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A CRITERION FOR THE NORMALITY OF A FAMILY OF MEROMORPHIC FUNCTIONS

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Let D be a domain in the complex plane. A family $\mathcal{F}$ of meromorphic functions on D is said to be normal if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to function f which is meromorphic or identically ∞, the convergence being with respect to the spherical metric $d\sigma=|dw|/(1+|w|^2)$. A well known criterion of Marty (cf. [1], p. 226) asserts that $\mathcal{F}$ is normal if, and only if, for each compact subset $K\subset D$ there is constant C_K such that

$$|f'(z)| \leqq C_K(1+|f(z)|^2)$$

for all $f\in\mathcal{F}$ and all $z\in K$.

Although this criterion is necessary and sufficient, it does not tell whether the family of functions in D satisfying, for example,

$$|f'| \leqq e^{|f|}$$

is normal or not. The purpose of the present note is to prove the following strengthening of the sufficiency part of Marty's criterion:

Theorem. *Let $\mathcal{F}$ be a family of meromorphic functions on D with the property that for each compact set $K\subset D$ there is a monotone increasing function h_K such that*

$$|f'(z)| \leqq h_K(|f(z)|)$$

for all $f\in\mathcal{F}$ and all $z\in K$. Then $\mathcal{F}$ is normal.

Proof. Since the property of being normal is a local one, it suffices to consider the case when $|f'(z)|\leqq h(|f(z)|)$ in a domain Δ. Since this inequality holds a fortiori for any larger h, we may assume that h is continuous and that $h(t)>1+t^2$. For a differentiable curve γ on the Riemann sphere $\hat{C}$ we define the length of γ by

$$l(\gamma) = \int_\gamma \frac{|dw|}{h(|w|)}.$$

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Then $l(\gamma)$ is less than the spherical length of γ. We define a new metric ϱ on $\boldsymbol{C}$ by setting

$$\varrho(w_1, w_2) = \inf \{l(\gamma)\colon \gamma \text{ connecting } w_1 \text{ to } w_2\}.$$

Since $h(|w|)$ is continuous and positive in $\boldsymbol{C}$, ϱ is a metric, i.e. $\varrho(w_1, w_2)=0$ implies $w_1=w_2$. It is smaller than the spherical metric σ and hence uniformly equivalent to σ because of the compactness of $\hat{\boldsymbol{C}}$ in σ.

Each $f\in\mathcal{F}$ now satisfies

$$\varrho(f(z_1), f(z_2)) \leqq |z_1-z_2|,$$

and so the family $\mathcal{F}$ is uniformly equicontinuous. By the Arzelà selection theorem (cf. [1], p. 222) each sequence from $\mathcal{F}$ contains a subsequence which converges uniformly (on compacta) in the metric ϱ to a continuous map f of Δ into $\boldsymbol{C}$. Since ϱ and the spherical metric are uniformly equivalent, the convergence is uniform (on compacta) in the spherical metric. Thus f is meromorphic (or $\equiv\infty$). This establishes the theorem.

References

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