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BOUNDARY OF A HOMOGENEOUS JORDAN DOMAIN

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A Jordan domain D in the extended complex plane $\bar{C}$ is called a quasidisk if D is the image of an open unit disk under a quasiconformal (abbreviated qc) mapping of $\bar{C}$ onto itself. For the basic properties of qc mappings we refer to [3]. Quasidisks can be characterized in several different ways which illustrate the multitude of the interesting properties of these domains [2].

One of the characteristic properties is homogeneity. Let $\mathscr{F}(K)$ be the family of all K-qc mappings $f\colon \bar{C}\to\bar{C}$ with $1\leqq K<\infty$. We say that a set $E\subset\bar{C}$ is homogeneous with respect to $\mathscr{F}(K)$ if for each $z_1, z_2\in E$ there is an $f\in\mathscr{F}(K)$ with $g(E)=E$ and $f(z_1)=z_2$. Erkama showed in [1] that a domain D is a quasidisk if and only if the boundary ∂D of D is a Jordan curve which is homogeneous with respect to $\mathscr{F}(K)$ for some K. This result raised the question if ∂D can be replaced by D in this characterization. In the present paper we answer the question affirmatively and prove the following result.

Theorem. *A Jordan domain D is a quasidisk if and only if D is homogeneous with respect to $\mathscr{F}(K)$ for some K, $1\leqq K<\infty$.*

Especially, this result shows that the boundary of a homogeneous Jordan domain is a quasicircle, i.e., the image of the unit circle under a qc mapping of $\bar{c}$ onto itself.

To prove the theorem we note that a quasidisk is always homogeneous with respect to $\mathscr{F}(K)$ for some K because a disk is homogeneous with respect to Möbius transformations of $\bar{C}$. It remains to prove the sufficiency. By an auxiliary Möbius transformation we may suppose that D lies in the finite complex plane C. Suppose that D is homogeneous with respect to $\mathscr{F}(K)$ with $1\leqq K<\infty$ but D is not a quasidisk. Then its boundary cannot be a quasicircle and cannot have the three point property, i.e., we can find a sequence (u_i, v_i, w_i), $i=1, 2, \ldots,$ of triples of distinct points $u_i, v_i, w_i\in\partial D$ such that

(i) if J_i and J_i' are the components of $\partial D\setminus\{u_i, v_i\}$ and $\operatorname{diam}(J_i)\leqq\operatorname{diam}(J_i')$, then $w_i\in J_i$ and

(ii) $$\lim_{i\to\infty}\frac{|w_i-v_i|}{|u_i-v_i|}=\infty.$$

In the rest of the paper we show that this situation leads to a contradiction.

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We may suppose that each open segment of line (u_i, v_i) is contained either in D or in $\operatorname{int}(C\setminus D)$, the interior of $C\setminus D$. To see this, suppose that $|v_i-w_i|/|v_i-u_i|>2$. Let $z_i\in J_i'$ with $2|v_i-z_i|\geqq|v_i-w_i|>2|v_i-u_i|=4r_i$. We denote by $B(z, r)$ the open disk with center at $z\in C$ and radius $r>0$. The closure of a set $E\subset\bar{C}$ is denoted by $\bar{E}$. Next we set $\bar{B}_i=\bar{B}((1/2)(u_i+v_i), r_i)$. Let I_i and I_i' be the components of $\partial D\setminus\{w_i, z_i\}$. Let $a_i\in I_i$, $b_i\in I_i'$ be two points such that

$$|a_i-b_i| = \operatorname{dist}(\bar{I}_i\cap\bar{B}_i, \bar{I}_i'\cap\bar{B}_i) > 0,$$

where we have denoted the distance between two sets $A, B\subset C$ by $\operatorname{dist}(A, B)=\inf\{|z_1-z_2|: z_1\in A, z_2\in B\}$. Then (a_i, b_i) lies entirely in D or $\operatorname{int}(C\setminus D)$, w_i and z_i are in different components of $\partial D\setminus\{a_i, b_i\}$ and $|z_i-b_i|/|a_i-b_i|$ and $|w_i-b_i|/|a_i-b_i|$ tend to ∞ as i tends to ∞. Finally, replace u_i and v_i by a_i and b_i (and replace w_i by z_i if necessary).

Note also that if a_i and b_i are as above and H_i and H_i' are the two open half disks of $B((1/2)(a_i+b_i), (1/2)|a_i-b_i|)$ with (a_i, b_i) as a common boundary, then one of them, say H_i, lies in $\bar{B}_i$, and, therefore, it lies in D or $C\setminus\bar{D}$ because it does not meet $\bar{I}_i\cup\bar{I}_i'=\partial D$. Recall that after the above replacement $u_i=a_i$ and $v_i=b_i$. By passing to a subsequence, if necessary, we can divide the proof into the following two cases:

Case A: Every $(u_i, v_i)\subset D$ and also the half disk $H_i\subset D$.

Case B: Every $(u_i, v_i)\subset\operatorname{int}(C\setminus D)$.

We first treat Case A. Fix $a\in D$. Let again $z_i\in J_i'$ with $2|z_i-v_i|\geqq|w_i-v_i|$. Let $y_i'=(1/2)(u_i+v_i)$ and let y_i be the middle point of the line segment which joins y_i' and the middle point of the circular arc in the boundary of H_i. Let $f_i\in\mathcal{F}(K)$ such that $f_i(a)=y_i$ and $f_iD=D$. Let $L_i\colon\bar{C}\to\bar{C}$ be an affine conformal mapping such that $L_iH_i=\{z\in C\colon |z|<1, \operatorname{Re} z<0\}=H$ and $L_i(u_i)=-e_2$, $L_i(v_i)=e_2=(0, 1)$, where H is the open half disk $\{z\in C\colon |z|<1, \operatorname{Re} z<0\}$. Let $g_i=L_i\circ f_i$, $i=1, 2, \ldots$. Then $b=g_i(a)=(-1/2, 0)$, $i=1, 2, \ldots$. Note also that $|L_i(w_i),|$ $|L_i(z_i)|$ tend to infinity as i tends to infinity because $|w_i-v_i|/|u_i-v_i|=|L_i(w_i)-e_2|/2$ tends to infinity as i tends to infinity.

The family $\{g_i|D\colon i=1, 2, \ldots\}$ is normal because every $g_i|D$ omits points e_2, $-e_2$ and ∞. Then we may suppose that $\lim_{i\to\infty} g_i=g\colon D\to\bar{C}$ locally uniformly in D. Because $H\subset g_iD$ and $g_i(a)=b\in H$ for all i, g is not constant, and therefore, g is a K-qc mapping from D onto $D'\supset H$. Because the sequence $\{g_i\}$ converges locally uniformly to g in D and g is injective in D, then $\{g_i\}$ is a normal family in $\bar{C}$, and we may assume that $\{g_i\}$ converges to a K-qc mapping $h\colon\bar{C}\to\bar{C}$ uniformly in $\bar{C}$. Then $h|D=g$, and we extend g to $\bar{C}$ by setting $g=h$. We may also suppose that

$$\lim_{i\to\infty} f_i^{-1}(u_i) = \tilde{u}\in\partial D,\quad \lim_{i\to\infty} f_i^{-1}(v_i) = \tilde{v}\in\partial D,$$

$$\lim_{i\to\infty} f_i^{-1}(w_i) = \tilde{w}\in\partial D,\quad \lim_{i\to\infty} f_i^{-1}(z_i) = \tilde{z}\in\partial D.$$

Then by the construction of $g, g(\tilde{u})=-e_2$, $g(\tilde{v})=e_2$ and $g(\tilde{w})=\infty=g(\tilde{z})$, and therefore, $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are distinct, but $\tilde{z}=\tilde{w}$.

On the other hand, let I_i and I_i' be the components of $\partial D\setminus\{f_i^{-1}(u_i), f_i^{-1}(v_i)\}$ such that $f_i^{-1}(w_i)\in I_i$ and $f_i^{-1}(z_i)\in I_i'$. Let I and I' be the components of $\partial D\setminus\{\tilde{u}, \tilde{v}\}$ labelled so that I_i tends to I and I_i' tends to I' as i tends to ∞. Then $\tilde{w}\in I$ (because $\tilde{w}\neq\tilde{u}, \tilde{v}$) and $\tilde{z}\in\bar{I}'$. But $I\cap\bar{I}'=\emptyset$ and thus $\tilde{z}\neq\tilde{w}$, which is a contradiction.

Next we treat Case B. Here every $(u_i, v_i)\subset C\setminus\bar{D}$. We may assume that $|w_i-v_i|=\sup\{|w-v_i|: w\in\bar{J}_i\}>|u_i-v_i|$ for all i. Then it is not difficult to see that for every i there is $r>0$ such that

$$B(y_i, r|w_i-v_i|)\subset D \quad \text{with} \quad y_i = w_i+r(w_i-v_i). \tag{1}$$

Next we observe that, by passing to a subsequence, we may assume that (1) is true for all i and for a fixed $r>0$. Namely, if this is not the case, we have a decreasing sequence r_i, $i\geqq i_0$, with $\lim_{i\to\infty} r_i=0$ and

$$B_i = B(y_i, r_i|w_i-v_i|)\subset D \quad \text{and} \quad w_i'\in(\partial D\setminus\{w_i\})\cap\bar{B}_i$$

for all $i\geqq i_0$. Here $w_i'\in J_i'$ because w_i is the furthest point on J_i from v_i. But now the points w_i', w_i, v_i, $i\geqq i_0$, form triples which reduce the situation to Case A because $(w_i', w_i)\subset D$ and if I_i and I_i' are the components of $\partial D\setminus\{w_i', w_i\}$, then

$$\frac{\min\{\operatorname{diam}(I_i), \operatorname{diam}(I_i')\}}{|w_i'-w_i|} \geqq \frac{\min\{|u_i-w_i|, |v_i-w_i|\}}{|w_i'-w_i|}$$
$$\geqq \frac{|v_i-w_i|-|u_i-v_i|}{2r_i|v_i-w_i|}\to\infty \quad \text{as} \quad i\to\infty.$$

Therefore, we may assume that (1) is true for a fixed $r>0$ and all $i\geqq 1$.

Fix $a\in D$. Let $f_i\in\mathcal{F}(K)$ such that $f_i(a)=y_i$ and $f_iD=D$, $i=1, 2, \ldots$, where y_i is as in (1). Let L_i be an affine conformal mapping such that $L_iB(y_i, r|w_i-v_i|)=B(0, 1)$ and $L_i(w_i)=e_2=(0, 1)$. Then $L_i(v_i)=(1+1/r)e_2$. Let $g_i=L_i\circ f_i$, $i=1, 2, \ldots$. The family $\{g_i|D\}$ is a normal family because every $g_i|D$ omits points e_2, $(1+1/r)e_2$ and ∞. Therefore, we may assume that the sequence $\{g_i\}$ converges to a mapping $g: D\to C$ locally uniformly in D. Here g cannot be constant because $B(0, 1)\subset g_iD$ and $0=g_i(a)$ for all $i\geqq 1$. Therefore, $g: D\to D'$ is K-qc, and we can extend it K-quasiconformally to $\bar{C}$ as in Case A, and we may assume that $\lim_{i\to\infty} g_i=g$ uniformly in $\bar{C}$.

Let z_i be the last point where the ray $\{w_i+t(w_i-v_i): t\geqq 0\}$ meets ∂D. Then $z_i\in J_i'$. We may suppose that

$$\lim_{i\to\infty} f_i^{-1}(u_i) = \tilde{u}\in\partial D, \quad \lim_{i\to\infty} f_i^{-1}(v_i) = \tilde{v}\in\partial D,$$
$$\lim_{i\to\infty} f_i^{-1}(w_i) = \tilde{w}\in\partial D, \quad \lim_{i\to\infty} f_i^{-1}(z_i) = \tilde{z}\in\partial D.$$

Because $|g_i(f_i^{-1}(u_i))-g_i(f_i^{-1}(v_i))|=|L_i(u_i)-L_i(v_i)|\to 0$ as $i\to\infty$, we have $g(\tilde{u})=g(\tilde{v})$, and thus $\tilde{u}=\tilde{v}$. Furthermore, $g_i(f_i^{-1}(v_i))=(1+1/r)e_2$, $g_i(f_i^{-1}(w_i))=e_2$

and $g_i(f_i^{-1}(z_i))=s_i e_2$ with $s_i<0$ for all $i\geqq 1$. Thus $g(\tilde{v})$, $g(\tilde{w})$ and $g(\tilde{z})$ are distinct points, and so are $\tilde{v}$, $\tilde{w}$ and $\tilde{z}$.

Finally, let I_i and I_i' be the components of $\partial D\setminus\{w_i, z_i\}$ such that $u_i\in I_i$ and $v_i\in I_i'$, and let I and I' be the components of $\partial D\setminus\{\tilde{w}, \tilde{z}\}$ such that $f_i^{-1}I_i\to I$ and $f_i^{-1}I_i'\to I'$ as $i\to\infty$. Then $f_i^{-1}(u_i)\in f_i^{-1}I_i$ and $f_i^{-1}I_i\to I$ as $i\to\infty$. This implies that $\tilde{u}\in\bar{I}$. Further, $f_i^{-1}(v_i)\in f_i^{-1}I_i'$ and $f_i^{-1}I_i'\to I'$ as $i\to\infty$. This implies that $\tilde{v}\in\bar{I}'$. Because $\tilde{v}\neq\tilde{w}, \tilde{z}$, we have $\tilde{v}\in I'$. Since $\bar{I}\cap I'=\emptyset$, we have $\tilde{u}\neq\tilde{v}$, which is a contradiction. The theorem is proved.

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