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HARMONIC AND RELATIVE HARMONIC DIMENSIONS

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Consider an open Riemann surface R of *Heins type*, i.e., a parabolic Riemann surface with a single ideal boundary component δR. Let K be a closed parametric disk on R and $HP(R-K;\partial(R-K))$ the class of nonnegative harmonic functions on $R-K$ with vanishing boundary values on the relative boundary $\partial(R-K)$ of $R-K$. The cardinal number of the set of nonproportional minimal functions in $HP(R-K;\partial(R-K))$, is, by definition, the *harmonic dimension* $\dim \delta R$ of the ideal boundary δR of R. It is at least 1 and independent of the choice of K. The notion of harmonic dimension of the ideal boundary of a surface of what we shall call Heins type was introduced by Heins in [4].

Let F be the union of a locally finite family of disjoint closed parametric disks on R and define $HP(R-F;\partial(R-F))$ as above. The cardinal number of the set of nonproportional minimal functions in $HP(R-F;\partial(R-F))$ is known as the *relative harmonic dimension* $\dim_F \delta R$ of the ideal boundary δR of R relative to F. Again it is at least 1, but this time it depends essentially on F unless F is compact, in which case $\dim_F \delta R=\dim \delta R$. A more detailed description of these concepts will be given in Section 1.

The first purpose of this paper is to compare $\dim \delta R$ and $\dim_F \delta R$. We shall see that all three cases can occur, depending on the choice of R and F: $\dim_F \delta R<\dim \delta R$, $\dim_F \delta R>\dim \delta R$, and $\dim_F \delta R=\dim \delta R$. In Section 2 we shall show that, on any R, there exists an F such that $\dim_F \delta R=1$; therefore, $\dim_F \delta R<\dim \delta R$ if the latter exceeds 1. In Section 3 an example will be given of an F in the complex plane $\boldsymbol{C}$ with $\dim_F \delta \boldsymbol{C}>1$; since $\dim \delta \boldsymbol{C}=1$, we have an R and an F with $\dim_F \delta R>\dim \delta R$. The most important task in this context is to characterize those F for which $\dim_F \delta R=\dim \delta R$. We shall give in Section 4 a useful sufficient condition.

The second purpose of this paper is to construct an example of a Riemann surface R of Heins type with $\dim \delta R=\mathfrak{a}$, the cardinal number of a countably infinite set. This will be done in Section 5, our method demonstrating the applicability of the sufficient conditions obtained in Section 4. An example of $\dim \delta R=\mathfrak{a}$ was first given by Kuramochi [5], and quite recently by Segawa [11], who used his duality theorem. It was Heins [4] who gave an example of a Riemann surface R of Heins type with $\dim \delta R=n$, an arbitrary positive integer. If one tries to follow the Heins

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construction for the case $\dim \delta R=\mathfrak{a}$ as well, he is led in a natural manner to the concept of relative harmonic dimension. Anybody who visualizes this is convinced of the fact that the example of R with $\dim \delta R=\mathfrak{a}$ already existed implicitly in the pioneering work of Heins. This observation is a motivation of our present study of the relative harmonic dimension.

1. Relative harmonic dimension

1.1. We denote by δR the *ideal boundary* of an open Riemann surface R. By this we mean that δR is an abstract set and $R\cup\delta R$ is topologized as a compact Hausdorff space containing R as its open and dense subset. Unless otherwise explicitly stated, we do not specify the topology of $R\cup\delta R$ beyond this requirement. However, we often write $\zeta\to\delta R$ $(\zeta\in R)$ to mean that ζ converges to the point at infinity of R with $R\cup\delta R$ then understood as the Alexandroff compactification of R.

We say that R has a *single ideal boundary component* δR if δR is connected when $R\cup\delta R$ is realized as the Kerékjártó–Stoïlow compactification (cf., e.g., [3]). We shall say that a Riemann surface is of *Heins type* if it is of the kind first systematically studied by Heins [4]: an open surface of parabolic type with a single ideal boundary component δR. The complex plane C and the punctured sphere $\hat{C}_0=\hat{C}-\{0\}$ are typically of Heins type.

If R is of Heins type, then there exists an exhaustion $\{R_n\}_0^\infty$ of R with the following properties: i) each R_n is a relatively compact region and ∂R_n is an analytic Jordan curve, ii) $R_n\subset\bar{R}_n\subset R_{n+1}$ $(n=0, 1, 2, \ldots)$, iii) $\bigcup_{n=0}^\infty R_n=R$, iv) the functions $w_n\in C(R)\cap H(R_n-\bar{R}_0)$ with $w_n|\bar{R}_0=0$ and $w_n|R-R_n=1$ satisfy $\lim_{n\to\infty}w_n=0$, uniformly on each compact subset of R. Conversely, if R has an exhaustion with properties i)—iv), then R is of Heins type. Here $H(S)$ is the class of harmonic functions on a Riemann surface S. We also denote by $HP(S)$ the class of nonnegative functions in $H(S)$. A function u in $HP(S)$ is said to be *minimal* in $HP(S)$ if $u>0$ and $u\geqq v\geqq 0$ for any v in $HP(S)$ implies that v/u is constant on S.

1.2. Let S be a subregion of an open Riemann surface R such that each point in ∂S is regular for the Dirichlet problem for the region S. We consider the *relative class*

$$HP(S;\partial S)=\{u\in C(R)\cap HP(S);\ u|R-S=0\}. \tag{1}$$

A function u in $HP(S;\partial S)$ is, by definition, *minimal* in $HP(S;\partial S)$ if u is not identically zero and $u\geqq v\geqq 0$ for any v in $HP(S;\partial S)$ implies that v/u is constant on S, i.e., there exists a constant c such that $v=cu$ on R. Clearly, if u is minimal in $HP(S;\partial S)$, then u is minimal in $HP(S)$, but not conversely.

With class (1), we associate two mappings, λ and μ, defined as follows. For each u in $HP(R)$, let λu be the upper envelope of the family of functions v in $HP(S;\partial S)$

with $v<u$. Then λ is a homogeneous, additive, and order preserving mapping of $HP(R)$ to $HP(S;\partial S)$. We denote by $HP(S;\partial S)_\mu$ the class of functions v in $HP(S;\partial S)$ with harmonic majorants on R. For each v in $HP(S;\partial S)_\mu$, let μv be the least harmonic majorant of v on R. Then μ is a homogeneous, additive, and order preserving mapping of $HP(S;\partial S)_\mu$ to $HP(R)$. We will use the following properties of λ and μ (cf., e.g., Noshiro [10], pp. 102—103):

(a) $\lambda\mu v=v$ for every v in $HP(S;\partial S)_\mu$ so that μ is injective, and λ is injective on $\mu(HP(S;\partial S)_\mu)$.

(b) If $u\in HP(R)$ satisfies $u\leqq\mu v$ for some v in $HP(S;\partial S)_\mu$, then u belongs to $\mu(HP(S;\partial S)_\mu)$.

(c) $v\in HP(S;\partial S)_\mu$ is minimal in $HP(S;\partial S)$ if and only if μv is minimal in $HP(R)$.

(d) If u is minimal in $HP(R)$ and $\lambda u>0$ on S, then λu is minimal in $HP(S;\partial S)$.

1.3. Unless otherwise explicitly stated, we consider henceforth exclusively open Riemann surfaces R of Heins type. Take a finite or countably infinite sequence $\{K_n\}=\{K_n\}_1^N$ $(1\leqq N\leqq\infty)$ of nondegenerate compact continua K_n in R with the following conditions: α) $K_n\cap K_m=\emptyset$ $(n\neq m)$, β) $\{K_n\}$ is locally finite, i.e., the set $\{n;\, K_n\cap X\neq\emptyset\}$ is finite for any compact subset X of R, γ) $R-\bigcup_1^N K_n$ is connected. Such a sequence $\{K_n\}$ will be called a *$\mathscr{K}$-sequence* in this paper. With a $\mathscr{K}$-sequence $\{K_n\}$ we associate a closed set F and a region W given by

(2)
$$F=\bigcup_1^N K_n,\quad W=R-F.$$

We also fix a reference point a in W and denote by $M(W)$ the class of minimal functions v in $HP(W;\partial W)$ with $v(a)=1$. We call the cardinal number $\#M(W)$ of $M(W)$ the *relative harmonic dimension* of δR with respect to F and denote it by $\dim_F\delta R$:

(3)
$$\dim_F\delta R=\#M(W).$$

Clearly it is independent of the choice of the reference point a.

Let $\{K_{in}\}_{n=1}^{N_i}$ $(i=1,2)$ be two $\mathscr{K}$-sequences on R and set $F_i=\bigcup_1^{N_i}K_{in}$ and $W_i=R-F_i$ $(i=1,2)$. Obviously F_i is compact on R if and only if $N_i<\infty$. Suppose F_1 and F_2 are compact on R. Then it is not difficult to prove that there exists a bijective, homogeneous, additive, and order preserving mapping $u_1\mapsto u_2$ of $HP(W_1;\partial W_1)$ to $HP(W_2;\partial W_2)$ such that u_1-u_2 is bounded near δR. Hence $\#M(W_1)=\#M(W_2)$ and a fortiori $\dim_{F_1}\delta R=\dim_{F_2}\delta R$. The quantity

(4)
$$\dim\delta R=\dim_F\delta R\quad(F\text{ compact})$$

is thus uniquely determined, with a compact F chosen at will. This quantity, an appropriate one attached to R, is called the *harmonic dimension* of δR.

1.4. Let $\{K_n\}$ be an arbitrary $\mathscr{K}$-sequence in R and let F and W be associated with $\{K_n\}$ as in (2). Denote by W^* the *Martin compactification* of W and by $k_W(z,\zeta)$

the *Martin kernel* on W^* with the reference point a in W (cf., e. g., [3]):

$$k_W(z,\zeta) = \frac{g_W(z,\zeta)}{g_W(a,\zeta)} \tag{5}$$

for (z,ζ) in $W\times W$, with $g_W(z,\zeta)$ the Green's function on W. For any q in W^*-W there exists a sequence $\{\zeta_n\}$ in W such that $\zeta_n\to q$ in W^* and either $\zeta_n\to\delta R$ or any subsequence of $\{\zeta_n\}$ contains a subsequence converging to a point of F. We denote by $Q(W)$ the class of the points q in W^*-W over δR, i.e., those q in W^*-W for which the first alternative occurs. One can easily see that $Q(W)$ is compact in W^*. We also denote by $Q_1(W)$ the class of *minimal points* q over δR, i.e., those points q in $Q(W)$ for which $k_W(\cdot,q)\in M(W)$. By the Martin theory (cf., e.g., [3], pp. 134—144),

$$M(W) = \{k_W(\cdot,q);\quad q\in Q_1(W)\}, \tag{6}$$

and there exists a bijective correspondence $u\leftrightarrow\nu$ between $HP(W;\partial W)$ and the class of positive Borel measures ν on $Q_1(W)$ such that

$$u = \int_{Q_1(W)} k_W(\cdot,q)\,d\nu(q). \tag{7}$$

As a consequence of (6) we have $\dim_F\delta R\geqq 1$ and $\dim\delta R\geqq 1$. Needless to say, $\dim_F\delta R$ and $\dim\delta R$ are at most $\mathfrak{c}$, the cardinal number of a continuum.

2. The smallest relative harmonic dimension

2.1. By a closed Jordan region on a Riemann surface we mean the closure of a Jordan region on it. One might feel that $\dim_F\delta R$ for compact F, i.e., $\dim\delta R$, never exceeds $\dim_F\delta R$ for any noncompact F. Contrary to this intuition we have the following

Theorem. *For any open Riemann surface R of Heins type, there always exists a $\mathcal{K}$-sequence $\{K_n\}_1^\infty$ of closed Jordan regions K_n on R such that* $\dim_F\delta R=1$ *for $F=\bigcup_1^\infty K_n$.*

The proof will be given in 2.2—2.4. What we need to show is that $HP(W;\partial W)$ is generated by a single nonzero element k in $HP(W;\partial W)$, i.e., $HP(W;\partial W)=\{\alpha k;\,\alpha\in\boldsymbol{R}^+\}$, where $\boldsymbol{R}^+$ is the set of nonnegative numbers in the set $\boldsymbol{R}$ of real numbers. It will be seen from the proof that we do not use the parabolicity of R. Therefore, what we can really assert is the following: *For any open Riemann surface R with a single ideal boundary component δR, there exists a sequence $\{K_n\}_1^\infty$ of disjoint closed Jordan regions K_n converging to δR such that $HP(W;\delta W)$ is generated by a single nonzero element.*

2.2. Since R has a single ideal boundary component, there exists an exhaustion $\{R_n\}_0^\infty$ of R with properties i), ii), and iii) stated in 1.1. We may choose R_0 as a parametric disk. For each $n \geqq 0$, take a regular subregion S_n of R such that $\bar{R}_n \subset S_n \subset \bar{S}_n \subset R_{n+1}$ and $S_n - \bar{R}_n$ is an annulus. For $n \geqq 1$, let φ_n be a conformal mapping of $S_n - \bar{R}_n$ onto the region $\{1 < |t| < a_n\}$ such that ∂S_n and ∂R_n correspond to the circles $\{|t|=1\}$ and $\{|t|=a_n\}$, respectively, under the mapping φ_n extended to $\bar{S}_n - R_n$. For a number δ_n in $(0, \pi)$ to be specified later, the set

$$K_n = \varphi_n^{-1}\{1 \leqq |t| \leqq a_n,\ \delta_n \leqq \arg t \leqq 2\pi - \delta_n\} \quad (n = 1, 2, \ldots)$$

is a closed Jordan region on R. Clearly, $K_n \cap K_m = \emptyset$ $(n \neq m)$, and $\{K_n\}$ converges to δR. For convenience we include $K_0 = \bar{R}_0$ in $\{K_n\}$ and set

$$F = \bigcup_0^\infty K_n, \quad W = R - F.$$

We shall prove that $HP(W; \partial W)$ is generated by a single nonzero element if $\{\delta_n\}$ is properly chosen in $(0, \pi)$.

2.3. For $n \geqq 1$ we consider the arc $I_n = \varphi_n^{-1}\{|t| = a_n,\ -\delta_n \leqq \arg t \leqq \delta_n\}$ on ∂R_n and fix a point ζ_n in I_n with $\varphi_n(\zeta_n) = a_n$ so that ζ_n is the midpoint of I_n. We denote by $g_n(\zeta, z)$ the Green's function on $R_n \cap W$. We use the same symbol ζ for a point of R and its image in a parametric disk. The inner normal derivative $\partial/\partial n_\zeta\, g_n(\zeta, z)$ of $g_n(\zeta, z)$ at ζ on ∂R_n depends on the choice of the parametric disk but, for a fixed point a in $R_1 \cap W$, the ratio

$$h_n(\zeta, z) = \left(\frac{\partial}{\partial n_\zeta} g_n(\zeta, z)\right) \Big/ \left(\frac{\partial}{\partial n_\zeta} g_n(\zeta, a)\right)$$

for (ζ, z) in $(\partial R_n) \times (R_n \cap W)$ does not. Since $h_n(\zeta, z)$ is continuous on $(\partial R_n) \times (R_n \cap W)$, the function $\zeta \to h_n(\zeta, z)$ is uniformly continuous on ∂R_n for each z in $\partial S_0 \subset R_n \cap W$. Hence the function

$$\psi_n(\zeta) = \sup_{z \in \partial S_0} |h_n(\zeta, z) - h_n(\zeta_n, z)|$$

is nonnegative and continuous on ∂R_n, with $\psi_n(\zeta_n) = 0$. Here ψ_n depends on $\delta_1, \ldots, \delta_{n-1}$ but does not depend on δ_n. If we take δ_n sufficiently small in $(0, \pi)$, then $\sup_{\zeta \in I_n} \psi_n(\zeta) < 2^{-n}$. Therefore we can and will choose $\delta_n (n = 1, 2, \ldots)$ successively in $(0, \pi)$ so small that

$$\sup_{\zeta \in I_n} \left(\sup_{z \in \partial S_0} |h_n(\zeta, z) - h_n(\zeta_n, z)| \right) < 2^{-n} \quad (n = 1, 2, \ldots). \tag{8}$$

2.4. We now determine the generator k of $HP(W; \partial W)$. To this end we consider functions $k_n(z) = h_n(\zeta_n, z)$ $(n = 1, 2, \ldots)$ which are in $HP(R_n \cap W; \partial(R_n \cap W) - \{\zeta_n\})$, the class of nonnegative harmonic functions on $R_n \cap W$ with vanishing boundary values on $\partial(R_n \cap W) - \{\zeta_n\}$. We recall that $k_n(a) = 1$. Since $\{k_n\}_1^\infty$ forms a normal family, there exists a subsequence $\{k_{\nu(n)}\}_{n=1}^\infty$ of $\{k_n\}_1^\infty$ such that

$$k(z) = \lim_{n \to \infty} k_{\nu(n)}(z) \tag{9}$$

exists on $\overline{W}$ and the convergence is uniform on each compact subset of $\overline{W}$. Clearly $k \in HP(W; \partial W)$ and $k(a)=1$.

Choose an arbitrary element v in $HP(W; \partial W)$ with $v(a)=1$. The proof will be complete if we can show that $v=k$. Since $\partial(R_n \cap W)$ is piecewise analytic, $*dg_n(\cdot, z)$ exists on $\partial(R_n \cap W)$ except at corner points. But since v vanishes on those components of $\partial(R_n \cap W)$ which contain these corner points, the Poisson-type formula is valid:

$$v(z) = -\frac{1}{2\pi}\int_{\partial(R_n \cap W)} v * dg_n(\cdot, z) \quad (z \in R_n \cap W).$$

Observe that the boundary function $v|\partial(R_n \cap W)$ is nonvanishing on $I_n (\subset \partial R_n)$. Therefore we can rewrite the above formula as

$$v(z) = \int_{I_n} \frac{1}{2\pi}\left(\frac{\partial}{\partial n_\zeta} g_n(\zeta, z)\right) v(\zeta)|d\zeta| \quad (z \in R_n \cap W).$$

Using a positive measure μ_n on I_n defined by

$$d\mu_n(\zeta) = \frac{1}{2\pi}\frac{\partial}{\partial n_\zeta}(g_n(\zeta, a))v(\zeta)\,|d\zeta|,$$

and recalling the definition of $h_n(\zeta, z)$ we obtain

$$v(z) = \int_{I_n} h_n(\zeta, z)\, d\mu_n(\zeta) \quad (z \in R_n \cap W).$$

For $z=a$ this implies that $\mu_n(I_n)=1$. Set $\Omega = S_0 - \overline{R}_0$. If $z \in \Omega$, then (8) gives

$$|v(z) - k_{\nu(n)}(z)| = \left|\int_{I_{\nu(n)}} (h_{\nu(n)}(\zeta, z) - h_{\nu(n)}(\zeta_{\nu(n)}, z))\, d\mu_{\nu(n)}(\zeta)\right|$$

$$\leqq \int_{I_{\nu(n)}} |h_{\nu(n)}(\zeta, z) - h_{\nu(n)}(\zeta_{\nu(n)}, z)|\, d\mu_{\nu(n)}(\zeta) < 2^{-\nu(n)}.$$

We have shown that $\sup_{z \in \Omega} |v(z) - k_{\nu(n)}(z)| \leqq 2^{-\nu(n)}$. On letting $n \to \infty$ and using (9) we see that $v=k$ on Ω and hence on W. □

2.5. Examples of Riemann surfaces R of Heins type with $\dim \delta R > 1$ are not lacking. As already mentioned in the introduction, Heins [4] constructed an R with $\dim \delta R = n$, an arbitrary positive integer, and Kuramochi [5] exhibited an R with $\dim \delta R = \mathfrak{a}$ $(= \# N$, with N the set of positive integers) (see also Segawa [11]). Constantinescu and Cornea [2] even constructed an R with $\dim \delta R = \mathfrak{c}$ $(= \# \boldsymbol{R})$. We shall also construct, in Section 5, an R with $\dim \delta R = \mathfrak{a}$. From Theorem 2.1 we thus conclude that *there exists an open Riemann surface* R *of Heins type and a* $\mathscr{K}$*-sequence* $\{K_n\}_1^\infty$ *on* R *such that for* $F = \bigcup_1^\infty K_n$,

$$\dim_F \delta R < \dim \delta R.$$

3. The finite complex plane

3.1. The classical Picard principle states that the harmonic dimension of the point at infinity, $\infty=\delta C$, of the finite complex plane C: $|z|<\infty$ is one : $\dim \delta C=1$. Let $\{K_n\}_1^\infty$ be a $\mathcal{K}$-sequence of *radial slits* K_n on C and $F=\bigcup_1^\infty K_n$. We can always find an F with $\dim_F \delta C=m$ for any cardinal number $m\geqq 1$ of a countable set or the cardinal number m of a continuum (cf., e.g., [6], [7], [9]). Therefore, the relation $\dim_F \delta R\geqq \dim \delta R$ occurs frequently. For the sake of completeness we append here an example of extreme simplicity (both in the example itself and its proof) of a $\mathcal{K}$-sequence $\{K_n\}_1^\infty$ in C such that $\dim_F C\geqq 2$ for $F=\bigcup_1^\infty K_n$; our example was inspired by Ancona [1]. In particular, we have a proof of the occurrence of the relation

$$\dim_F \partial R > \dim \partial R.$$

3.2. Consider a nondegenerate continuum K in C with the following four properties: (K.1) K is symmetric about the real axis $\operatorname{Im} z=0$, (K.2) K is symmetric about the imaginary axis $\operatorname{Re} z=0$, (K.3) $(K+1)\cap K=\emptyset$, (K.4) $C-K$ is connected. Here $K+c=\{z+c;\, z\in K\}$ for any given $c\in C$. The slit $[-a, a]$ $(0<a<1/2)$, the disk $\{|z|\leqq a\}$ $(0<a<1/2)$, and the rectangle $\{|\operatorname{Re} z|\leqq a, |\operatorname{Im} z|\leqq b\}$ $(0<a<1/2)$, $(b>0)$ are examples of K. Let $Z=\{0, \pm 1, \pm 2, \dots\}$ and

$$K_n = K+n \quad (n\in Z), \quad F=\bigcup_{-\infty}^{\infty} K_n. \tag{10}$$

Theorem. *Let K be a nondegenerate continuum K in C with properties* (K.1)—(K.4), *and let F be given by* (10). *Then* $\dim_F \delta C\geqq 2$.

It can be seen that, in reality, $\dim_F \delta C=2$ but our main interest here is in constructing an F with $\dim_F \delta R>\dim \delta R$ and also in giving a proof as simple and elementary as possible. Hence we only establish $\dim_F \delta C\geqq 2$ and omit the proof for $\dim_F \delta C\leqq 2$, which requires rather elaborate reasoning.

3.3. Our proof of Theorem 3.2 is by contradiction. Suppose $\dim_F \delta C=1$, so that there exists a nonzero function u in $HP(W; \partial W)$ with $W=C-F$ such that $HP(W; \partial W)=R^+u$. Since v defined by $v(z)=2^{-1}(u(z)+u(\bar z))$ for z in W is also a function in $HP(W; \partial W)$ by (K.1), we may assume that $u(\bar z)=u(z)$ for z in W. Similarly, the function v given by $v(z)=2^{-1}(u(z)+u(-\bar z))$ for z in W is again in $HP(W; \partial W)$ by (K.2). Hence we may and will assume that $u(z)=u(\bar z)=u(-\bar z)$ for any z in W.

Let I_n be the line segment contained in $\{\operatorname{Im} z=0\}\cap \overline{W}$ connecting the rightmost point of $\{\operatorname{Im} z=0\}\cap K_n$ to the leftmost point of $\{\operatorname{Im} z=0\}\cap K_{n+1}$ $(n\in Z)$. We maintain that

$$(u|I_k)(z) = (u|I_{k+1})(2(k+1)-z) \tag{11}$$

for $z\in I_k$, i.e., $u|I_k$ and $u|I_{k+1}$ are symmetric about $\{\operatorname{Re} z=k+1\}$. In fact, let

$v\in HP(W;\ \partial W)$ be given by $v(z)=u(z+(k+1))$ for z in W. Then (11) for k follows from $v(z)=v(-\bar{z})$ for z in W. Since $v\in HP(W;\partial W)=\boldsymbol{R}^+u$, there exists a constant c in $\boldsymbol{R}^+$ with $v=cu$. Therefore, $u(z)=u(-\bar{z})$ implies $v(z)=v(-\bar{z})$ for any z in W.

Let $W^+=W\cap\{\mathrm{Im}\, z>0\}$. From (11) it follows that $u|\partial W^+$ is bounded and a fortiori the solution $H_u^{W^+}$ of the Dirichlet problem on W^+ with boundary values $u|\partial W^+$ on ∂W^+ (cf., e.g., [3], p. 21) is bounded on W^+. Note that $H_u^{W^+}$ is continuous on $\overline{W^+}$ with $H_u^{W^+}|\partial W^+=u|\partial W^+$. Define the harmonic function h on W by

$$h(z)=\begin{cases} u(z)-H_u^{W^+}(z) & (z\in\overline{W^+}),\\ -(u(\bar{z})-H_u^{W^+}(\bar{z})) & (z\in W-\overline{W^+}).\end{cases}$$

Since $u(z)=u(\bar{z})$, the function w defined by $w=u+h$ belongs to $HP(W;\partial W)$ and is unbounded (bounded, respectively) on $\overline{W^+}$ ($W-\overline{W^+}$, respectively). Thus u is not symmetric about the real axis, in violation of the fact that w is a constant multiple of u. In the above proof we took it for granted that u is unbounded on W. If this were not the case, the parabolic character of δC would imply $u\equiv 0$. □

4. Identity

4.1. Take a $\mathcal{K}$-sequence $\{K_n\}_1^N$ on R and set $F=\bigcup_1^N K_n$ and $W=R-F$. We now take up the most intriguing case: when is $\dim_F \delta R=\dim \delta R$? If F is compact, the identity holds by definition. This suggests that, for a noncompact F, the identity occurs whenever $\{K_n\}$ is distributed on R sparsely in some sense. We shall give here a condition to assure such sparseness. Let $g_W(z,\zeta)$ be the Green's function on W. A curve γ in R is said to converge to δR, $\gamma\to\delta R$, if $\lim_{t\to 1} p(t)=\delta R$, with $p=p(t)$ $(0\leqq t<1)$ a parametric representation of γ. The curvewise superior limit of $g_W(z,\zeta)$ along γ is, by definition,

$$\limsup_{z\in\gamma, z\to\delta R} g_W(z,\zeta)=\limsup_{t\to 1} g_W(p(t),\zeta),$$

where we define $g_W(\cdot,\zeta)=0$ on $R-W=F$. We say that $\{K_n\}$ is *sparse* on R if the curvewise superior limit of $g_W(\cdot,\zeta)$ along any curve γ in R converging to δR is positive. The condition is clearly independent of ζ in W. If F is compact, then, since R is parabolic, we have

$$\liminf_{z\to\delta R} g_W(z,\zeta)>0$$

and therefore $\{K_n\}_1^N$ $(F=\bigcup_1^N K_n)$ is sparse on R.

Theorem. *If* $\{K_n\}$ *is sparse on* R, *then the relative harmonic dimension of* δR *relative to* $F=\cup K_n$ *coincides with the harmonic dimension of* δR,

$$\dim_F\partial R=\dim\partial R.$$

The proof will be given in 4.2—4.4.

4.2. For $W_1=R-K_1$, we have $\dim \delta R = \# M(W_1)$. Denote by ∂X ($\partial_1 Y$, respectively) the relative boundary of the subset X (Y, respectively) of R (W_1, respectively) relative to R (W_1, respectively). Since $W\subset W_1\subset R$, we can consider both ∂W and $\partial_1 W$, with $\partial_1 W \subsetneqq \partial W$ and, in fact, $\partial W=(\partial_1 W)\cup(\partial K_1)=(\partial_1 W)\cup(\partial W_1)$. We consider the mapping λ_1 of $HP(W_1)$ to $HP(W;\partial_1 W)$ and the mapping μ_1 of $HP(W;\partial_1 W)_{\mu_1}$ to $HP(W_1)$ introduced in 1.2. Here λ_1 defines a mapping $\lambda=\lambda_1|HP(W_1;\partial W_1)$ of $HP(W_1;\partial W_1)$ to $HP(W;\partial W)$ and similarly μ_1 defines a mapping $\mu=\mu_1|HP(W;\partial W)_\mu$ of $HP(W;\partial W)_\mu$ to $HP(W_1;\partial W_1)$, where $HP(W;\partial W)_\mu = HP(W;\partial W)\cap HP(W;\partial_1 W)_{\mu_1}$. The properties corresponding to (a), (b), (c), and (d) in 1.2 are readily verified to hold for the present λ and μ. We shall refer to these properties again as (a), (b), (c), and (d).

4.3. Let u be minimal in $HP(W_1;\partial W_1)$. We normalize u by $\beta u(a)=1$, i.e., $\beta u\in M(W_1)$. Then $\beta u=k_{W_1}(\cdot,p)$ for some p in $Q_1(W_1)$. The Brelot theorem (cf., e.g., [3], p. 139) states that any minimal point p in $W_1^*-W_1$ is accessible from W_1 in the topology of W_1^*, so that for $p\in Q_1(W_1)$ there exists a curve γ in W_1 converging to δR, and to p in W_1^*. Since $\{K_n\}$ is sparse on R, there exists a sequence $\{\zeta_n\}\subset\gamma$ such that $\zeta_n\to\delta R$ and also $\zeta_n\to p$ in W_1^* and $\lim_{n\to\infty} g_W(\zeta_n,z)=v(z)>0$ for every $z\in W_1$. We can assume, moreover, that $\lim_{n\to\infty} g_W(\zeta_n,z)$ exists for z in R. Since $g_{W_1}(\zeta_n,z)\geqq g_W(\zeta_n,z)$, $\alpha=\lim_{n\to\infty} g_{W_1}(\zeta_n,a)>0$, and

$$\beta u(z) = k_{W_1}(z,p) = \lim_{n\to\infty}\left(g_{W_1}(\zeta_n,z)/g_{W_1}(\zeta_n,a)\right)$$

$$\geqq \frac{1}{\alpha}\lim_{n\to\infty} g_W(\zeta_n,z) = \frac{1}{\alpha}v(z)>0.$$

We conclude that $\lambda u\geqq(\alpha\beta)^{-1}v>0$ because $v\in HP(W;\partial W)$, and, by (d), λu is minimal in $HP(W;\partial W)$. Since $\lambda u\leqq u$ for $u\in HP(W_1;\partial W_1)$, we have $\lambda u\in HP(W;\partial W)_\mu$ and $\mu\lambda u\leqq u$. There exists a positive constant c with $\mu\lambda u=cu$, because u is minimal in $HP(W_1;\partial W_1)$. Thus $c\lambda u=\lambda(cu)=\lambda(\mu\lambda u)=(\lambda\mu)(\lambda u)=\lambda u$ by (a). Since λu is minimal and, in particular, $\lambda u>0$ on W, we have $c=1$. A fortiori $\mu\lambda u=u$ for minimal u in $HP(W_1;\partial W_1)$. Suppose u_1 and u_2 are minimal in $HP(W_1;\partial W_1)$, and $\lambda u_1=\lambda u_2$. Then $u_1=\mu\lambda u_1=\mu\lambda u_2=u_2$. Therefore, we can define an injective mapping of $M(W_1)$ to $M(W)$ and infer that $\dim\delta R\leqq\dim_F \delta R$.

4.4. Conversely, let v be minimal in $HP(W;\partial W)$. We again normalize v by $\beta v(a)=1$ for a reference point a in W. Then $\beta v=k_W(\cdot,q)$ for some $q\in Q_1(W)$. Again by the Brelot theorem there exists a curve γ in W converging to δR such that ζ_n in γ converges to δR and also to q in W^*, and $\lim_{n\to\infty} g_W(\zeta_n,z)=w(z)>0$ for every z in R; this is possible since $\{K_n\}$ is sparse on R. Hence

$$\beta v = k_W(\cdot,q) = \lim_{n\to\infty}\frac{g_W(\zeta_n,\cdot)}{g_W(\zeta_n,a)} = \frac{w}{w(a)}$$

and $w=\beta w(a)v$ is also minimal in $HP(W; \partial W)$. We may assume, moreover, that $\lim_{n\to\infty} g_{W_1}(\zeta_n, z)=u(z)$ exists for every $z\in W_1$ by choosing a subsequence if necessary. Since $g_W(\zeta_n, z)\leqq g_{W_1}(\zeta_n, z)$, we obtain $w\leqq u$ on passing to the limit. Thus $w\in HP(W; \partial W)_\mu$. The function μw is minimal in $HP(W_1; \partial W_1)$ by (c). Since μ is injective, we can define an injective mapping of $M(W)$ to $M(W_1)$, and obtain $\dim_F \delta R \leqq \dim \delta R$. In view of the result in 4.3, we have shown that $\dim_F \delta R=\dim \delta R$. □

5. Countably infinite harmonic dimension

5.1. As an application of the identity theorem established in 4.1 we shall give a new proof of the following theorem originally obtained by Kuramochi [5] (see also Segawa [11]):

Theorem. *There exists an open Riemann surface* R *of Heins type such that* $\dim \delta R=\mathfrak{a}$, *the cardinal number of a countably infinite set.*

The surface R we are going to construct will be an infinitely sheeted unlimited covering surface of the punctured sphere $\hat{C}_0: 0<|z|\leqq\infty$ whose projections of branch points are all in the punctured disk $\Delta_0: 0<|z|<1$. From each sheet of R we remove a disk $1\leqq|z|\leqq\infty$ and obtain $F=\bigcup_1^\infty K_n$, where the K_n are duplicates of $1\leqq|z|\leqq\infty$ lying in each sheet of R. By a judicious choice of the branch points of R we can see to it that $\{K_n\}$ is sparse on R, and $\dim_F \delta R=\mathfrak{a}$. Then we apply Theorem 4.1 to conclude that $\dim \delta R=\dim_F \delta R=\mathfrak{a}$. This is a rough sketch of the construction and reasoning we are going to develop in 5.2—5.7.

5.2. Let $\{a_n\}_1^\infty$ be a strictly decreasing zero sequence in (0,1), and $\{\theta_m\}_1^\infty$ a strictly increasing sequence in $(-\pi/2, \pi/2)$. We then choose a decreasing zero sequence $\{d_n\}$ of positive numbers d_n as follows. Let $D_{nm}=\{|z-a_n e^{i\theta_m}|\leqq d_n\}$. We make $\{d_n\}$ converge to zero so rapidly that any two closed disks in the family $\bigcup_{m=1}^\infty \{D_{nm};\, n\geqq m\}$ are disjoint. We set

$$\begin{cases} D(m) = \bigcup\limits_{n\geqq m} D_{nm} \quad (m = 1, 2, \ldots), \\ D(0) = \bigcup\limits_{m=1}^{\infty} D(m) = \bigcup\limits_{m=1}^{\infty} (\bigcup\limits_{n\geqq m} D_{nm}). \end{cases}$$

We fix sequences $\{a_n\}$, $\{\theta_m\}$, and an auxiliary sequence $\{d_n\}$ once and for all. We then choose a strictly decreasing zero sequence $\{b_n\}_1^\infty$ in (0, 1) such that $a_{n+1}<b_n<a_n$ $(n=1, 2, \ldots)$. Let $I_{nm}=\{b_n\leqq|z|\leqq a_n,\ \arg z=\theta_m\}$, a radial line segment. First of all we require that each I_{nm} is contained in the interior of $D_{nm}(n\geqq m)$. Set

$$\begin{cases} I(m) = \bigcup\limits_{n\geqq m} I_{nm} \quad (m = 1, 2, \ldots), \\ I(0) = \bigcup\limits_{m=1}^{\infty} I(m) = \bigcup\limits_{m=1}^{\infty} (\bigcup\limits_{n\geqq m} I_{nm}). \end{cases}$$

Actually we will choose each b_n so close to a_n that it satisfies not only the above requirement but also the conditions (A) and (B) to be specified later.

5.3. Using a countably infinite number of duplicates of the punctured sphere $\hat{C}_0: 0<|z|\leqq\infty$, and the slits $I(0)$ and $I(m)$, we form the disjoint sheets

$$R_m = \hat{C}_0 - I(m) \quad (m = 0, 1, \ldots).$$

Then we join each R_m $(m=1, 2, \ldots)$ to R_0 crosswise along the slits $I(m)$ and denote the resulting surface by R. It is a covering surface of $\hat{C}_0$ with the natural projection mapping π. It is not difficult to see that R *is an open Riemann surface of Heins type.*

In each R_m $(m=0, 1, \ldots)$, take the closed disk $K_m=\{1\leqq|z|\leqq\infty\}$. Clearly $\{K_m\}_0^\infty$ is a $\mathscr{K}$-sequence on R. We set

$$F = \bigcup_0^\infty K_m, \quad W = R - F,$$

and

$$W_m = R_m - K_m \quad (m = 0, 1, \ldots).$$

Then W is also obtained by joining W_m to W_0 crosswise along the slits $I(m)$ $(m=1, 2, \ldots)$. Thus it is a covering surface of the punctured disk $\Delta_0: 0<|z|<1$ with the natural projection π.

5.4. Fix a number c in $(a_1+d_1, 1)$. The circle $C_m=\{|z|=c\}$ is contained in W_m and so is the annulus $\{c<|z|<1\}$ $(m=0, 1, \ldots)$. Let w be the harmonic function on $\{0<|z|<c\}-I(0)$ with boundary values 1 on $|z|=c$ and 0 on $I(0)$. We now choose each b_n so close to a_n that the following condition is satisfied:

(A) $$\eta_A = \inf\{w(z);\ z\in\{0<|z|<c\}-D(0)\} > 0.$$

As a consequence of this choice of $\{b_n\}_1^\infty$, *the $\mathscr{K}$-sequence $\{K_m\}_0^\infty$ in R is sparse on R.*

To prove this we set

$$G_m = \{0<|z|<c\} - I(m) \subset W_m \subset R_m \quad (m = 0, 1, \ldots).$$

Then w can be considered subharmonic on each G_m $(m=0, 1, \ldots)$ by defining $w=0$ on $I(0)$. Let γ be an arbitrary curve in R tending to δR, and denote by $g_W(\cdot, \zeta)$ the Green's function on W with pole ζ in W and extended as zero to $R-W$. We are to show that

$$\limsup_{z\in\gamma, z\to\delta R} g_W(z, \zeta) > 0$$

for one and hence for every ζ in W. Observe that $\pi(\gamma)$ is a curve in $\hat{C}_0$ tending to the origin 0.

First we consider the case in which there exists a single G_m such that $\gamma\subset G_m$. Choose a sequence $\{z_n\}$ in $\gamma\cap(G_m-D(m))$ such that $z_n\to\delta R$. Let $\alpha=\inf_{C_m} g_W(\cdot, \zeta)>0$. Then clearly $g_W(\cdot, \zeta)\geqq\alpha w$ on G_m. In view of (A) we have

$$\limsup_{z\in\gamma, z\to\delta R} g_W(z, \zeta) \geqq \limsup_{n\to\infty} g_W(z_n, \zeta) \geqq \alpha \limsup_{n\to\infty} w(z_n) \geqq \alpha\eta_A > 0.$$

Next we consider the case in which the above alternative does not occur, so that there exists a sequence $\{z_n\}$ in $\gamma\cap(G_{m(n)}-D(m(n))$ such that $z_n\to\delta R$ and $m(n)\neq m(n')$ $(n\neq n')$. Let γ_n be that part of γ which starts from z_n and ends at z_{n+1}. In view of the construction of R, γ_n must pass through $G_0-D(0)$ and, therefore, we can choose a point w_n in $\gamma_n\cap(G_0-D(0))$. Then $w_n\to\delta R$, and in the same fashion as above we conclude that

$$\limsup_{z\in\gamma,\, z\to\delta R} g_W(z,\zeta)>0. \qquad \Box$$

5.5. It is readily seen that there exists on $0<|z|\leqq 1$ a unique smallest function l in the family of continuous functions v on $0<|z|\leqq 1$ which are harmonic on $\{0<|z|<1\}-I(0)$ and satisfy $v(z)=0$ on $|z|=1$, and $v(z)=\log(2/|z|)$ on $I(0)$. We now impose upon the closeness of b_n to a_n the additional condition

$$\text{(B)} \qquad \eta_B=\sup\{l(z);\ z\in(-1,0)\}<+\infty.$$

The function l may be viewed as being defined and superharmonic on each W_m $(m=0, 1, \ldots)$.

Denote by L_m the segment $(-1, 0)$ in W_m $(m=0, 1, \ldots)$. Fix an m for the time being and choose a sequence $\{-t_n\}_1^\infty\subset L_m$ such that $-t_n\to 0$ and $g_W(z, -t_n)$ is convergent for each z in W. On setting $\alpha=\inf_{C_m} g_W(z,\cdot)$, we see that $g_W(z,-t_n)\geqq \alpha w(-t_n)\geqq\alpha\eta_A>0$. Therefore,

$$u_m(z)=\lim_{n\to\infty} g_W(z,-t_n)>0$$

for $z\in W$, and $u_m\in HP(W;\partial W)$.

We now study the growth of u_m. Let

$$h_\zeta(z)=\log\left|\frac{1-\bar\zeta z}{z-\zeta}\right| \qquad (|z|, |\zeta|<1).$$

We can lift $h_\cdot(\cdot)$ to $W\times W$ from $\Delta_0\times\Delta_0$ by

$$h_\zeta(z)=h_{\pi(\zeta)}(\pi(z)) \quad ((z,\gamma)\in W\times W).$$

The discussion in what follows will be based on the inequality

$$\text{(12)} \qquad g_W(\cdot,\zeta)\leqq h_\zeta \quad (=h_{\pi(\zeta)}\circ\pi)$$

on W for any ζ in W. If $z\in I(m')$ $(m'=0, 1, \ldots)$, then $\operatorname{Re} z>0$ and

$$g_W(z,-t_n)\leqq h_{-t_n}(z)=\log\left|\frac{1+t_n z}{z+t_n}\right|\leqq \log\frac{2}{|z|}=l(z).$$

By the maximum principle,

$$g_W(z,-t_n)\leqq l(z) \quad (z\in W_{m'},\ m'\neq m).$$

Observe that $h_{-t_n}(z)-l(z)\leqq 0$ on $I(m)$. Therefore,

$$g_W(z,-t_n)\geqq h_{-t_n}(z)-l(z) \quad (z\in W_m),$$

since the same is true of the boundary values on ∂W_m. On passing to the limit we conclude that

$$u_m(z) \leqq l(z) \quad (z \in W_{m'}, m' \neq m),$$
$$u_m(z) \geqq h_0(z) - l(z) \quad (z \in W_m).$$

Here $h_0(z) = \log(1/|z|)$. Therefore, by (B), we have

$$\text{(13)} \qquad \sup_{L_{m'}} u_m < +\infty \quad (m' \neq m), \qquad \sup_{L_m} u_m = +\infty.$$

With each m we associate a u_m as above. By using the Martin representation (7) for each function in $\{u_m\}_1^\infty$, we can easily see from (13) that $\# M(W) \geqq \mathfrak{a}$, i.e., $\dim_F \delta R \geqq \mathfrak{a}$.

5.6. Take an arbitrary u in $M(W)$, so that $u = k_W(\cdot, q)$ for some $q \in Q_1(W)$. By the Brelot theorem there exists a curve γ in W tending to δR and to q in W^*. Since $\{K_n\}$ is sparse on R, there exists a sequence $\{\zeta_n\}$ in γ tending to δR such that $\lim_{n\to\infty} g_W(\zeta_n, z)$ exists and is positive for any $z \in W$. In view of

$$\lim_{n\to\infty} \frac{g_W(\zeta_n, z)}{g_W(\zeta_n, a)} = \lim_{n\to\infty} k_W(z, \zeta_n) = k(z, q) = u(z),$$

we set

$$\beta = \lim_{n\to\infty} g_W(\zeta_n, a) > 0,$$

and obtain $\lim_{n\to\infty} g_W(z, \zeta_n) = \beta u(z)$ for any $z \in W$. By (12), $\beta u \leqq h_0$ on W. Set

$$\beta_u = \sup\{\beta;\ \beta u \leqq h_0 \quad \text{on} \quad W\}.$$

We have obtained a mapping $u \to \beta_u u = v$ from $M(W)$ onto $M'(W) = \{\beta_u u;\ u \in M(W)\}$, which is bijective. Thus $\# M(W) = \# M'(W)$.

Set

$$M_k'(W) = \left\{v \in M(W);\ v(a) \geqq \frac{1}{k} h_0(a)\right\} \quad (k = 1, 2, \ldots).$$

Take different elements $v_1, \ldots, v_n$ in $M_k'(W)$. By the Kjellberg lemma (cf., e. g., [3], p. 18), the relations $v_j \leqq h_0$ $(j = 1, \ldots, n)$ imply that $v_1 + \ldots + v_n \leqq h_0$. Considering this at a we see that

$$\frac{n}{k} h_0(a) \leqq v_1(a) + \ldots + v_n(a) \leqq h_0(a),$$

or $n/k \leqq 1$. Therefore, $n \leqq k$ and $\# M_k'(W) \leqq k$. Since $M'(W) = \bigcup_{k=1}^\infty M_k'(W)$, we obtain $\# M'(W) \leqq \mathfrak{a}$, i.e., $\dim_F \delta R \leqq \mathfrak{a}$.

5.7. From 5.5 and 5.6 it follows that $\dim_F \delta R = \mathfrak{a}$. Since $\{K_n\}$ is sparse on R, Theorem 4.1 implies that $\dim \delta R = \dim_F \delta R = \mathfrak{a}$. The proof of Theorem 5.1 is complete.

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