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CONFORMAL REFLECTIONS AND MEROMORPHIC SLIT MAPPINGS

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1. In an earlier paper [3] I showed that if f is a rational function in $\hat{C}=C\cup\{\infty\}$ and if f maps $\Delta=\{z\in C\colon |z|<1\}$ injectively onto $\hat{C}-E$ where E is a continuum in $\hat{C}$ with void interior, then E is a circular arc or a line segment in $\hat{C}$, f is of degree 2 and has the form

(2)
$$f(z)=\varphi\left(\left(\frac{z-a}{z-b}\right)^2\right)$$

where a and b are distinct points on $\partial\Delta$ and φ is a Möbius transformation with $\varphi(0)=f(a)$ and $\varphi(\infty)=f(b)$.

As an answer to a question raised by Y. Domar about the possibility of extending this result to meromorphic functions in C, we now prove the following

3. Theorem. *Let f be a meromorphic, single-valued function in $\hat{C}-F$, where F is a compact set in $\hat{C}-\bar{\Delta}$ of zero linear measure. If $f_0=f|\Delta$ maps Δ injectively onto a dense set in C, then*

(i) *$E=\hat{C}-f_0(\Delta)$ is a circular arc or a line segment in $\hat{C}$.*

(ii) *The function f is the restriction of a rational function of degree* 2 *and has the form* (2) *with distinct points, $a, b\in\partial\Delta$.*

The proof of the theorem is based on a lemma about conformal reflections; see Sections 5 and 6 below.

4. An immediate corollary of the theorem applies to slit functions in the class S of all univalent functions $f(z)=z+a_2z^2+\ldots$ in Δ. A function $f\in S$ is called a *slit function* if $\Gamma=\hat{C}-f(\Delta)$ is a Jordan arc in $\hat{C}$ with a tip at ∞. In the following sections $\Lambda_1(F)$ denotes the linear measure of F.

Corollary. *Let f be a slit function in S. If f has a single valued meromorphic continuation on $\hat{C}-F$ where F is a compact set in $\hat{C}-\bar{\Delta}$ with $\Lambda_1(F)=0$, then the slit of f is straight and $f=\varphi\circ k\circ\psi$ where $k(z)=z(1-z)^{-2}$ is the Koebe function and φ and ψ are Möbius transformations.*

5. *Conformal reflections and a counterexample.* The assumption in the theorem and in the corollary that f or its continuation is single-valued is indispensible as can

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be seen from examples which we will construct here by using the *generalized reflection principle*. According to this principle, cf. [2, p. 187], a continuous function $f\colon \bar{\Delta}\to\hat{C}$, which is meromorphic in Δ can be continued meromorphically across every point of $\partial\Delta$, provided that there is a conformal reflection with respect to $f(\partial\Delta)$. A function φ is said to be a *conformal reflection* with respect to an infinite closed set E in $\hat{C}$, if φ is antimeromorphic in some domain containing E and $\varphi(z)=z$ for all $z\in E$. Obviously, not every set admits a conformal reflection and if an infinite closed set E has a conformal reflection, then the conformal reflection is unique. One can show by considering $\varphi\circ\varphi$ that φ is a (single-valued) involution in some domain containing E.

We now turn to the construction of a (multivalued) function f (mentioned above) which is meromorphic everywhere in $\hat{C}$ except for a finite set of points, such that f maps Δ conformally onto $\hat{C}-\Gamma$ where Γ is a compact simple arc in C which is neither a line segment nor a circular arc. In this example Γ is part of an algebraic curve.

Let $F(u, v)$ be a (real) polynomial in u and v and Γ a compact simple arc in C, which satisfies the equation $F(u, v)=0$ and the condition $|F_u(u, v)|+|F_v(u, v)|\neq 0$ for all $(u, v)\in\Gamma$. By setting $u=(1/2)(w+\bar{w})$ and $v=(1/2)\,(w-\bar{w})$, $F(u, v)$ reduces to a polynomial $G(w, \bar{w})$ and since $G_{\bar{w}}=(1/2)(F_u+iF_v)\neq 0$ on Γ, it follows that $G(w, \bar{w})=0$ can be solved for $\bar{w}$ yielding a meromorphic function g which is locally univalent at every point of Γ and such that $g(w)=\bar{w}$ for all $w\in\Gamma$. Evidently, the function $\varphi=\bar{g}$ is a conformal reflection in Γ. Note that g is algebraic and hence it is meromorphic everywhere except for a finite set in $\hat{C}-\Gamma$.

Now let f be a conformal mapping of Δ onto $\hat{C}-\Gamma$. Then f can be continued by letting $f(z)=\varphi\big(f(1/\bar{z})\big)$ for $z\in\hat{C}-\Delta$. Since $\bar{\varphi}$ is algebraic, it follows that f is meromorphic everywhere in $\hat{C}$ except for a finite set. It is clear that in the above construction Γ need not be a line segment of a circular arc.

6. Lemma. *Let φ be a conformal reflection with respect to an infinite closed set E. If φ is single-valued (antimeromorphic) in $D=\hat{C}-F$, where F is a compact set in $C-E$ with $\Lambda_1(F)=0$, then*

(i) *E is contained in a straight line or in a circle.*

(ii) *The function φ is injective and $\bar{\varphi}$ is the restriction of a Möbius transformation.*

Proof of the lemma. $\Lambda_1(F)=0$ and φ is meromorphic, hence by using injective branches of φ^{-1} one sees that $\Lambda_1\big(\varphi^{-1}(F)\big)=0$. Consequently, $\Lambda_1\big(F\cup\varphi^{-1}(F)\big)=0$ and $D_0=D-\varphi^{-1}(F)$ is a domain. Let $\varphi_0=\varphi|D_0$. Then $h=\varphi\circ\varphi_0$ is a well defined single-valued meromorphic function in D_0. Now, $\varphi(w)=w$ for all $w\in E$, hence $h(w)=w$ for all $w\in E$, and since E is infinite and closed in $\hat{C}$, it follows that E clusters in the domain of h and thus $h(w)=w$ for all $w\in D_0$. Consequently, φ_0 is injective in D_0. Indeed, $\varphi(w_1)=\varphi(w_2)$ for $w_1, w_2\in D_0$ implies

$$w_1=\varphi\big(\varphi(w_1)\big)=\varphi\big(\varphi(w_2)\big)=w_2.$$

Next $\Lambda_1(F\cup\varphi^{-1}(F))=0$ and $\bar{\varphi}_0$ is conformal, hence, by a known removability theorem (*cf.* [1]) applied to $\bar{\varphi}_0$ it follows that $\bar{\varphi}_0$ has a conformal extension on $\hat{\boldsymbol{C}}$ and thus (ii) follows.

Finally, note that the fixed set of an anti-Möbius transformation is either finite or a circle or a line, thus (i) follows by (ii) and the fact that φ fixes every point of E.

7. *Proof of the theorem.* We first show that E admits a conformal reflection which satisfies the assumptions of the lemma. This will imply part (i) of the theorem.

Let $\psi(z)=1/\bar{z}$ and $F'=f_0\circ\psi(F)$. Evidently, F' is compact and $F'\cap E=\emptyset$, and since $f_0\circ\psi$ is a diffeomorphism in $\hat{\boldsymbol{C}}-\bar{\Delta}$ and $\Lambda_1(F)=0$, it follows that $\Lambda_1(F')=0$. Let $f_1=f|\hat{\boldsymbol{C}}-\Delta-F$, then

$$\varphi=f_1\circ\psi\circ(f_0^{-1}|\hat{\boldsymbol{C}}-E-F')$$

is antimeromorphic in $\hat{\boldsymbol{C}}-E-F'$. Now, f has only finitely many branch points on $\partial\Delta, f_0$ is a homeomorphism and int $E=\emptyset$, therefore

$$E=\partial f_0(\Delta)=f(\partial\Delta)$$

is a finite union of analytic arcs. Furthermore, φ has a continuous extension on $\hat{\boldsymbol{C}}-F'$ denoted again by φ, with $\varphi(w)=w$ for all $w\in E$. Hence, by applying Morera's theorem to $\bar{\varphi}$, or by [2, p. 183] it follows that φ is antimeromorphic in $\hat{\boldsymbol{C}}-F'$. Therefore, φ is a conformal reflection in E and satisfies the assumptions of the lemma. Thus, by part (i) of the lemma, E is either a circular arc or a line segment in $\hat{\boldsymbol{C}}$ as stated in part (i) of the theorem.

We now turn to part (ii) of the theorem. Since $E=f(\partial\Delta)$ is either a circular arc or a line segment in $\hat{\boldsymbol{C}}$, we may apply the standard reflection principle to f_0. By doing so noting that f_0 is univalent, we conclude that f is a rational function of degree two and that f maps its two branch points to the two tips of E. By using an auxiliary Möbius transformation φ which maps 0 and ∞ to the tips of E it is not hard to verify that f has the form (2) (*cf.* [3]). This completes the proof of the theorem.

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References

[1] BESICOVITCH, A. S.: On sufficient conditions for a function to be analytic, and on behaviour of analytic functions in the neighbourhood of non-isolated singular points. - Proc. London Math. Soc. (2) 32, 1931, 1—9.

[2] NEHARI, Z.: Conformal mapping. - McGraw-Hill Book Company, Inc., New York—Toronto—London, 1952.

[3] SREBRO, U.: Is the slit of rational slit mappings in *S* straight? - Proc. Amer. Math. Soc. (to appear).

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