 is also valid in this more general context, provided one understands m as the infimum of $m(w', S_0')$, where w' is freely homotopic to w, and S_0' is any annulus containing w', and contained in S_0.

Remark 2. If L_A has elliptic fixed points on it, but is otherwise simple, the results are slightly different. We outline these below.

Let A be a hyperbolic element of the finitely generated Fuchsian group G, where for every $B\in G$, either $B(L_A)=L_A$, or $B(L_A)\cap L_A=\emptyset$. Assume that there is an element $E\in G$, where E is not a power of A, and $E(L_A)=L_A$. Then E is necessarily elliptic of order 2, and $p(L_A)$ is a path from one ramification point of order 2 to another. Call these ramification points x and x', and let w be a simple loop which separates S' into two subsurfaces, where one of these is a disc with the two punctures, x and x'. We have already defined $l(A)=l(w)$, and we set $m(A)=m(w)$.

We normalize G so that $A(z)=e^l z$, and so that E has fixed points at $\pm i$; then $A\circ E$ has its fixed points at $\pm ie^{l/2}$.

If $\{\pi/2-\theta_1<\arg z<\pi/2+\theta_2\}$ is a collar about L_A (i.e., it is precisely invariant under the stability subgroup of L_A in G), then either $\theta_1=0$, or $\theta_2=0$; we assume without loss of generality that $\theta_2=0$.

Inequality (1) still holds, and inequality (2) can be replaced with

(2′) $$l\leqq m\pi/2.$$

The proof is the same, except that to prove (2′), observe that f conjugates E into the transformation $z\to -z$; hence if k is the measure of the intersection of $f(T)$ with the vertical line $\operatorname{Re}(z)=a$, $a<l/2$, and k' is the measure of the intersection of $f(T)$ with the line $\operatorname{Re}(z)=-a$, then $k+k'\leqq\pi$.

If w is a boundary loop, then G is elementary, and $l=m\pi/2$, as can be verified directly.

The proof of Proposition 2 is essentially unchanged, but the statement is different.

Proposition 2′. *If A represents a simple loop w, and L_A has elliptic fixed points on it, then L_A has a collar of angle width θ, where* $\sin\theta=e^{-l/2}$.

Corollary 2′. *If A is as in Proposition 2′, then l and m go to zero together and* $\lim_{l\to 0} l/m=\pi/2$.

Corollary 3′. *If A is as in Proposition 2′, then* $m\leqq le^{l/2}$.

The group G_α has a Z_2 extension H_α obtained by adjoining the transformation $j(z)=-z$. Then $l(A_\alpha, H_\alpha)=l(A_\alpha, G_\alpha)$, $l(B_\alpha, H_\alpha)=l(B_\alpha, G_\alpha)$, and $m(A_\alpha, H_\alpha)=$ $=2m(A_\alpha, G_\alpha)$, $m(B_\alpha, H_\alpha)=2m(B_\alpha, G_\alpha)$. The first two equalities are trivial, and the second two follow from the fact that the map φ commutes with j, and φ maps the axes of A and B onto Euclidean lines which are parallel to the sides and bisect the rectangle $\varphi(D_\alpha)$.

In this case we obtain that for α sufficiently small, $m(A, H_\alpha)\geqq(1-\varepsilon)e^{l(A, H_\alpha)/2}$.

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